CHAPTER 1

FIRST-ORDER DIFFERENTIAL EQUATIONS

SECTION 1.1

DIFFERENTIAL EQUATIONS AND MATHEMATICAL MODELS

The main purpose of Section 1.1 is simply to introduce the basic notation and terminology of differential equations, and to show the student what is meant by a solution of a differential equation. Also, the use of differential equations in the mathematical modeling of real-world phenomena is outlined.

Problems 1–12 are routine verifications by direct substitution of the suggested solutions into the given differential equations. We include here just some typical examples of such verifications.

3. If $y_1 = \cos 2x$ and $y_2 = \sin 2x$, then $y_1' = -2\sin 2x$ and $y_2' = 2\cos 2x$ so

$$y_1'' = -4\cos 2x = -4y_1 \quad \text{and} \quad y_2'' = -4\sin 2x = -4y_2.$$

Thus $y_1'' + 4y_1 = 0$ and $y_2'' + 4y_2 = 0$.

4. If $y_1 = e^{3x}$ and $y_2 = e^{-3x}$, then $y_1 = 3e^{3x}$ and $y_2 = -3e^{-3x}$ so

$$y_1'' = 9e^{3x} = 9y_1 \quad \text{and} \quad y_2'' = 9e^{-3x} = 9y_2.$$

5. If $y = e^x - e^{-x}$, then $y' = e^x + e^{-x}$ so $y' - y = \left(e^x + e^{-x}\right) - \left(e^x - e^{-x}\right) = 2e^{-x}$. Thus $y' = y + 2e^{-x}$.

6. If $y_1 = e^{-2x}$ and $y_2 = xe^{-2x}$, then $y_1' = -2e^{-2x}$, $y_1'' = 4e^{-2x}$, $y_2' = e^{-2x} - 2xe^{-2x}$, and $y_2'' = -4e^{-2x} + 4xe^{-2x}$. Hence

$$y_1'' + 4y_1' + 4y_1 = \left(4e^{-2x}\right) + 4\left(-2e^{-2x}\right) + 4\left(e^{-2x}\right) = 0$$

and

$$y_2'' + 4y_2' + 4y_2 = \left(-4e^{-2x} + 4xe^{-2x}\right) + 4\left(e^{-2x} - 2xe^{-2x}\right) + 4\left(xe^{-2x}\right) = 0.$$

8. If $y_1 = \cos x - \cos 2x$ and $y_2 = \sin x - \cos 2x$, then $y_1' = -\sin x + 2\sin 2x$, $y_1'' = -\cos x + 4\cos 2x$, and $y_2' = \cos x + 2\sin 2x$, $y_2'' = -\sin x + 4\cos 2x$. Hence

$$y_1'' + y_1 = (-\cos x + 4\cos 2x) + (\cos x - \cos 2x) = 3\cos 2x$$

and

$$y_2'' + y_2 = (-\sin x + 4\cos 2x) + (\sin x - \cos 2x) = 3\cos 2x.$$

11. If $y = y_1 = x^{-2}$ then $y' = -2x^{-3}$ and $y'' = 6x^{-4}$, so

$$x^2 y'' + 5x\,y' + 4y = x^2\left(6x^{-4}\right) + 5x\left(-2x^{-3}\right) + 4\left(x^{-2}\right) = 0.$$

If $y = y_2 = x^{-2}\ln x$ then $y' = x^{-3} - 2x^{-3}\ln x$ and $y'' = -5x^{-4} + 6x^{-4}\ln x$, so

$$\begin{aligned} x^2 y'' + 5x\,y' + 4y &= x^2\left(-5x^{-4} + 6x^{-4}\ln x\right) + 5x\left(x^{-3} - 2x^{-3}\ln x\right) + 4\left(x^{-2}\ln x\right) \\ &= \left(-5x^{-2} + 5x^{-2}\right) + \left(6x^{-2} - 10x^{-2} + 4x^{-2}\right)\ln x = 0. \end{aligned}$$

13. Substitution of $y = e^{rx}$ into $3y' = 2y$ gives the equation $3re^{rx} = 2e^{rx}$ that simplifies to $3r = 2$. Thus $r = 2/3$.

14. Substitution of $y = e^{rx}$ into $4y'' = y$ gives the equation $4r^2 e^{rx} = e^{rx}$ that simplifies to $4r^2 = 1$. Thus $r = \pm 1/2$.

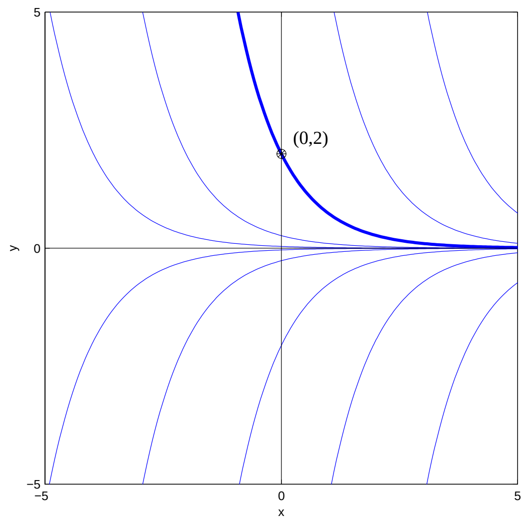
15. Substitution of $y = e^{rx}$ into $y'' + y' - 2y = 0$ gives the equation $r^2 e^{rx} + re^{rx} - 2e^{rx} = 0$ that simplifies to $r^2 + r - 2 = (r+2)(r-1) = 0$. Thus $r = -2$ or $r = 1$.

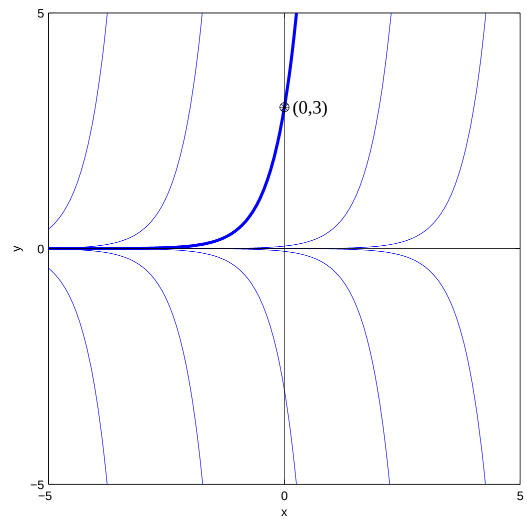
16. Substitution of $y = e^{rx}$ into $3y'' + 3y' - 4y = 0$ gives the equation $3r^2 e^{rx} + 3re^{rx} - 4e^{rx} = 0$ that simplifies to $3r^2 + 3r - 4 = 0$. The quadratic formula then gives the solutions $r = \left(-3 \pm \sqrt{57}\right)/6$.

The verifications of the suggested solutions in Problems 17–26 are similar to those in Problems 1–12. We illustrate the determination of the value of C only in some typical cases. However, we illustrate typical solution curves for each of these problems.

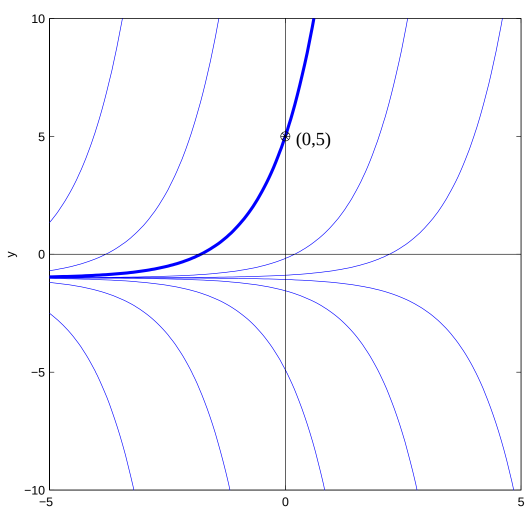
17. $C = 2$

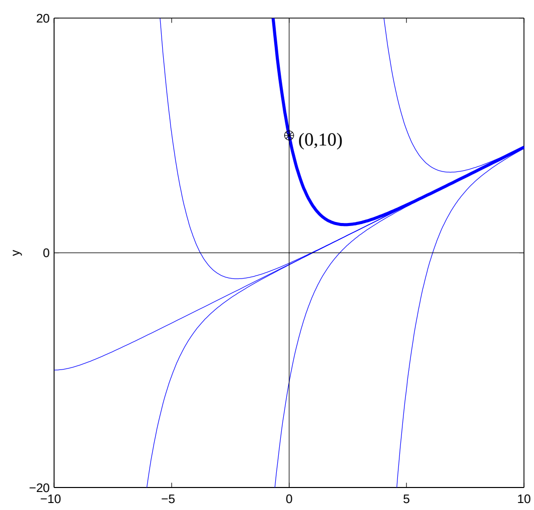
18. $C = 3$





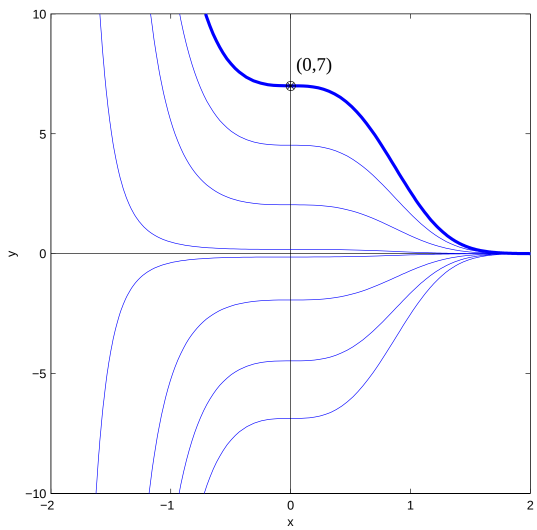
19. If $y(x) = Ce^x - 1$ then $y(0) = 5$ gives $C - 1 = 5$, so $C = 6$. The figure is on the left below.

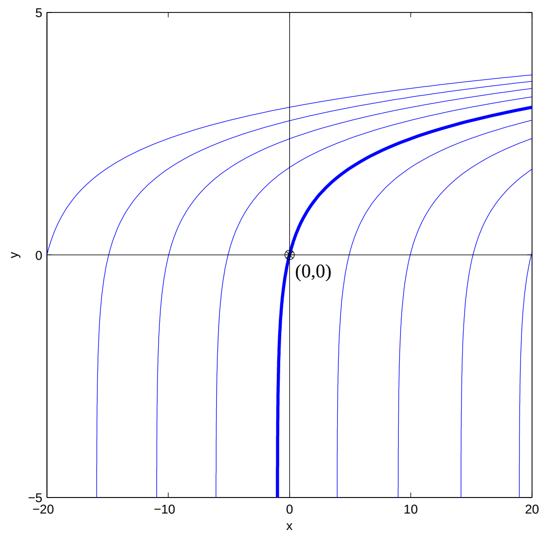




20. If $y(x) = Ce^{-x} + x - 1$ then $y(0) = 10$ gives $C - 1 = 10$, so $C = 11$. The figure is on the right above.

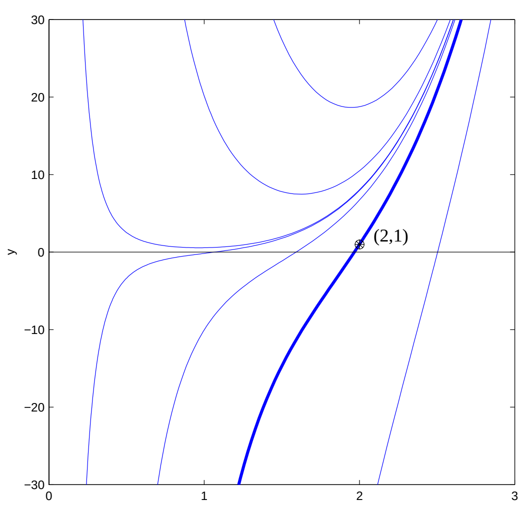
21. $C = 7$. The figure is on the left at the top of the next page.

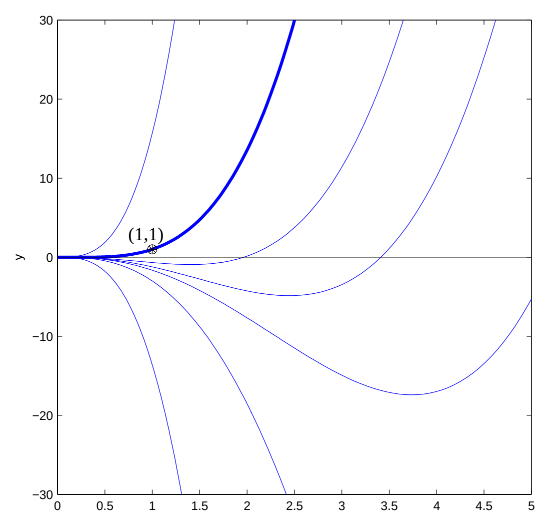




22. If $y(x) = \ln(x+C)$ then $y(0)=0$ gives $\ln C = 0$, so $C = 1$. The figure is on the right above.

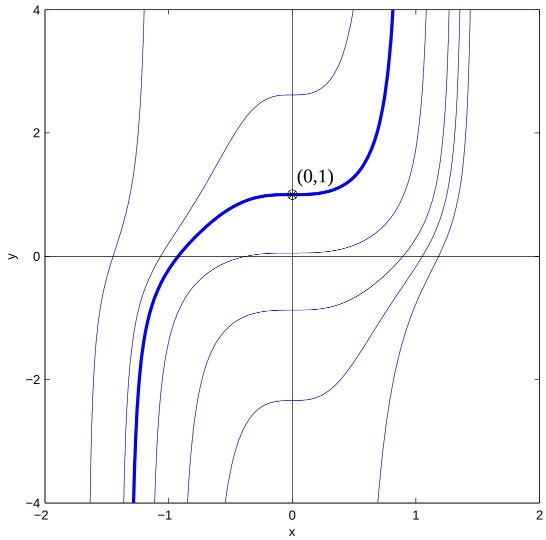
23. If $y(x) = \frac{1}{4}x^5 + C\,x^{-2}$ then $y(2)=1$ gives the equation $\frac{1}{4}\cdot 32 + C\cdot\frac{1}{8} = 1$ with solution $C = -56$. The figure is on the left below.



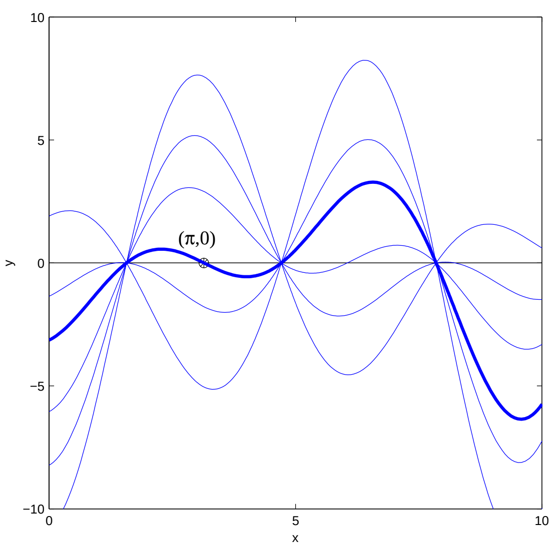


24. $C = 17$. The figure is on the right above.

25. If $y(x) = \tan(x^2 + C)$ then $y(0) = 1$ gives the equation $\tan C = 1$. Hence one value of C is $C = \pi/4$ (as is this value plus any integral multiple of π).



26. Substitution of $x = \pi$ and $y = 0$ into $y = (x + C)\cos x$ yields the equation $0 = (\pi + C)(-1)$, so $C = -\pi$.



27. $y' = x + y$

28. The slope of the line through (x, y) and $(x/2, 0)$ is $y' = (y - 0)/(x - x/2) = 2y/x$, so the differential equation is $x\,y' = 2\,y$.

29. If $m = y'$ is the slope of the tangent line and m' is the slope of the normal line at (x, y), then the relation $m\,m' = -1$ yields $m' = 1/y' = (y-1)/(x-0)$. Solution for y' then gives the differential equation $(1-y)y' = x$.

30. Here $m = y'$ and $m' = D_x(x^2+k) = 2x$, so the orthogonality relation $m\,m' = -1$ gives the differential equation $2x\,y' = -1$.

31. The slope of the line through (x, y) and $(-y, x)$ is $y' = (x-y)/(-y-x)$, so the differential equation is $(x+y)y' = y-x$.

In Problems 32–36 we get the desired differential equation when we replace the "time rate of change" of the dependent variable with its derivative, the word "is" with the = sign, the phrase "proportional to" with k, and finally translate the remainder of the given sentence into symbols.

32. $dP/dt = k\sqrt{P}$

33. $dv/dt = k\,v^2$

34. $dv/dt = k(250-v)$

35. $dN/dt = k(P-N)$

36. $dN/dt = k\,N(P-N)$

37. The second derivative of any linear function is zero, so we spot the two solutions $y(x) \equiv 1$ or $y(x) = x$ of the differential equation $y'' = 0$.

38. A function whose derivative equals itself, and hence a solution of the differential equation $y' = y$ is $y(x) = e^x$.

39. We reason that if $y = kx^2$, then each term in the differential equation is a multiple of x^2. The choice $k = 1$ balances the equation, and provides the solution $y(x) = x^2$.

40. If y is a constant, then $y' \equiv 0$ so the differential equation reduces to $y^2 = 1$. This gives the two constant-valued solutions $y(x) \equiv 1$ and $y(x) \equiv -1$.

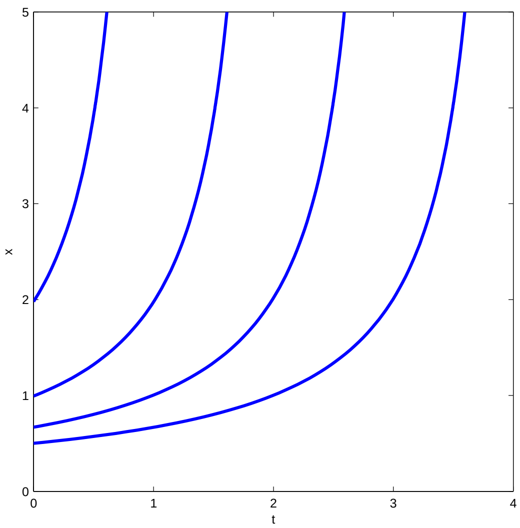
41. We reason that if $y = ke^x$, then each term in the differential equation is a multiple of e^x. The choice $k = \frac{1}{2}$ balances the equation, and provides the solution $y(x) = \frac{1}{2}e^x$.

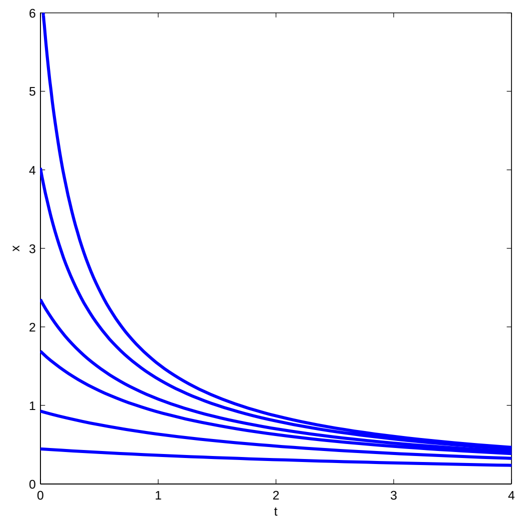
42. Two functions, each equaling the negative of its own second derivative, are the two solutions $y(x) = \cos x$ and $y(x) = \sin x$ of the differential equation $y'' = -y$.

43. **(a)** We need only substitute $x(t) = 1/(C - kt)$ in both sides of the differential equation $x' = kx^2$ for a routine verification.

(b) The zero-valued function $x(t) \equiv 0$ obviously satisfies the initial value problem $x' = kx^2,\ x(0) = 0$.

44. **(a)** The figure on the left below shows typical graphs of solutions of the differential equation $x' = \frac{1}{2}x^2$.





(b) The figure on the right above shows typical graphs of solutions of the differential equation $x' = -\frac{1}{2}x^2$. We see that — whereas the graphs with $k = \frac{1}{2}$ appear to "diverge to infinity" — each solution with $k = -\frac{1}{2}$ appears to approach 0 as $t \to \infty$. Indeed, we see from the Problem 43(a) solution $x(t) = 1/(C - \frac{1}{2}t)$ that $x(t) \to \infty$ as $t \to 2C$. However, with $k = -\frac{1}{2}$ it is clear from the resulting solution $x(t) = 1/(C + \frac{1}{2}t)$ that $x(t)$ remains bounded on any finite interval, but $x(t) \to 0$ as $t \to +\infty$.

45. Substitution of $P' = 1$ and $P = 10$ into the differential equation $P' = kP^2$ gives $k = \frac{1}{100}$, so Problem 43(a) yields a solution of the form $P(t) = 1/(C - t/100)$. The initial condition $P(0) = 2$ now yields $C = \frac{1}{2}$, so we get the solution

$$P(t) = \frac{1}{\frac{1}{2} - \frac{t}{100}} = \frac{100}{50 - t}.$$

We now find readily that $P = 100$ when $t = 49$, and that $P = 1000$ when $t = 49.9$. It appears that P grows without bound (and thus "explodes") as t approaches 50.

46. Substitution of $v' = -1$ and $v = 5$ into the differential equation $v' = kv^2$ gives $k = -\frac{1}{25}$, so Problem 43(a) yields a solution of the form $v(t) = 1/(C + t/25)$. The initial condition $v(0) = 10$ now yields $C = \frac{1}{10}$, so we get the solution

$$v(t) = \frac{1}{\frac{1}{10} + \frac{t}{25}} = \frac{50}{5 + 2t}.$$

We now find readily that $v = 1$ when $t = 22.5$, and that $v = 0.1$ when $t = 247.5$. It appears that v approaches 0 as t increases without bound. Thus the boat gradually slows, but never comes to a "full stop" in a finite period of time.

47. **(a)** $y(10) = 10$ yields $10 = 1/(C - 10)$, so $C = 101/10$.

(b) There is no such value of C, but the constant function $y(x) \equiv 0$ satisfies the conditions $y' = y^2$ and $y(0) = 0$.

(c) It is obvious visually (in Fig. 1.1.8 of the text) that one and only one solution curve passes through each point (a, b) of the *xy*-plane, so it follows that there exists a unique solution to the initial value problem $y' = y^2$, $y(a) = b$.

48. **(b)** Obviously the functions $u(x) = -x^4$ and $v(x) = +x^4$ both satisfy the differential equation $xy' = 4y$. But their derivatives $u'(x) = -4x^3$ and $v'(x) = +4x^3$ match at $x = 0$, where both are zero. Hence the given piecewise-defined function $y(x)$ is differentiable, and therefore satisfies the differential equation because $u(x)$ and $v(x)$ do so (for $x \le 0$ and $x \ge 0$, respectively).

(c) If $a \ge 0$ (for instance), choose C_+ fixed so that $C_+ a^4 = b$. Then the function

$$y(x) = \begin{cases} C_- x^4 & \text{if } x \le 0, \\ C_+ x^4 & \text{if } x \ge 0 \end{cases}$$

satisfies the given differential equation for every real number value of C_-.

SECTION 1.2

INTEGRALS AS GENERAL AND PARTICULAR SOLUTIONS

This section introduces **general solutions** and **particular solutions** in the very simplest situation — a differential equation of the form $y' = f(x)$ — where only direct integration and evaluation of the constant of integration are involved. Students should review carefully the elementary concepts of velocity and acceleration, as well as the fps and mks unit systems.

1. Integration of $y' = 2x+1$ yields $y(x) = \int (2x+1)\,dx = x^2+x+C$. Then substitution of $x=0$, $y=3$ gives $3 = 0+0+C = C$, so $y(x) = x^2+x+3$.

2. Integration of $y' = (x-2)^2$ yields $y(x) = \int (x-2)^2\,dx = \frac{1}{3}(x-2)^3+C$. Then substitution of $x=2$, $y=1$ gives $1 = 0+C = C$, so $y(x) = \frac{1}{3}(x-2)^3$.

3. Integration of $y' = \sqrt{x}$ yields $y(x) = \int \sqrt{x}\,dx = \frac{2}{3}x^{3/2}+C$. Then substitution of $x=4$, $y=0$ gives $0 = \frac{16}{3}+C$, so $y(x) = \frac{2}{3}(x^{3/2}-8)$.

4. Integration of $y' = x^{-2}$ yields $y(x) = \int x^{-2}\,dx = -1/x+C$. Then substitution of $x=1$, $y=5$ gives $5 = -1+C$, so $y(x) = -1/x+6$.

5. Integration of $y' = (x+2)^{-1/2}$ yields $y(x) = \int (x+2)^{-1/2}\,dx = 2\sqrt{x+2}+C$. Then substitution of $x=2$, $y=-1$ gives $-1 = 2\cdot 2+C$, so $y(x) = 2\sqrt{x+2}-5$.

6. Integration of $y' = x(x^2+9)^{1/2}$ yields $y(x) = \int x(x^2+9)^{1/2}\,dx = \frac{1}{3}(x^2+9)^{3/2}+C$. Then substitution of $x=-4$, $y=0$ gives $0 = \frac{1}{3}(5)^3+C$, so $y(x) = \frac{1}{3}\left[(x^2+9)^{3/2}-125\right]$.

7. Integration of $y' = 10/(x^2+1)$ yields $y(x) = \int 10/(x^2+1)\,dx = 10\tan^{-1}x+C$. Then substitution of $x=0$, $y=0$ gives $0 = 10\cdot 0+C$, so $y(x) = 10\tan^{-1}x$.

8. Integration of $y' = \cos 2x$ yields $y(x) = \int \cos 2x\,dx = \frac{1}{2}\sin 2x+C$. Then substitution of $x=0$, $y=1$ gives $1 = 0+C$, so $y(x) = \frac{1}{2}\sin 2x+1$.

9. Integration of $y' = 1/\sqrt{1-x^2}$ yields $y(x) = \int 1/\sqrt{1-x^2}\,dx = \sin^{-1}x+C$. Then substitution of $x=0$, $y=0$ gives $0 = 0+C$, so $y(x) = \sin^{-1}x$.

10. Integration of $y' = x e^{-x}$ yields

$$y(x) = \int x e^{-x} dx = \int u e^{u} du = (u-1)e^{u} = -(x+1)e^{-x} + C$$

(when we substitute $u = -x$ and apply Formula #46 inside the back cover of the textbook). Then substitution of $x = 0$, $y = 1$ gives $1 = -1 + C$, so $y(x) = -(x+1)e^{-x} + 2.$

11. If $a(t) = 50$ then $v(t) = \int 50\, dt = 50t + v_0 = 50t + 10.$ Hence

$$x(t) = \int (50t + 10)\, dt = 25t^2 + 10t + x_0 = 25t^2 + 10t + 20.$$

12. If $a(t) = -20$ then $v(t) = \int (-20)\, dt = -20t + v_0 = -20t - 15.$ Hence

$$x(t) = \int (-20t - 15)\, dt = -10t^2 - 15t + x_0 = -10t^2 - 15t + 5.$$

13. If $a(t) = 3t$ then $v(t) = \int 3t\, dt = \frac{3}{2}t^2 + v_0 = \frac{3}{2}t^2 + 5.$ Hence

$$x(t) = \int (\tfrac{3}{2}t^2 + 5)\, dt = \tfrac{1}{2}t^3 + 5t + x_0 = \tfrac{1}{2}t^3 + 5t.$$

14. If $a(t) = 2t + 1$ then $v(t) = \int (2t+1)\, dt = t^2 + t + v_0 = t^2 + t - 7.$ Hence

$$x(t) = \int (t^2 + t - 7)\, dt = \tfrac{1}{3}t^3 + \tfrac{1}{2}t - 7t + x_0 = \tfrac{1}{3}t^3 + \tfrac{1}{2}t - 7t + 4.$$

15. If $a(t) = 4(t+3)^2.$ then $v(t) = \int 4(t+3)^2\, dt = \frac{4}{3}(t+3)^3 + C = \frac{4}{3}(t+3)^3 - 37$ (taking $C = -37$ so that $v(0) = -1$). Hence

$$x(t) = \int \left[\tfrac{4}{3}(t+3)^3 - 37\right] dt = \tfrac{1}{3}(t+3)^4 - 37t + C = \tfrac{1}{3}(t+3)^4 - 37t - 26.$$

16. If $a(t) = 1/\sqrt{t+4}$ then $v(t) = \int 1/\sqrt{t+4}\, dt = 2\sqrt{t+4} + C = 2\sqrt{t+4} - 5$ (taking $C = -5$ so that $v(0) = -1$). Hence

$$x(t) = \int (2\sqrt{t+4} - 5)\, dt = \tfrac{4}{3}(t+4)^{3/2} - 5t + C = \tfrac{4}{3}(t+4)^{3/2} - 5t - \tfrac{29}{3}$$

(taking $C = -29/3$ so that $x(0) = 1$).

17. If $a(t) = (t+1)^{-3}$ then $v(t) = \int (t+1)^{-3}\,dt = -\frac{1}{2}(t+1)^{-2} + C = -\frac{1}{2}(t+1)^{-2} + \frac{1}{2}$ (taking $C = \frac{1}{2}$ so that $v(0) = 0$). Hence

$$x(t) = \int \left[-\tfrac{1}{2}(t+1)^{-2} + \tfrac{1}{2}\right] dt = \tfrac{1}{2}(t+1)^{-1} + \tfrac{1}{2}t + C = \tfrac{1}{2}\left[(t+1)^{-1} + t - 1\right]$$

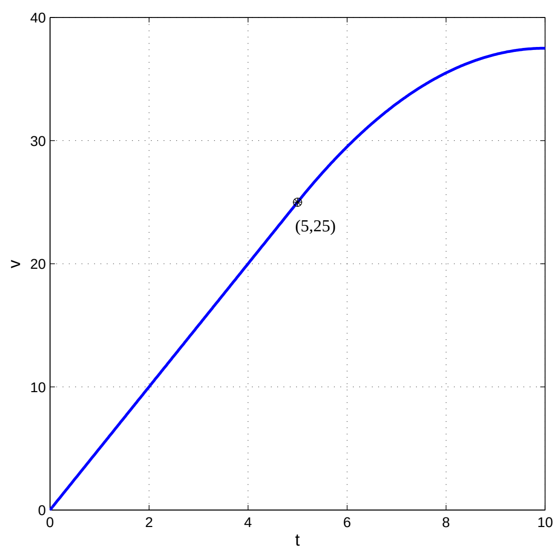
(taking $C = -\frac{1}{2}$ so that $x(0) = 0$).

18. If $a(t) = 50\sin 5t$ then $v(t) = \int 50\sin 5t\,dt = -10\cos 5t + C = -10\cos 5t$ (taking $C = 0$ so that $v(0) = -10$). Hence

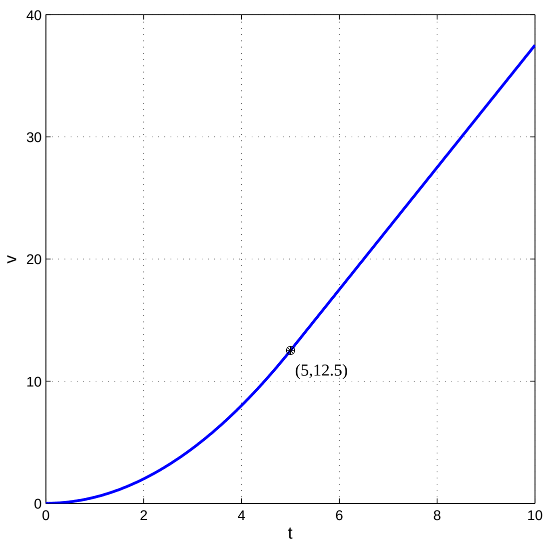
$$x(t) = \int(-10\cos 5t)\,dt = -2\sin 5t + C = -2\sin 5t + 10$$

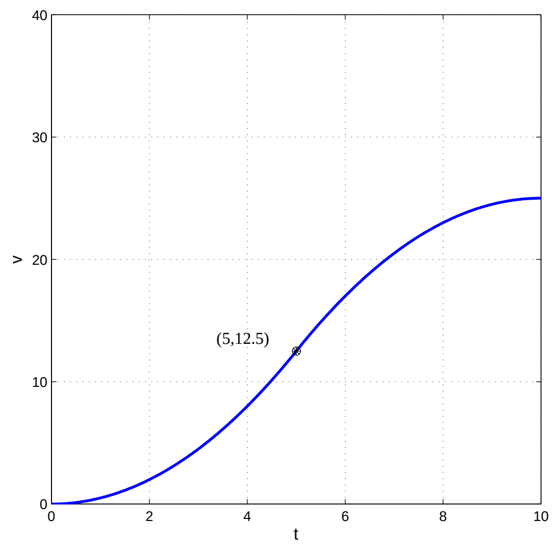
(taking $C = -10$ so that $x(0) = 8$).

19. Note that $v(t) = 5$ for $0 \le t \le 5$ and that $v(t) = 10 - t$ for $5 \le t \le 10$. Hence $x(t) = 5t + C_1$ for $0 \le t \le 5$ and $x(t) = 10t - \frac{1}{2}t^2 + C_2$ for $5 \le t \le 10$. Now $C_1 = 0$ because $x(0) = 0$, and continuity of $x(t)$ requires that $x(t) = 5t$ and $x(t) = 10t - \frac{1}{2}t^2 + C_2$ agree when $t = 5$. This implies that $C_2 = -\frac{25}{2}$, and we get the following graph.



20. Note that $v(t)=t$ for $0 \le t \le 5$ and that $v(t)=5$ for $5 \le t \le 10$. Hence $x(t)=\frac{1}{2}t^2+C_1$ for $0 \le t \le 5$ and $x(t)=5t+C_2$ for $5 \le t \le 10$. Now $C_1=0$ because $x(0)=0$, and continuity of $x(t)$ requires that $x(t)=\frac{1}{2}t^2$ and $x(t)=5t+C_2$ agree when $t=5$. This implies that $C_2=-\frac{25}{2}$, and we get the graph on the left below.



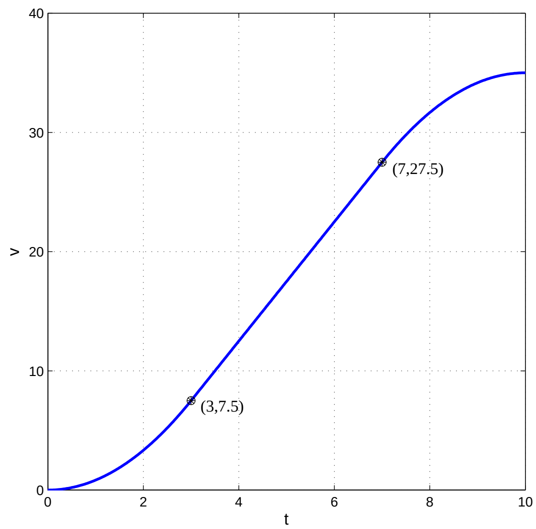


21. Note that $v(t)=t$ for $0 \le t \le 5$ and that $v(t)=10-t$ for $5 \le t \le 10$. Hence $x(t)=\frac{1}{2}t^2+C_1$ for $0 \le t \le 5$ and $x(t)=10t-\frac{1}{2}t^2+C_2$ for $5 \le t \le 10$. Now $C_1=0$ because $x(0)=0$, and continuity of $x(t)$ requires that $x(t)=\frac{1}{2}t^2$ and $x(t)=10t-\frac{1}{2}t^2+C_2$ agree when $t=5$. This implies that $C_2=-25$, and we get the graph on the right above.

22. For $0 \le t \le 3$: $v(t)=\frac{5}{3}t$ so $x(t)=\frac{5}{6}t^2+C_1$. Now $C_1=0$ because $x(0)=0$, so $x(t)=\frac{5}{6}t^2$ on this first interval, and its right endpoint value is $x(3)=7\frac{1}{2}$.

For $3 \le t \le 7$: $v(t)=5$ so $x(t)=5t+C_2$. Now $x(3)=7\frac{1}{2}$ implies that $C_2=-7\frac{1}{2}$, so $x(t)=5t-7\frac{1}{2}$ on this second interval, where its right endpoint value is $x(7)=27\frac{1}{2}$.

For $7 \le t \le 10$: $v-5=-\frac{5}{3}(t-7)$, so $v(t)=-\frac{5}{3}t+\frac{50}{3}$. Hence $x(t)=-\frac{5}{6}t^2+\frac{50}{3}t+C_3$, and $x(7)=27\frac{1}{2}$ implies that $C_3=-\frac{290}{6}$. Finally, $x(t)=\frac{1}{6}(-5t^2+100t-290)$ on this third interval, and we get the graph at the top of the next page.



23. $v = -9.8t + 49$, so the ball reaches its maximum height $(v = 0)$ after $t = 5$ seconds. Its maximum height then is $y(5) = -4.9(5)^2 + 49(5) = 122.5$ meters.

24. $v = -32t$ and $y = -16t^2 + 400$, so the ball hits the ground $(y = 0)$ when $t = 5$ sec, and then $v = -32(5) = -160$ ft/sec.

25. $a = -10\text{ m/s}^2$ and $v_0 = 100\text{ km/h} \approx 27.78\text{ m/s}$, so $v = -10t + 27.78$, and hence $x(t) = -5t^2 + 27.78t$. The car stops when $v = 0$, $t \approx 2.78$, and thus the distance traveled before stopping is $x(2.78) \approx 38.59$ meters.

26. $v = -9.8t + 100$ and $y = -4.9t^2 + 100t + 20$.

(a) $v = 0$ when $t = 100/9.8$ so the projectile's maximum height is $y(100/9.8) = -4.9(100/9.8)^2 + 100(100/9.8) + 20 \approx 530$ meters.

(b) It passes the top of the building when $y(t) = -4.9t^2 + 100t + 20 = 20$, and hence after $t = 100/4.9 \approx 20.41$ seconds.

(c) The roots of the quadratic equation $y(t) = -4.9t^2 + 100t + 20 = 0$ are $t = -0.20,\ 20.61$. Hence the projectile is in the air 20.61 seconds.

27. $a = -9.8\text{ m/s}^2$ so $v = -9.8\,t - 10$ and

$$y = -4.9\,t^2 - 10\,t + y_0.$$

The ball hits the ground when $y = 0$ and

$$v = -9.8\,t - 10 = -60,$$

so $t \approx 5.10$ s. Hence

$$y_0 = 4.9(5.10)^2 + 10(5.10) \approx 178.57\text{ m}.$$

28. $v = -32t - 40$ and $y = -16t^2 - 40t + 555$. The ball hits the ground ($y = 0$) when $t \approx 4.77$ sec, with velocity $v = v(4.77) \approx -192.64$ ft/sec, an impact speed of about 131 mph.

29. Integration of $dv/dt = 0.12\, t^3 + 0.6\, t$, $v(0) = 0$ gives $v(t) = 0.3\, t^2 + 0.04\, t^3$. Hence $v(10) = 70$. Then integration of $dx/dt = 0.3\, t^2 + 0.04\, t^3$, $x(0) = 0$ gives $x(t) = 0.1\, t^3 + 0.04\, t^4$, so $x(10) = 200$. Thus after 10 seconds the car has gone 200 ft and is traveling at 70 ft/sec.

30. Taking $x_0 = 0$ and $v_0 = 60$ mph $= 88$ ft/sec, we get

$$v = -at + 88,$$

and $v = 0$ yields $t = 88/a$. Substituting this value of t and $x = 176$ in

$$x = -at^2/2 + 88t,$$

we solve for $a = 22$ ft/sec^2. Hence the car skids for $t = 88/22 = 4$ sec.

31. If $a = -20$ m/sec^2 and $x_0 = 0$ then the car's velocity and position at time t are given by

$$v = -20t + v_0, \quad x = -10\, t^2 + v_0 t.$$

It stops when $v = 0$ (so $v_0 = 20t$), and hence when

$$x = 75 = -10\, t^2 + (20t)t = 10\, t^2.$$

Thus $t = \sqrt{7.5}$ sec so

$$v_0 = 20\sqrt{7.5} \approx 54.77 \text{ m/sec} \approx 197 \text{ km/hr}.$$

32. Starting with $x_0 = 0$ and $v_0 = 50$ km/h $= 5\times10^4$ m/h, we find by the method of Problem 30 that the car's deceleration is $a = (25/3)\times10^7$ m/h^2. Then, starting with $x_0 = 0$ and $v_0 = 100$ km/h $= 10^5$ m/h, we substitute $t = v_0/a$ into

$$x = -at^2/2 + v_0 t$$

and find that $x = 60$ m when $v = 0$. Thus doubling the initial velocity quadruples the distance the car skids.

33. If $v_0 = 0$ and $y_0 = 20$ then

$$v = -at \text{ and } y = -\tfrac{1}{2}at^2 + 20.$$

Substitution of $t = 2$, $y = 0$ yields $a = 10$ ft/sec^2. If $v_0 = 0$ and $y_0 = 200$ then

$$v = -10t \quad \text{and} \quad y = -5t^2 + 200.$$

Hence $y = 0$ when $t = \sqrt{40} = 2\sqrt{10}$ sec and $v = -20\sqrt{10} \approx -63.25$ ft/sec.

34. **On Earth:** $v = -32t + v_0$, so $t = v_0/32$ at maximum height (when $v = 0$). Substituting this value of t and $y = 144$ in

$$y = -16t^2 + v_0 t,$$

we solve for $v_0 = 96$ ft/sec as the initial speed with which the person can throw a ball straight upward.

On Planet Gzyx: From Problem 27, the surface gravitational acceleration on planet Gzyx is $a = 10$ ft/sec^2, so

$$v = -10t + 96 \quad \text{and} \quad y = -5t^2 + 96t.$$

Therefore $v = 0$ yields $t = 9.6$ sec, and thence $y_{max} = y(9.6) = 460.8$ ft is the height a ball will reach if its initial velocity is 96 ft/sec.

35. If $v_0 = 0$ and $y_0 = h$ then the stone′s velocity and height are given by

$$v = -gt, \quad y = -0.5\, gt^2 + h.$$

Hence $y = 0$ when $t = \sqrt{2h/g}$ so

$$v = -g\sqrt{2h/g} = -\sqrt{2gh}\,.$$

36. The method of solution is precisely the same as that in Problem 30. We find first that, on Earth, the woman must jump straight upward with initial velocity $v_0 = 12$ ft/sec to reach a maximum height of 2.25 ft. Then we find that, on the Moon, this initial velocity yields a maximum height of about 13.58 ft.

37. We use units of miles and hours. If $x_0 = v_0 = 0$ then the car′s velocity and position after t hours are given by

$$v = at, \quad x = \tfrac{1}{2}t^2.$$

Since $v = 60$ when $t = 5/6$, the velocity equation yields $a = 72$ mi/hr^2. Hence the distance traveled by 12:50 pm is

$$x = (0.5)(72)(5/6)^2 = 25 \text{ miles}.$$

38. Again we have

$$v = at, \qquad x = \tfrac{1}{2}t^2.$$

But now $v = 60$ when $x = 35$. Substitution of $a = 60/t$ (from the velocity equation) into the position equation yields

$$35 = (0.5)(60/\text{t})(t^2) = 30t,$$

whence $t = 7/6$ hr, that is, 1:10 p.m.

39. Integration of $y' = (9/v_S)(1 - 4x^2)$ yields

$$y = (3/v_S)(3x - 4x^3) + C,$$

and the initial condition $y(-1/2) = 0$ gives $C = 3/v_S$. Hence the swimmer′s trajectory is

$$y(x) = (3/v_S)(3x - 4x^3 + 1).$$

Substitution of $y(1/2) = 1$ now gives $v_S = 6$ mph.

40. Integration of $y' = 3(1 - 16x^4)$ yields

$$y = 3x - (48/5)x^5 + C,$$

and the initial condition $y(-1/2) = 0$ gives $C = 6/5$. Hence the swimmer′s trajectory is

$$y(x) = (1/5)(15x - 48x^5 + 6),$$

so his downstream drift is $y(1/2) = 2.4$ miles.

41. The bomb equations are $a = -32$, $v = -32$, and $s_B = s = -16t^2 + 800$, with $t = 0$ at the instant the bomb is dropped. The projectile is fired at time $t = 2$, so its corresponding equations are $a = -32$, $v = -32(t-2) + v_0$, and

$$s_P = s = -16(t-2)^2 + v_0(t-2)$$

for $t \geq 2$ (the arbitrary constant vanishing because $s_P(2) = 0$). Now the condition $s_B(t) = -16t^2 + 800 = 400$ gives $t = 5$, and then the requirement that $s_P(5) = 400$ also yields $v_0 = 544/3 \approx 181.33$ ft/s for the projectile's needed initial velocity.

42. Let $x(t)$ be the (positive) altitude (in miles) of the spacecraft at time t (hours), with $t = 0$ corresponding to the time at which the its retrorockets are fired; let $v(t) = x'(t)$ be the velocity of the spacecraft at time t. Then $v_0 = -1000$ and $x_0 = x(0)$ is unknown. But the (constant) acceleration is $a = +20000$, so

$$v(t) \;=\; 20000t - 1000 \quad \text{and} \quad x(t) = 10000t^2 - 1000t + x_0.$$

Now $v(t) \;=\; 20000t - 1000 \;=\; 0$ (soft touchdown) when $t = \frac{1}{20}$ hr (that is, after exactly 3 minutes of descent. Finally, the condition

$$0 \;=\; x(\tfrac{1}{20}) = 10000(\tfrac{1}{20})^2 - 1000(\tfrac{1}{20}) + x_0$$

yields $x_0 = 25$ miles for the altitude at which the retrorockets should be fired.

43. The velocity and position functions for the spacecraft are $v_S(t) = 0.0098t$ and $x_S(t) = 0.0049t^2$, and the corresponding functions for the projectile are $v_P(t) = \frac{1}{10}c = 3\times 10^7$ and $x_P(t) = 3\times 10^7 t$. The condition that $x_S = x_P$ when the spacecraft overtakes the projectile gives $0.0049t^2 = 3\times 10^7 t$, whence

$$\begin{aligned} t \;&=\; \frac{3\times 10^7}{0.0049} \;\approx\; 6.12245\times 10^9 \;\text{ sec} \\ &\approx\; \frac{6.12245\times 10^9}{(3600)(24)(365.25)} \;\approx\; 194 \;\text{ years.} \end{aligned}$$

Since the projectile is traveling at $\frac{1}{10}$ the speed of light, it has then traveled a distance of about 19.4 light years, which is about 1.8367×10^{17} meters.

44. Let $a > 0$ denote the constant deceleration of the car when braking, and take $x_0 = 0$ for the cars position at time $t = 0$ when the brakes are applied. In the police experiment with $v_0 = 25$ ft/s, the distance the car travels in t seconds is given by

$$x(t) \;=\; -\frac{1}{2}at^2 + \frac{88}{60}\cdot 25t$$

(with the factor $\frac{88}{60}$ used to convert the velocity units from mi/hr to ft/s). When we solve simultaneously the equations $x(t) = 45$ and $x'(t) = 0$ we find that $a = \frac{1210}{81} \approx 14.94$ ft/s^2. With this value of the deceleration and the (as yet) unknown velocity v_0 of the car involved in the accident, it position function is

$$x(t) \;=\; -\frac{1}{2}\cdot\frac{1210}{81}t^2 + v_0 t.$$

The simultaneous equations $x(t) = 210$ and $x'(t) = 0$ finally yield $v_0 = \frac{110}{9}\sqrt{42} \approx 79.21$ ft/s, almost exactly 54 miles per hour.

SECTION 1.3

SLOPE FIELDS AND SOLUTION CURVES

The instructor may choose to delay covering Section 1.3 until later in Chapter 1. However, before proceeding to Chapter 2, it is important that students come to grips at some point with the question of the existence of a unique solution of a differential equation — and realize that it makes no sense to look for the solution without knowing in advance that it exists. It may help some students to simplify the statement of the existence-uniqueness theorem as follows:

> Suppose that the function $f(x, y)$ and the partial derivative $\partial f / \partial y$ are both continuous in some neighborhood of the point (a, b). Then the initial value problem
>
> $$\frac{dy}{dx} = f(x, y), \qquad y(a) = b$$
>
> has a unique solution in some neighborhood of the point a.

Slope fields and geometrical solution curves are introduced in this section as a concrete aid in visualizing solutions and existence-uniqueness questions. Instead, we provide some details of the construction of the figure for the Problem 1 answer, and then include without further comment the similarly constructed figures for Problems 2 through 9.

1. The following sequence of *Mathematica* commands generates the slope field and the solution curves through the given points. Begin with the differential equation $dy/dx = f(x, y)$ where

```
f[x_, y_] := -y - Sin[x]
```

Then set up the viewing window

```
a = -3; b = 3; c = -3; d = 3;
```

The components (u, v) of unit vectors corresponding to the short slope field line segments are given by

```
u[x_, y_] := 1/Sqrt[1 + f[x, y]^2]
v[x_, y_] := f[x, y]/Sqrt[1 + f[x, y]^2]
```

The slope field is then constructed by the commands

```
Needs["Graphics`PlotField`"]
dfield = PlotVectorField[{u[x, y], v[x, y]}, {x, a, b}, {y, c, d},
         HeadWidth -> 0, HeadLength -> 0, PlotPoints -> 19,
         PlotRange -> {{a, b}, {c, d}}, Axes -> True, Frame -> True,
         FrameLabel -> {"x", "y"}, AspectRatio -> 1];
```

The original curve shown in Fig. 1.3.12 of the text (and its initial point not shown there) are plotted by the commands

```
x0 = -1.9; y0 = 0;
point0 = Graphics[{PointSize[0.025], Point[{x0, y0}]}];
soln = NDSolve[{Derivative[1][y][x] == f[x, y[x]], y[x0] == y0},
                y[x], {x, a, b}];
soln[[1,1,2]];
curve0 = Plot[soln[[1,1,2]], {x, a, b},
             PlotStyle -> {Thickness[0.0065], RGBColor[0, 0, 1]}];
```

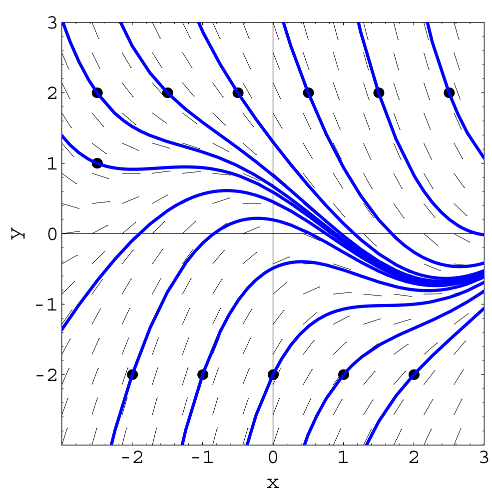
The *Mathematica* **NDSolve** command carries out an approximate numerical solution of the given differential equation. Numerical solution techniques are discussed in Sections 2.4–2.6 of the textbook.

The coordinates of the 12 points are marked in Fig. 1.3.12 in the textbook. For instance the 7th point is (–2.5, 1). It and the corresponding solution curve are plotted by the commands

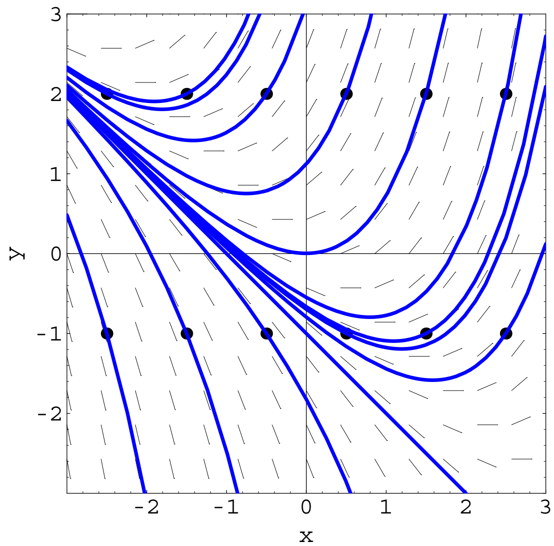
```
x0 = -2.5; y0 = 1;
point7 = Graphics[{PointSize[0.025], Point[{x0, y0}]}];
soln = NDSolve[{Derivative[1][y][x] == f[x, y[x]], y[x0] == y0},
                y[x], {x, a, b}];
soln[[1,1,2]];
curve7 = Plot[soln[[1,1,2]], {x, a, b},
             PlotStyle -> {Thickness[0.0065], RGBColor[0, 0, 1]}];
```

Finally, the desired figure is assembled by the *Mathematica* command

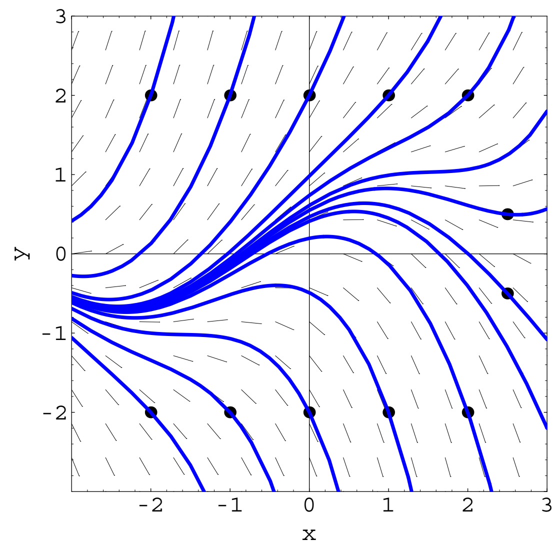
```
Show[ dfield, point0,curve0,
       point1,curve1, point2,curve2, point3,curve3,
       point4,curve4, point5,curve5, point6,curve6,
       point7,curve7, point8,curve8, point9,curve9,
       point10,curve10, point11,curve11, point12,curve12];
```



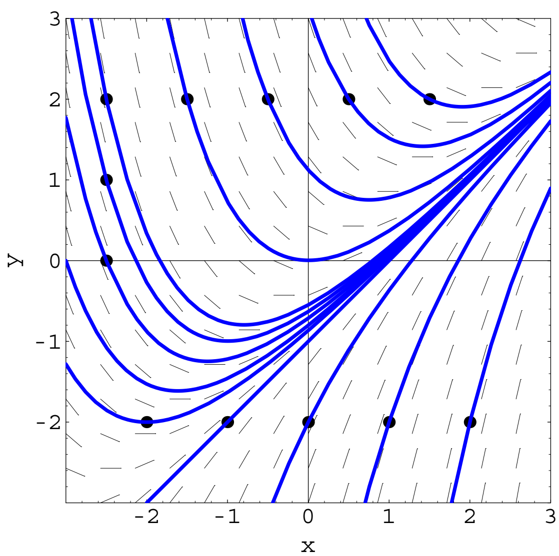
2.



3.



4.



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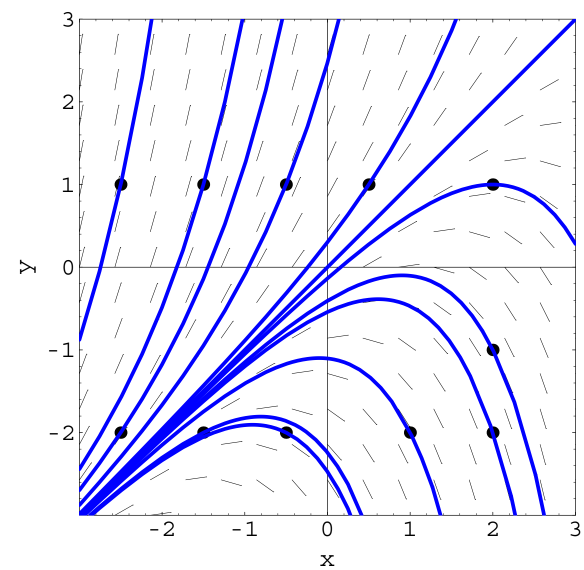

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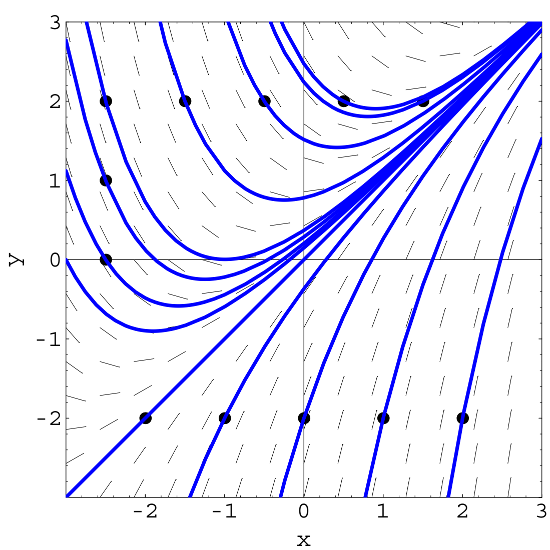
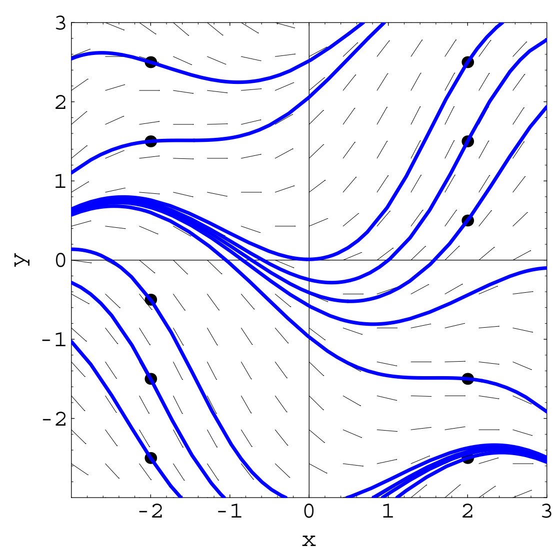
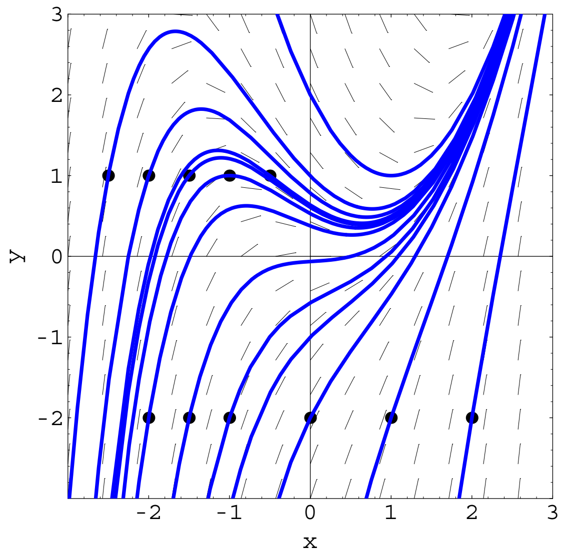


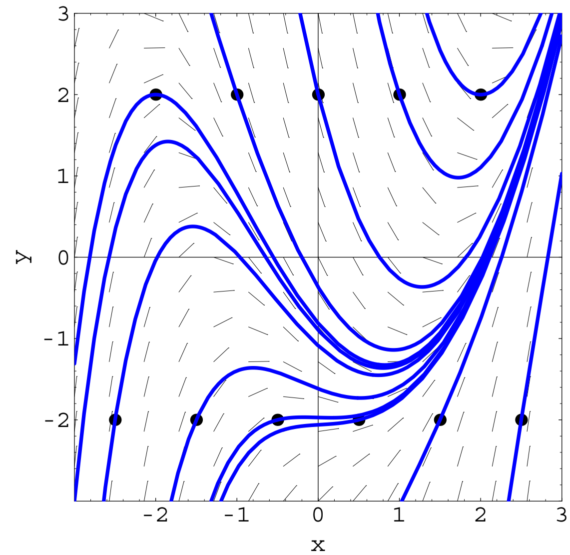
7.



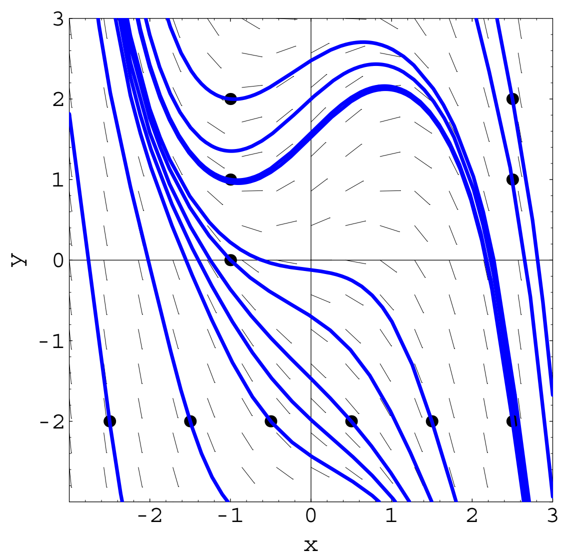
8.



9.



10.



11. Because both $f(x, y) = 2x^2y^2$ and $\partial f / \partial y = 4x^2y$ are continuous everywhere, the existence-uniqueness theorem of Section 1.3 in the textbook guarantees the existence of a unique solution in some neighborhood of $x = 1$.

12. Both $f(x, y) = x \ln y$ and $\partial f / \partial y = x/y$ are continuous in a neighborhood of $(1, 1)$, so the theorem guarantees the existence of a unique solution in some neighborhood of $x = 1$.

13. Both $f(x, y) = y^{1/3}$ and $\partial f / \partial y = (1/3)y^{-2/3}$ are continuous near $(0, 1)$, so the theorem guarantees the existence of a unique solution in some neighborhood of $x = 0$.

14. $f(x,y) = y^{1/3}$ is continuous in a neighborhood of $(0, 0)$, but $\partial f/\partial y = (1/3)y^{-2/3}$ is not, so the theorem guarantees existence but not uniqueness in some neighborhood of $x = 0$.

15. $f(x,y) = (x - y)^{1/2}$ is not continuous at $(2, 2)$ because it is not even defined if $y > x$. Hence the theorem guarantees neither existence nor uniqueness in any neighborhood of the point $x = 2$.

16. $f(x,y) = (x - y)^{1/2}$ and $\partial f/\partial y = -(1/2)(x - y)^{-1/2}$ are continuous in a neighborhood of $(2, 1)$, so the theorem guarantees both existence and uniqueness of a solution in some neighborhood of $x = 2$.

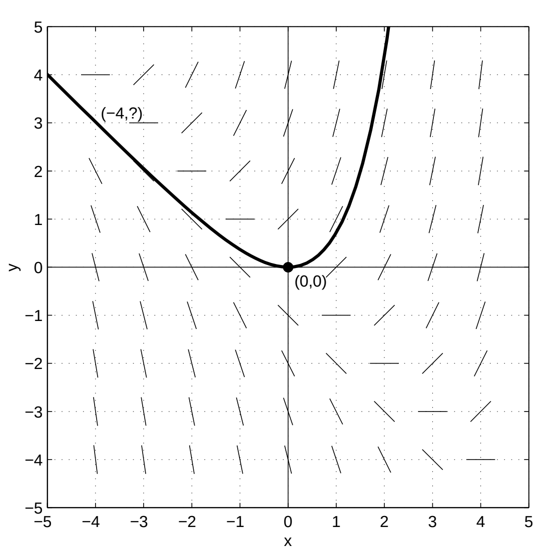
17. Both $f(x,y) = (x - 1/y$ and $\partial f/\partial y = -(x - 1)/y^2$ are continuous near $(0, 1)$, so the theorem guarantees both existence and uniqueness of a solution in some neighborhood of $x = 0$.

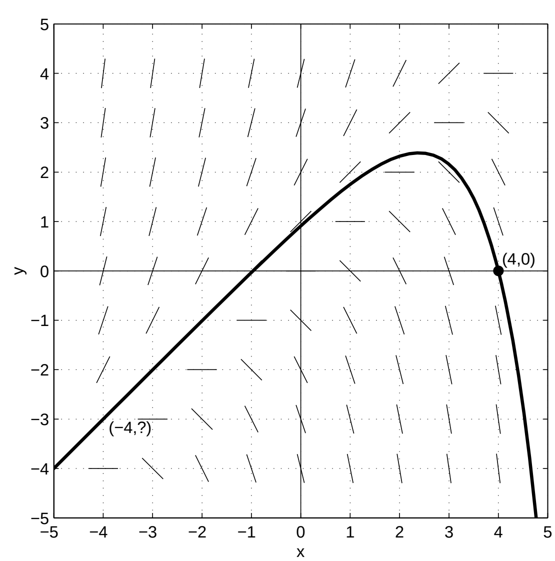
18. Neither $f(x,y) = (x - 1)/y$ nor $\partial f/\partial y = -(x - 1)/y^2$ is continuous near $(1, 0)$, so the existence-uniqueness theorem guarantees nothing.

19. Both $f(x,y) = \ln(1 + y^2)$ and $\partial f/\partial y = 2y/(1 + y^2)$ are continuous near $(0, 0)$, so the theorem guarantees the existence of a unique solution near $x = 0$.

20. Both $f(x,y) = x^2 - y^2$ and $\partial f/\partial y = -2y$ are continuous near $(0, 1)$, so the theorem guarantees both existence and uniqueness of a solution in some neighborhood of $x = 0$.

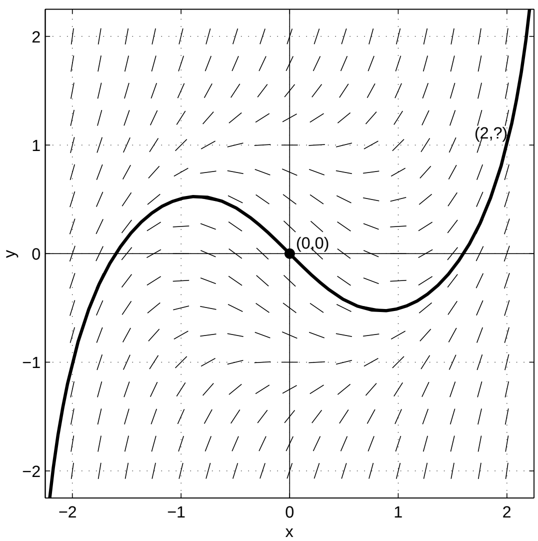
21. The curve in the figure on the left below can be constructed using the commands illustrated in Problem 1 above. Tracing this solution curve, we see that $y(-4) \approx 3$. An exact solution of the differential equation yields the more accurate approximation $y(-4) \approx 3.0183$.

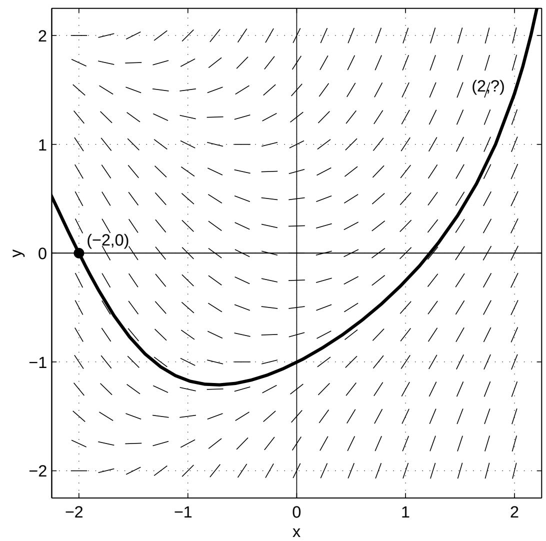




22. Tracing the curve in the figure on the right at the bottom of the preceding page , we see that $y(-4) \approx -3$. An exact solution of the differential equation yields the more accurate approximation $y(-4) \approx -3.0017$.

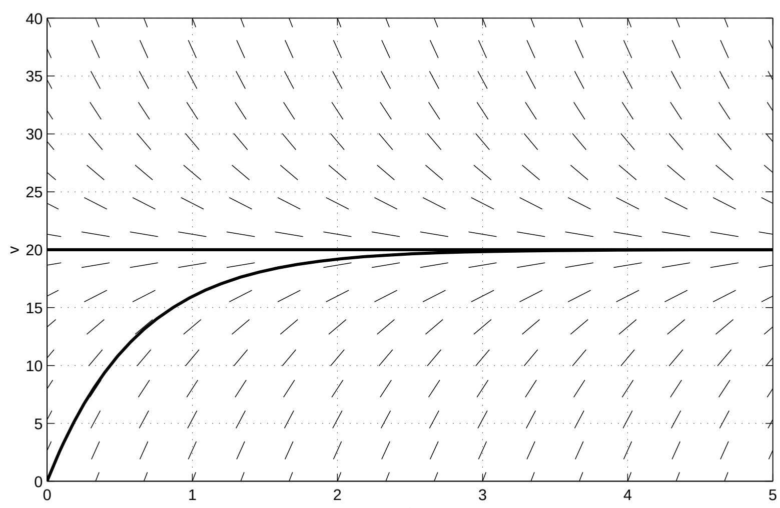
23. Tracing the curve in figure on the left below, we see that $y(2) \approx 1$. A more accurate approximation is $y(2) \approx 1.0044$.



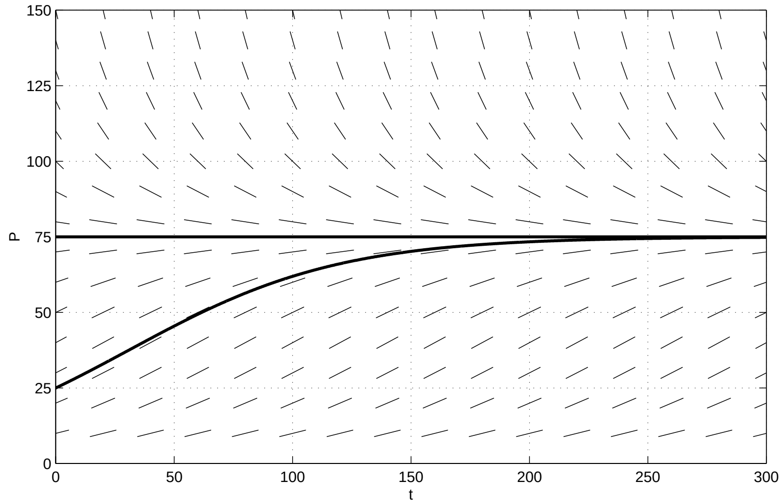


24. Tracing the curve in the figure on the right above, we see that $y(2) \approx 1.5$. A more accurate approximation is $y(2) \approx 1.4633$.

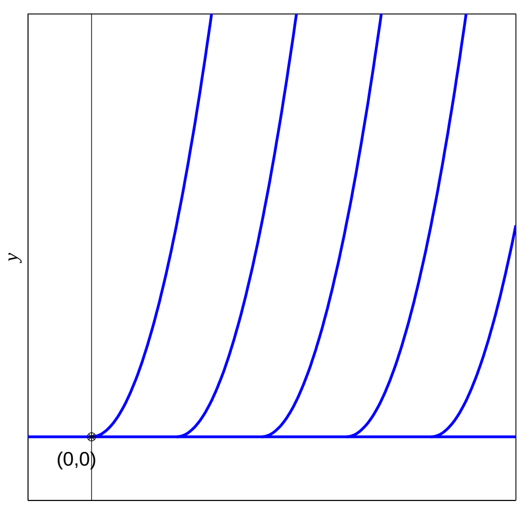
25. The figure below indicates a limiting velocity of 20 ft/sec — about the same as jumping off a $6\frac{1}{4}$-foot wall, and hence quite survivable. Tracing the curve suggests that $v(t) = 19$ ft/sec when t is a bit less than 2 seconds. An exact solution gives $t \approx 1.8723$ then.

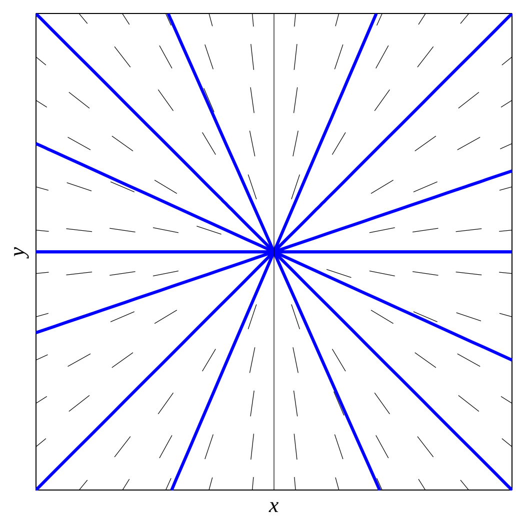


26. The figure below suggests that there are 40 deer after about 60 months; a more accurate value is $t \approx 61.61$. And it's pretty clear that the limiting population is 75 deer.



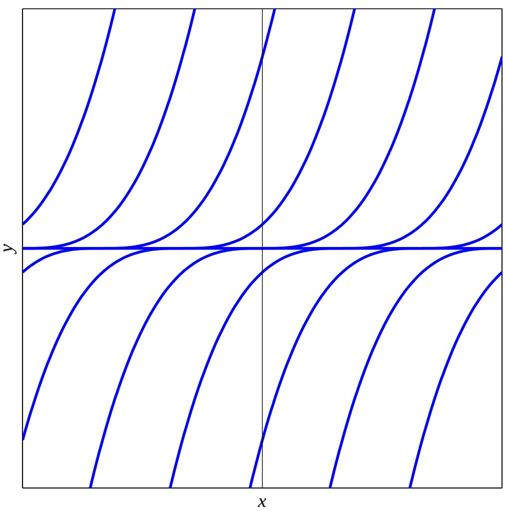
27. If $b < 0$ then the initial value problem $y' = 2\sqrt{y},\ y(0) = b$ has no solution, because the square root of a negative number would be involved. If $b > 0$ we get a unique solution curve through $(0, b)$ defined for all x by following a parabola — in the figure on the left below — down (and leftward) to the x-axis and then following the x-axis to the left. But starting at (0,0) we can follow the positive x-axis to the point $(c, 0)$ and then branching off on the parabola $y = (x - c)^2$. This gives infinitely many different solutions if $b = 0$.

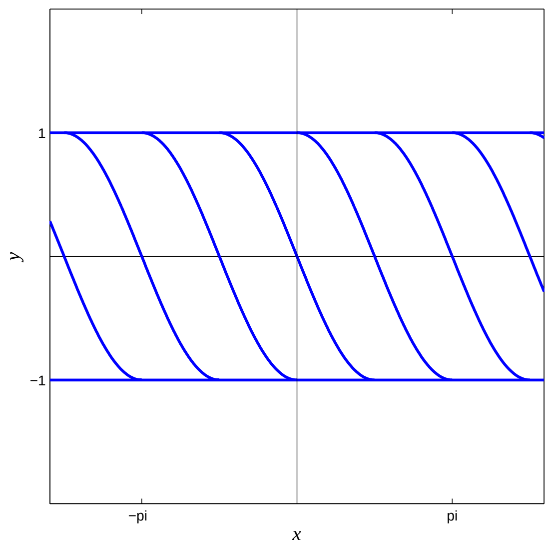




28. The figure on the right above makes it clear initial value problem $xy' = y,\ y(a) = b$ has a unique solution off the y-axis where $a \neq 0$; infinitely many solutions through the origin where $a = b = 0$; no solution if $a = 0$ but $b \neq 0$ (so the point (a, b) lies on the positive or negative y-axis).

29. Looking at the figure on the left below, we see that we can start at the point (a,b) and follow a branch of a cubic up or down to the x-axis, then follow the x-axis an arbitrary distance before branching off (down or up) on another cubic. This gives infinitely many solutions of the initial value problem $y' = 3y^{2/3}$, $y(a) = b$ that are defined for all x. However, if $b \neq 0$ there is only a single cubic $y = (x-c)^3$ passing through (a,b), so the solution is unique near $x = a$.





30. The function $y(x) = \cos(x-c)$, with $y'(x) = -\sin(x-c)$, satisfies the differential equation $y' = -\sqrt{1-y^2}$ on the interval $c < x < c + \pi$ where $\sin(x-c) > 0$, so it follows that

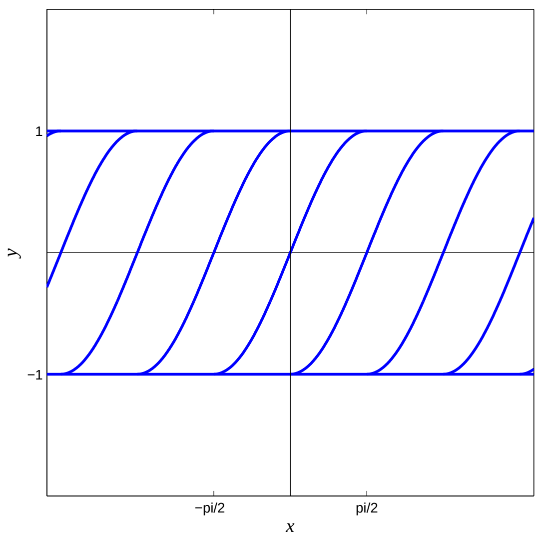
$$-\sqrt{1-y^2} \;=\; -\sqrt{1-\cos^2(x-c)} \;=\; -\sqrt{\sin^2(x-c)} \;=\; -\sin(x-c) \;=\; y.$$

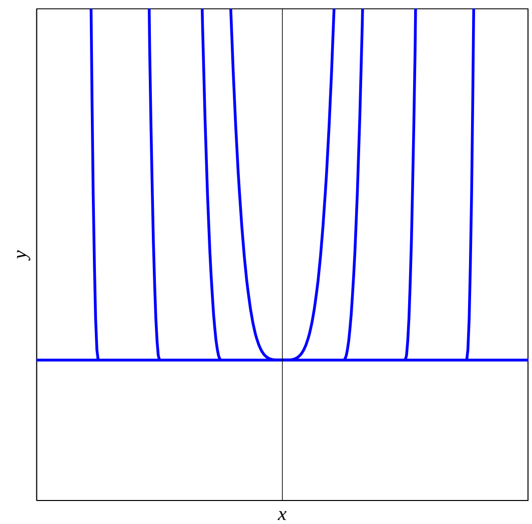
If $|b| > 1$ then the initial value problem $y' = -\sqrt{1-y^2}$, $y(a) = b$ has no solution because the square root of a negative number would be involved. If $|b| < 1$ then there is only one curve of the form $y = \cos(x-c)$ through the point (a,b); this give a unique solution. But if $b = \pm 1$ then we can combine a left ray of the line $y = +1$, a cosine curve from the line $y = +1$ to the line $y = -1$, and then a right ray of the line $y = -1$. Looking at the figure on the right above, we see that this gives infinitely many solutions (defined for all x) through any point of the form $(a, \pm 1)$.

31. The function $y(x) = \sin(x-c)$, with $y'(x) = \cos(x-c)$, satisfies the differential equation $y' = \sqrt{1-y^2}$ on the interval $c - \pi/2 < x < c + \pi/2$ where $\cos(x-c) > 0$, so it follows that

$$\sqrt{1-y^2} \;=\; \sqrt{1-\sin^2(x-c)} \;=\; \sqrt{\cos^2(x-c)} \;=\; -\sin(x-c) \;=\; y.$$

If $|b|>1$ then the initial value problem $y'=\sqrt{1-y^2}$, $y(a)=b$ has no solution because the square root of a negative number would be involved. If $|b|<1$ then there is only one curve of the form $y=\sin(x-c)$ through the point (a,b); this give a unique solution. But if $b=\pm 1$ then we can combine a left ray of the line $y=-1$, a sine curve from the line $y=-1$ to the line $y=+1$, and then a right ray of the line $y=+1$. Looking at the figure on the left below, we see that this gives infinitely many solutions (defined for all x) through any point of the form $(a,\pm 1)$.





32. Looking at the figure on the right above, we see that we can piece together a "left half" of a quartic for x negative, an interval along the x-axis, and a "right half" of a quartic curve for x positive. This makes it clear he initial value problem $y'=4x\sqrt{y}$, $y(a)=b$ has infinitely many solutions (defined for all x) if $b\geq 0$; there is no solution if $b<0$ because this would involve the square root of a negative number.

33. Looking at the figure provided in the answers section of the textbook, it suffices to observe that, among the pictured curves $y=x/(cx-1)$ for all possible values of c,

- there is a unique one of these curves through any point not on either coordinate axis;
- there is no such curve through any point on the y-axis other than the origin; and
- there are infinitely many such curves through the origin (0,0).

But in addition we have the constant-valued solution $y(x)\equiv 0$ that "covers" the x-axis. It follows that the given differential equation has near (a,b)

- a unique solution if $a\neq 0$;
- no solution if $a=0$ but $b\neq 0$;
- infinitely many different solutions if $a=b=0$.

SECTION 1.4

SEPARABLE EQUATIONS AND APPLICATIONS

Of course it should be emphasized to students that the possibility of separating the variables is the first one you look for. The general concept of natural growth and decay is important for all differential equations students, but the particular applications in this section are optional. Torricelli's law in the form of Equation (24) in the text leads to some nice concrete examples and problems.

1. $\displaystyle\int \frac{dy}{y} = -\int 2x\,dx; \quad \ln y = -x^2 + c; \quad y(x) = e^{-x^2+c} = Ce^{-x^2}$

2. $\displaystyle\int \frac{dy}{y^2} = -\int 2x\,dx; \quad -\frac{1}{y} = -x^2 - C; \quad y(x) = \frac{1}{x^2 + C}$

3. $\displaystyle\int \frac{dy}{y} = \int \sin x\,dx; \quad \ln y = -\cos x + c; \quad y(x) = e^{-\cos x + c} = Ce^{-\cos x}$

4. $\displaystyle\int \frac{dy}{y} = \int \frac{4\,dx}{1+x}; \quad \ln y = 4\ln(1+x) + \ln C; \quad y(x) = C(1+x)^4$

5. $\displaystyle\int \frac{dy}{\sqrt{1-y^2}} = \int \frac{dx}{2\sqrt{x}}; \quad \sin^{-1} y = \sqrt{x} + C; \quad y(x) = \sin\left(\sqrt{x} + C\right)$

6. $\displaystyle\int \frac{dy}{\sqrt{y}} = \int 3\sqrt{x}\,dx; \quad 2\sqrt{y} = 2x^{3/2} + 2C; \quad y(x) = \left(x^{3/2} + C\right)^2$

7. $\displaystyle\int \frac{dy}{y^{1/3}} = \int 4x^{1/3}\,dx; \quad \tfrac{3}{2}y^{2/3} = 3x^{4/3} + \tfrac{3}{2}C; \quad y(x) = \left(2x^{4/3} + C\right)^{3/2}$

8. $\displaystyle\int \cos y\,dy = \int 2x\,dx; \quad \sin y = x^2 + C; \quad y(x) = \sin^{-1}\left(x^2 + C\right)$

9. $\displaystyle\int \frac{dy}{y} = \int \frac{2\,dx}{1-x^2} = \int \left(\frac{1}{1+x} + \frac{1}{1-x}\right)dx$ (partial fractions)

$\displaystyle\ln y = \ln(1+x) - \ln(1-x) + \ln C; \quad y(x) = C\frac{1+x}{1-x}$

10. $\displaystyle\int \frac{dy}{(1+y)^2} = \int \frac{dx}{(1+x)^2}; \quad -\frac{1}{1+y} = -\frac{1}{1+x} - C = -\frac{1+C(1+x)}{1+x}$

$\displaystyle 1+y = \frac{1+x}{1+C(1+x)}; \quad y(x) = \frac{1+x}{1+C(1+x)} - 1 = \frac{x-C(1+x)}{1+C(1+x)}$

11. $\displaystyle\int \frac{dy}{y^3} = \int x\,dx; \quad -\frac{1}{2y^2} = \frac{x^2}{2} - \frac{C}{2}; \quad y(x) = \left(C - x^2\right)^{-1/2}$

12. $\displaystyle\int \frac{y\,dy}{y^2+1} = \int x\,dx; \quad \tfrac{1}{2}\ln\left(y^2+1\right) = \tfrac{1}{2}x^2 + \tfrac{1}{2}\ln C; \quad y^2+1 = C\,e^{x^2}$

13. $\displaystyle\int \frac{y^3\,dy}{y^4+1} = \int \cos x\,dx; \quad \tfrac{1}{4}\ln\left(y^4+1\right) = \sin x + C$

14. $\displaystyle\int\left(1+\sqrt{y}\right)dy = \int\left(1+\sqrt{x}\right)dx; \quad y + \tfrac{2}{3}y^{3/2} = x + \tfrac{2}{3}x^{3/2} + C$

15. $\displaystyle\int\left(\frac{2}{y^2} - \frac{1}{y^4}\right)dy = \int\left(\frac{1}{x} - \frac{1}{x^2}\right)dx; \quad -\frac{2}{y} + \frac{1}{3y^3} = \ln|x| + \frac{1}{x} + C$

16. $\displaystyle\int \frac{\sin y\,dy}{\cos y} = \int \frac{x\,dx}{1+x^2}; \quad -\ln(\cos x) = \tfrac{1}{2}\ln\left(1+x^2\right) + \ln C$

$\displaystyle \sec y = C\sqrt{1+x^2}; \quad y(x) = \sec^{-1}\left(C\sqrt{1+x^2}\right)$

17. $y' = 1 + x + y + xy = (1+x)(1+y)$

$\displaystyle\int \frac{dy}{1+y} = \int (1+x)\,dx; \quad \ln|1+y| = x + \tfrac{1}{2}x^2 + C$

18. $x^2 y' = 1 - x^2 + y^2 - x^2 y^2 = (1-x^2)(1+y^2)$

$\displaystyle\int \frac{dy}{1+y^2} = \int\left(\frac{1}{x^2} - 1\right)dx; \quad \tan^{-1} y = -\frac{1}{x} - x + C; \quad y(x) = \tan\left(C - \frac{1}{x} - x\right)$

19. $\displaystyle\int \frac{dy}{y} = \int e^x\,dx; \quad \ln y = e^x + \ln C; \quad y(x) = C\exp(e^x)$

$y(0) = 2e$ implies $C = 2$ so $y(x) = 2\exp(e^x)$.

20. $\displaystyle\int \frac{dy}{1+y^2} = \int 3x^2\,dx; \quad \tan^{-1} y = x^3 + C; \quad y(x) = \tan\left(x^3 + C\right)$

$y(0) = 1$ implies $C = \tan^{-1} 1 = \pi/4$ so $y(x) = \tan\left(x^3 + \pi/4\right)$.

21. $\displaystyle\int 2y\,dy = \int \frac{x\,dx}{\sqrt{x^2-16}}; \quad y^2 = \sqrt{x^2-16} + C$

$y(5) = 2$ implies $C = 1$ so $y^2 = 1 + \sqrt{x^2-16}$.

22. $\displaystyle\int \frac{dy}{y} = \int \left(4x^3 - 1\right) dx; \quad \ln y = x^4 - x + \ln C; \quad y(x) = C\exp(x^4 - x)$

$y(1) = -3$ implies $C = -3$ so $y(x) = -3\exp(x^4 - x)$.

23. $\displaystyle\int \frac{dy}{2y-1} = \int dx; \quad \tfrac{1}{2}\ln(2y-1) = x + \tfrac{1}{2}\ln C; \quad 2y - 1 = Ce^{2x}$

$y(1) = 1$ implies $C = e^{-2}$ so $y(x) = \tfrac{1}{2}\left(1 + e^{2x-2}\right)$.

24. $\displaystyle\int \frac{dy}{y} = \int \frac{\cos x\,dx}{\sin x}; \quad \ln y = \ln(\sin x) + \ln C; \quad y(x) = C\sin x$

$y(\tfrac{\pi}{2}) = \tfrac{\pi}{2}$ implies $C = \tfrac{\pi}{2}$ so $y(x) = \tfrac{\pi}{2}\sin x$.

25. $\displaystyle\int \frac{dy}{y} = \int \left(\frac{1}{x} + 2x\right); \quad \ln y = \ln x + x^2 + \ln C; \quad y(x) = Cx\exp(x^2)$

$y(1) = 1$ implies $C = e^{-1}$ so $y(x) = x\exp(x^2 - 1)$.

26. $\displaystyle\int \frac{dy}{y^2} = \int \left(2x + 3x^2\right); \quad -\frac{1}{y} = x^2 + x^3 + C; \quad y(x) = \frac{-1}{x^2 + x^3 + C}$

$y(1) = -1$ implies $C = -1$ so $\displaystyle y(x) = \frac{1}{1 - x^2 - x^3}$.

27. $\displaystyle\int e^y\,dy = \int 6e^{2x}\,dx; \quad e^y = 3e^{2x} + C; \quad y(x) = \ln\left(3e^{2x} + C\right)$

$y(0) = 0$ implies $C = -2$ so $y(x) = \ln\left(3e^{2x} - 2\right)$.

28. $\displaystyle\int \sec^2 y\,dy = \int \frac{dx}{2\sqrt{x}}; \quad \tan y = \sqrt{x} + C; \quad y(x) = \tan^{-1}\left(\sqrt{x} + C\right)$

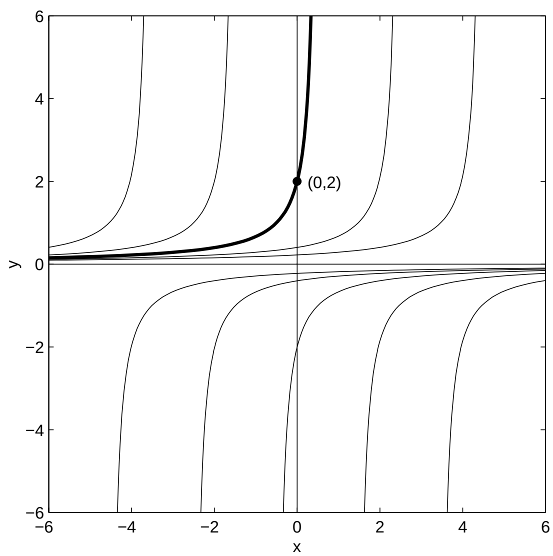
$y(4) = \frac{\pi}{4}$ implies $C = -1$ so $y(x) = \tan^{-1}\left(\sqrt{x} - 1\right)$.

29. (a) Separation of variables gives the general solution

$$\int \left(-\frac{1}{y^2}\right) dy = -\int x\,dx; \quad \frac{1}{y} = -x + C; \quad y(x) = -\frac{1}{x - C}.$$

(b) Inspection yields the singular solution $y(x) \equiv 0$ that corresponds to *no* value of the constant C.

(c) In the figure below we see that there is a unique solution curve through every point in the *xy*-plane.



30. When we take square roots on both sides of the differential equation and separate variables, we get

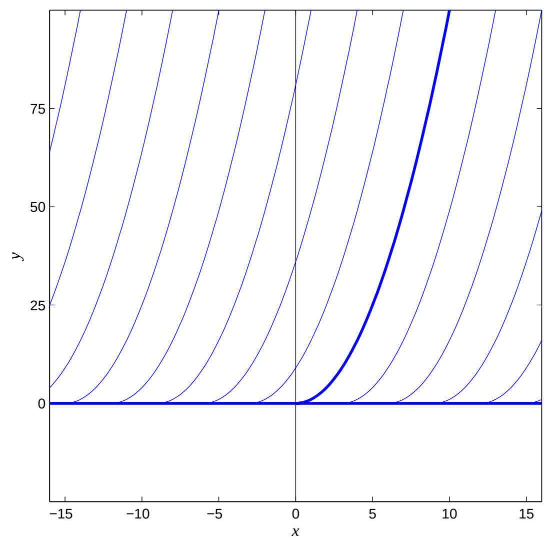
$$\int \frac{dy}{2\sqrt{y}} = \int dx; \quad \sqrt{y} = x - C; \quad y(x) = (x - C)^2.$$

This general solution provides the parabolas illustrated in Fig. 1.4.5 in the textbook. Observe that $y(x)$ is always nonnegative, consistent with both the square root and the original differential equation. We spot also the singular solution $y(x) \equiv 0$ that corresponds to *no* value of the constant C.

(a) Looking at Fig. 1.4.5, we see immediately that the differential equation $(y')^2 = 4y$ has no solution curve through the point (a,b) if $b < 0$.

(b) But if $b \geq 0$ we obviously can combine branches of parabolas with segments along the *x*-axis to form infinitely many solution curves through (a,b).

(c) Finally, if $b > 0$ then on a interval containing (a,b) there are exactly *two* solution curves through this point, corresponding to the two indicated parabolas through (a,b), one ascending and one descending from left to right.

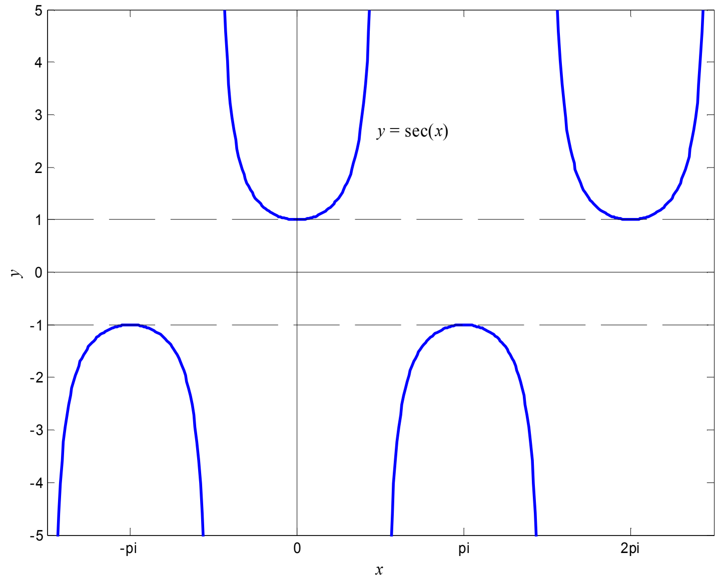


Problem 31 Figure

31. The formal separation-of-variables process is the same as in Problem 30 where, indeed, we started by taking square roots in $(y')^2 = 4y$ to get the differential equation $y' = 2\sqrt{y}$. But whereas y' can be either positive or negative in the original equation, the latter equation requires that y' be *nonnegative*. This means that only the *right half* of each parabola $y = (x - C)^2$ qualifies as a solution curve. Inspecting the figure above, we therefore see that through the point (a,b) there passes

(a) No solution curve if $b < 0$,
(b) A unique solution curve if $b > 0$,
(c) Infinitely many solution curves if $b = 0$, because in this case we can pick any $c > a$ and define the solution $y(x) = 0$ if $x \leq c$, $y(x) = (x - c)^2$ if $x \geq c$.

Problem 32 Figure (a)



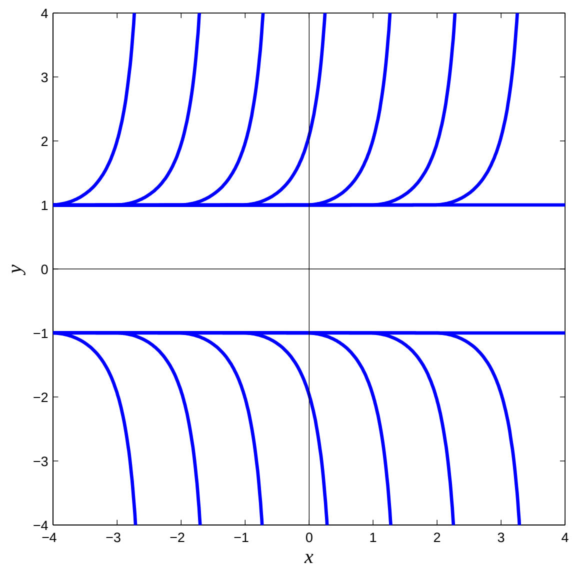
32. Separation of variables gives

$$x = \int \frac{dy}{y\sqrt{y^2-1}} = \sec^{-1}|y| + C$$

if $|y|>1,$ so the general solution has the form $y(x) = \pm\sec(x-C)$. But the original differential equation $y' = y\sqrt{y^2-1}$ implies that $y'>0$ if $y>1,$ while $y'<0$ if $y<-1.$ Consequently, only the *right halves* of translated branches of the curve $y=\sec x$ (figure above) qualify as general solution curves. This explains the plotted general solution curves we see in the figure at the top of the next page. In addition, we spot the two singular solutions $y(x)\equiv 1$ and $y(x)\equiv -1.$ It follows upon inspection of this figure that the initial value problem $y' = y\sqrt{y^2-1},$ $y(a)=b$ has a unique solution if $|b|>1$ and has no solution if $|b|<1.$ But if $b=1$ (and similarly if $b=-1$) then we can pick any $c>a$ and define the solution $y(x)=1$ if $x\le c,$ $y(x)=|\sec(x-c)|$ if $c\le x<c+\frac{\pi}{2}.$ So we see that if $b=\pm1,$ then the initial value problem $y' = y\sqrt{y^2-1},$ $y(a)=b$ has infinitely many solutions.

33. The population growth rate is $k = \ln(30000/25000)/10 \approx 0.01823,$ so the population of the city t years after 1960 is given by $P(t) = 25000e^{0.01823t}.$ The expected year 2000 population is then $P(40) = 25000e^{0.01823\times 40} \approx 51840.$

Problem 32 Figure (b)



34. The population growth rate is $k = \ln(6)/10 \approx 0.17918$, so the population after t hours is given by $P(t) = P_0 e^{0.17918t}$. To find how long it takes for the population to double, we therefore need only solve the equation $2P = P_0 e^{0.17918t}$ for $t = (\ln 2)/0.17918 \approx 3.87$ hours.

35. As in the textbook discussion of radioactive decay, the number of ^{14}C atoms after t years is given by $N(t) = N_0 e^{-0.0001216\,t}$. Hence we need only solve the equation $\frac{1}{6}N_0 = N_0 e^{-0.0001216\,t}$ for $t = (\ln 6)/0.0001216 \approx 14735$ years to find the age of the skull.

36. As in Problem 35, the number of ^{14}C atoms after t years is given by $N(t) = 5.0\times10^{10}\, e^{-0.0001216\,t}$. Hence we need only solve the equation $4.6\times10^{10} = 5.0\times10^{10}\, e^{-0.0001216\,t}$ for the age $t = \left(\ln(5.0/4.6)\right)/0.0001216 \approx 686$ years of the relic. Thus it appears not to be a genuine relic of the time of Christ 2000 years ago.

37. The amount in the account after t years is given by $A(t) = 5000\,e^{0.08t}$. Hence the amount in the account after 18 years is given by $A(18) = 5000\,e^{0.08\times18} \approx 21{,}103.48$ dollars.

38. When the book has been overdue for t years, the fine owed is given in dollars by $A(t) = 0.30\,e^{0.05t}$. Hence the amount owed after 100 years is given by $A(100) = 0.30\,e^{0.05\times100} \approx 44.52$ dollars.

39. To find the decay rate of this drug in the dog's blood stream, we solve the equation $\frac{1}{2} = e^{-5k}$ (half-life 5 hours) for $k = (\ln 2)/5 \approx 0.13863$. Thus the amount in the dog's bloodstream after t hours is given by $A(t) = A_0 e^{-0.13863t}$. We therefore solve the equation $A(1) = A_0 e^{-0.13863} = 50 \times 45 = 2250$ for $A_0 \approx 2585$ mg, the amount to anesthetize the dog properly.

40. To find the decay rate of radioactive cobalt, we solve the equation $\frac{1}{2} = e^{-5.27k}$ (half-life 5.27 years) for $k = (\ln 2)/5.27 \approx 0.13153$. Thus the amount of radioactive cobalt left after t years is given by $A(t) = A_0 e^{-0.13153t}$. We therefore solve the equation $A(t) = A_0 e^{-0.13153t} = 0.01 A_0$ for $t = (\ln 100)/0.13153 \approx 35.01$ and find that it will be about 35 years until the region is again inhabitable.

41. Taking $t = 0$ when the body was formed and $t = T$ now, the amount $Q(t)$ of ^{238}U in the body at time t (in years) is given by $Q(t) = Q_0 e^{-kt}$, where $k = (\ln 2)/(4.51 \times 10^9)$. The given information tells us that

$$\frac{Q(T)}{Q_0 - Q(T)} = 0.9.$$

After substituting $Q(T) = Q_0 e^{-kT}$, we solve readily for $e^{kT} = 19/9$, so $T = (1/k)\ln(19/9) \approx 4.86 \times 10^9$. Thus the body was formed approximately 4.86 billion years ago.

42. Taking $t = 0$ when the rock contained only potassium and $t = T$ now, the amount $Q(t)$ of potassium in the rock at time t (in years) is given by $Q(t) = Q_0 e^{-kt}$, where $k = (\ln 2)/(1.28 \times 10^9)$. The given information tells us that the amount $A(t)$ of argon at time t is

$$A(t) = \tfrac{1}{9}[Q_0 - Q(t)]$$

and also that $A(T) = Q(T)$. Thus

$$Q_0 - Q(T) = 9\,Q(T).$$

After substituting $Q(T) = Q_0 e^{-kT}$ we readily solve for

$$T = (\ln 10/\ln 2)(1.28 \times 10^9) \approx 4.25 \times 10^9.$$

Thus the age of the rock is about 1.25 billion years.

43. Because $A = 0$ the differential equation reduces to $T' = kT$, so $T(t) = 25e^{-kt}$. The fact that $T(20) = 15$ yields $k = (1/20)\ln(5/3)$, and finally we solve

$$5 = 25e^{-kt} \quad \text{for} \quad t = (\ln 5)/k \approx 63 \text{ min.}$$

44. The amount of sugar remaining undissolved after t minutes is given by $A(t) = A_0 e^{-kt}$; we find the value of k by solving the equation $A(1) = A_0 e^{-k} = 0.75 A_0$ for $k = -\ln 0.75 \approx 0.28768$. To find how long it takes for half the sugar to dissolve, we solve the equation $A(t) = A_0 e^{-kt} = \frac{1}{2} A_0$ for $t = (\ln 2)/0.28768 \approx 2.41$ minutes.

45. **(a)** The light intensity at a depth of x meters is given by $I(x) = I_0 e^{-1.4x}$. We solve the equation $I(x) = I_0 e^{-1.4x} = \frac{1}{2} I_0$ for $x = (\ln 2)/1.4 \approx 0.495$ meters.

(b) At depth 10 meters the intensity is $I(10) = I_0 e^{-1.4\times 10} \approx (8.32\times 10^{-7}) I_0$.

(c) We solve the equation $I(x) = I_0 e^{-1.4x} = 0.01 I_0$ for $x = (\ln 100)/1.4 \approx 3.29$ meters.

46. **(a)** The pressure at an altitude of x miles is given by $p(x) = 29.92\, e^{-0.2x}$. Hence the pressure at altitude 10000 ft is $p(10000/5280) \approx 20.49$ inches, and the pressure at altitude 30000 ft is $p(30000/5280) \approx 9.60$ inches.

(b) To find the altitude where $p = 15$ in., we solve the equation $29.92\, e^{-0.2x} = 15$ for $x = (\ln 29.92/15)/0.2 \approx 3.452$ miles $\approx 18{,}200$ ft.

47. If $N(t)$ denotes the number of people (in thousands) who have heard the rumor after t days, then the initial value problem is

$$N' = k(100 - N), \quad N(0) = 0$$

and we are given that $N(7) = 10$. When we separate variables ($dN/(100 - N) = k\,dt$) and integrate, we get $\ln(100 - N) = -kt + C$, and the initial condition $N(0) = 0$ gives $C = \ln 100$. Then $100 - N = 100\, e^{-kt}$, so $N(t) = 100\left(1 - e^{-kt}\right)$. We substitute $t = 7$, $N = 10$ and solve for the value $k = \ln(100/90)/7 \approx 0.01505$. Finally, 50 thousand people have heard the rumor after $t = (\ln 2)/k \approx 46.05$ days.

48. Let $N_8(t)$ and $N_5(t)$ be the numbers of ^{238}U and ^{235}U atoms, respectively, at time t (in billions of years after the creation of the universe). Then $N_8(t) = N_0 e^{-kt}$ and $N_5(t) = N_0 e^{-ct}$, where N_0 is the initial number of atoms of each isotope. Also, $k = (\ln 2)/4.51$ and $c = (\ln 2)/0.71$ from the given half-lives. We divide the equations for N_8 and N_5 and find that when t has the value corresponding to "now",

$$e^{(c-k)t} = \frac{N_8}{N_5} = 137.7.$$

Finally we solve this last equation for $t = (\ln 137.7)/(c-k) \approx 5.99$. Thus we get an estimate of about 6 billion years for the age of the universe.

49. The cake's temperature will be 100° after 66 min 40 sec; this problem is just like Example 6 in the text.

50. (a) $A(t) = 10e^{kt}$. Also $30 = A(\tfrac{15}{2}) = 10e^{15k/2}$, so so

$$e^{15k/2} = 3; \quad k = \frac{2}{15}\ln 3 = \ln\left(3^{2/15}\right).$$

Therefore $A(t) = 10(e^k)^t = 10 \cdot 3^{2t/15}$.

(b) After 5 years we have $A(5) = 10 \cdot 3^{2/3} \approx 20.80$ pu.

(c) $A(t) = 100$ when $A(t) = 10 \cdot 3^{2t/15}$; $t = \dfrac{15}{2} \cdot \dfrac{\ln(10)}{\ln(3)} \approx 15.72$ years.

51. (a) $A(t) = 15e^{-kt}$; $10 = A(5) = 15e^{-kt}$, so

$$\frac{3}{2} = e^{kt}; \quad k = \frac{1}{5}\ln\frac{3}{2}.$$

Therefore

$$A(t) = 15\exp\left(-\frac{t}{5}\ln\frac{3}{2}\right) = 15 \cdot \left(\frac{3}{2}\right)^{-t/5} = 15 \cdot \left(\frac{2}{3}\right)^{t/5}.$$

(b) After 8 months we have

$$A(8) = 15 \cdot \left(\frac{2}{3}\right)^{8/5} \approx 7.84 \text{ su.}$$

(c) $A(t) = 1$ when

$$A(t) = 15 \cdot \left(\frac{2}{3}\right)^{t/5} = 1; \quad t = 5 \cdot \frac{\ln(\frac{1}{15})}{\ln(\frac{2}{3})} \approx 33.3944.$$

Thus it will be safe to return after about 33.4 months.

52. If $L(t)$ denotes the number of human language families at time t (in years), then $L(t) = e^{kt}$ for some constant k. The condition that $L(6000) = e^{6000k} = 1.5$ gives

$k = \frac{1}{6000}\ln\frac{3}{2}$. If "now" corresponds to time $t = T$, then we are given that $L(T) = e^{kT} = 3300$, so $T = \frac{1}{k}\ln 3300 = \frac{6000\ln 3300}{\ln(3/2)} \approx 119887.18$. This result suggests that the original human language was spoken about 120 thousand years ago.

53. If $L(t)$ denotes the number of Native America language families at time t (in years), then $L(t) = e^{kt}$ for some constant k. The condition that $L(6000) = e^{6000k} = 1.5$ gives $k = \frac{1}{6000}\ln\frac{3}{2}$. If "now" corresponds to time $t = T$, then we are given that $L(T) = e^{kT} = 150$, so $T = \frac{1}{k}\ln 150 = \frac{6000\ln 150}{\ln(3/2)} \approx 74146.48$. This result suggests that the ancestors of today's Native Americans first arrived in the western hemisphere about 74 thousand years ago.

54. With $A(y)$ constant, Equation (19) in the text takes the form

$$\frac{dy}{dt} = k\sqrt{y}$$

We readily solve this equation for $2\sqrt{y} = kt + C$. The condition $y(0) = 9$ yields $C = 6$, and then $y(1) = 4$ yields $k = 2$. Thus the depth at time t (in hours) is $y(t) = (3 - t)^2$, and hence it takes 3 hours for the tank to empty.

55. With $A = \pi(3)^2$ and $a = \pi(1/12)^2$, and taking $g = 32$ ft/sec^2, Equation (20) reduces to $162\,y' = -\sqrt{y}$. The solution such that $y = 9$ when $t = 0$ is given by $324\sqrt{y} = -t + 972$. Hence $y = 0$ when $t = 972$ sec $= 16$ min 12 sec.

56. The radius of the cross-section of the cone at height y is proportional to y, so $A(y)$ is proportional to y^2. Therefore Equation (20) takes the form

$$y^2 y' = -k\sqrt{y},$$

and a general solution is given by

$$2y^{5/2} = -5kt + C.$$

The initial condition $y(0) = 16$ yields $C = 2048$, and then $y(1) = 9$ implies that $5k = 1562$. Hence $y = 0$ when

$$t = C/5k = 2048/1562 \approx 1.31 \text{ hr}.$$

57. The solution of $y' = -k\sqrt{y}$ is given by

$$2\sqrt{y} = -kt + C.$$

The initial condition $y(0) = h$ (the height of the cylinder) yields $C = 2\sqrt{h}$. Then substitution of $t = T$, $y = 0$ gives $k = (2\sqrt{h})/T$. It follows that

$$y = h(1 - t/T)^2.$$

If r denotes the radius of the cylinder, then

$$V(y) = \pi r^2 y = \pi r^2 h(1-t/T)^2 = V_0(1-t/T)^2.$$

58. Since $x = y^{3/4}$, the cross-sectional area is $A(y) = \pi x^2 = \pi y^{3/2}$. Hence the general equation $A(y)\,y' = -a\sqrt{2gy}$ reduces to the differential equation $yy' = -k$ with general solution

$$(1/2)y^2 = -kt + C.$$

The initial condition $y(0) = 12$ gives $C = 72$, and then $y(1) = 6$ yields $k = 54$. Upon separating variables and integrating, we find that the the depth at time t is

$$y(t) = \sqrt{144-108t}\,y(t).$$

Hence the tank is empty after $t = 144/108$ hr, that is, at 1:20 p.m.

59. **(a)** Since $x^2 = by$, the cross-sectional area is $A(y) = \pi x^2 = \pi by$. Hence the equation $A(y)y' = -a\sqrt{2gy}$ reduces to the differential equation

$$y^{1/2}y' = -k = -(a/\pi b)\sqrt{2g}$$

with the general solution

$$(2/3)y^{3/2} = -kt + C.$$

The initial condition $y(0) = 4$ gives $C = 16/3$, and then $y(1) = 1$ yields $k = 14/3$. It follows that the depth at time t is

$$y(t) = (8 - 7t)^{2/3}.$$

(b) The tank is empty after $t = 8/7$ hr, that is, at 1:08:34 p.m.

(c) We see above that $k = (a/\pi b)\sqrt{2g} = 14/3$. Substitution of $a = \pi r^2$, $b = 1$,

$g = (32)(3600)^2$ ft/hr^2 yields $r = (1/60)\sqrt{7/12}$ ft ≈ 0.15 in for the radius of the bottom-hole.

60. With $g = 32$ ft/sec^2 and $a = \pi(1/12)^2$, Equation (24) simplifies to

$$A(y)\frac{dy}{dt} = -\frac{\pi}{18}\sqrt{y}.$$

If z denotes the distance from the center of the cylinder down to the fluid surface, then $y = 3 - z$ and $A(y) = 10(9 - z^2)^{1/2}$. Hence the equation above becomes

$$10(9-z^2)^{1/2}\frac{dz}{dt} = \frac{\pi}{18}(3-z)^{1/2},$$
$$180(3+z)^{1/2}dz = \pi\, dt,$$

and integration yields

$$120(3+z)^{1/2} = \pi t + C.$$

Now $z = 0$ when $t = 0$, so $C = 120(3)^{3/2}$. The tank is empty when $z = 3$ (that is, when $y = 0$) and thus after

$$t = (120/\pi)(6^{3/2} - 3^{3/2}) \approx 362.90 \text{ sec.}$$

It therefore takes about 6 min 3 sec for the fluid to drain completely.

61. $A(y) = \pi(8y - y^2)$ as in Example 7 in the text, but now $a = \pi/144$ in Equation (24), so the initial value problem is

$$18(8y - y^2)y' = -\sqrt{y}, \qquad y(0) = 8.$$

We seek the value of t when $y = 0$. The answer is $t \approx 869$ sec $=$ 14 min 29 sec.

62. The cross-sectional area function for the tank is $A = \pi(1 - y^2)$ and the area of the bottom-hole is $a = 10^{-4}\pi$, so Eq. (24) in the text gives the initial value problem

$$\pi(1-y^2)\frac{dy}{dt} = -10^{-4}\pi\sqrt{2\times 9.8y}, \qquad y(0) = 1.$$

Simplification gives

$$\left(y^{-1/2} - y^{3/2}\right)\frac{dy}{dt} = -1.4\times 10^{-4}\sqrt{10}$$

so integration yields

$$2y^{1/2} - \frac{2}{5}y^{5/2} = -1.4\times 10^{-4}\sqrt{10}\,t + C.$$

The initial condition $y(0) = 1$ implies that $C = 2 - 2/5 = 8/5$, so $y = 0$ after $t = (8/5)/(1.4\times 10^{-4}\sqrt{10}) \approx 3614$ seconds. Thus the tank is empty at about 14 seconds after 2 pm.

63. **(a)** As in Example 8, the initial value problem is

$$\pi(8y - y^2)\frac{dy}{dt} = -\pi k\sqrt{y}, \qquad y(0) = 4$$

where $k = 0.6r^2\sqrt{2g} = 4.8r^2$. Integrating and applying the initial condition just in the Example 8 solution in the text, we find that

$$\frac{16}{3}y^{3/2} - \frac{2}{5}y^{5/2} = -kt + \frac{448}{15}.$$

When we substitute $y = 2$ (ft) and $t = 1800$ (sec, that is, 30 min), we find that $k \approx 0.009469$. Finally, $y = 0$ when

$$t = \frac{448}{15k} \approx 3154 \text{ sec} = 53 \text{ min } 34 \text{ sec}.$$

Thus the tank is empty at 1:53:34 pm.

(b) The radius of the bottom-hole is

$$r = \sqrt{k/4.8} \approx 0.04442 \text{ ft} \approx 0.53 \text{ in, thus about a half inch.}$$

64. The given rate of fall of the water level is $dy/dt = -4$ in/hr $= -(1/10800)$ ft/sec. With $A = \pi x^2$ and $a = \pi r^2$, Equation (24) is

$$(\pi x^2)(1/10800) = -(\pi r^2)\sqrt{2gy} = -8\pi r^2\sqrt{y}.$$

Hence the curve is of the form $y = kx^4$, and in order that it pass through $(1, 4)$ we must have $k = 4$. Comparing $\sqrt{y} = 2x^2$ with the equation above, we see that

$$(8r^2)(10800) = 1/2,$$

so the radius of the bottom hole is $r = 1/(240\sqrt{3})$ ft $\approx 1/35$ in.

65. Let $t = 0$ at the time of death. Then the solution of the initial value problem

$$T' = k(70 - T), \qquad T(0) = 98.6$$

is

$$T(t) = 70 + 28.6e^{-kt}.$$

If $t = a$ at 12 noon, then we know that

$$T(t) = 70 + 28.6e^{-ka} = 80,$$

$$T(a+1) = 70 + 28.6e^{-k(a+1)} = 75.$$

Hence

$$28.6e^{-ka} = 10 \quad \text{and} \quad 28.6e^{-ka}e^{-k} = 5.$$

It follows that $e^{-k} = 1/2$, so $k = \ln 2$. Finally the first of the previous two equations yields

$$a = (\ln 2.86)/(\ln 2) \approx 1.516 \text{ hr} \approx 1 \text{ hr } 31 \text{ min},$$

so the death occurred at 10:29 a.m.

66. Let $t = 0$ when it began to snow, and $t = t_0$ at 7:00 a.m. Let x denote distance along the road, with $x = 0$ where the snowplow begins at 7:00 a.m. If $y = ct$ is the snow depth at time t, w is the width of the road, and $v = dx/dt$ is the plow′s velocity, then "plowing at a constant rate" means that the product wyv is constant. Hence our differential equation is of the form

$$k\frac{dx}{dt} = \frac{1}{t}.$$

The solution with $x = 0$ when $t = t_0$ is

$$t = t_0 e^{kx}.$$

We are given that $x = 2$ when $t = t_0 + 1$ and $x = 4$ when $t = t_0 + 3$, so it follows that

$$t_0 + 1 = t_0 \mathrm{e}^{2k} \quad \text{and} \quad t_0 + 3 = t_0 e^{4k}.$$

Elimination of t_0 yields the equation

$$e^{4k} - 3e^{2k} + 2 = (e^{2k} - 1)(e^{2k} - 2) = 0,$$

so it follows (since $k > 0$) that $e^{2k} = 2$. Hence $t_0 + 1 = 2t_0$, so $t_0 = 1$. Thus it began to snow at 6 a.m.

67. We still have $t = t_0 e^{kx}$, but now the given information yields the conditions

$$t_0 + 1 = t_0 e^{4k} \qquad \text{and} \qquad t_0 + 2 = t_0 \, \mathrm{e}^{7k}$$

at 8 a.m. and 9 a.m., respectively. Elimination of t_0 gives the equation

$$2e^{4k} - e^{7k} - 1 = 0,$$

which we solve numerically for $k = 0.08276$. Using this value, we finally solve one of the preceding pair of equations for $t_0 = 2.5483$ hr $\approx$ 2 hr 33 min. Thus it began to snow at 4:27 a.m.

68. (a) Note first that if θ denotes the angle between the tangent line and the horizontal, then $\alpha = \frac{\pi}{2} - \theta$ so $\cot\alpha = \cot(\frac{\pi}{2} - \theta) = \tan\theta = y'(x)$. It follows that

$$\sin\alpha = \frac{\sin\alpha}{\sqrt{\sin^2\alpha + \cos^2\alpha}} = \frac{1}{\sqrt{1 + \cot^2\alpha}} = \frac{1}{\sqrt{1 + y'(x)^2}}.$$

Therefore the mechanical condition $(\sin\alpha)/v =$ constant (positive) with $v = \sqrt{2gy}$ translates to

$$\frac{1}{\sqrt{2gy}\sqrt{1 + (y')^2}} = \text{constant}, \quad \text{so} \quad y[1 + (y')^2] = 2a$$

for some positive constant a. We readily solve the latter equation for the differential equation

$$y' = \frac{dy}{dx} = \sqrt{\frac{2a - y}{y}}.$$

(b) The substitution $y = 2a\sin^2 t$, $dy = 4a\sin t\cos t\, dt$ now gives

$$4a\sin t\cos t\, dt = \sqrt{\frac{2a - 2a\sin^2 t}{2a\sin^2 t}}\, dx = \frac{\cos t}{\sin t}\, dx,$$
$$dx = 4a\sin^2 t\, dt.$$

Integration now gives

$$x = \int 4a\sin^2 t\, dt = 2a\int (1 - \cos 2t)\, dt$$
$$= 2a(t - \tfrac{1}{2}\sin 2t) + C = a(2t - \sin 2t) + C,$$

and we recall that $y = 2a\sin^2 t = a(1 - \cos 2t)$. The requirement that $x = 0$ when $t = 0$

implies that $C = 0$. Finally, the substitution $\theta = 2t$ (nothing to do with the previously mentioned angle θ of inclination from the horizontal) yields the desired parametric equations

$$x = a(\theta - \sin\theta), \qquad y = a(1 - \cos\theta)$$

of the cycloid that is generated by a point on the rim of a circular wheel of radius a as it rolls along the x-axis. [See Example 5 in Section 10.4 of Edwards and Penney, *Calculus*, 6th edition (Upper Saddle River, NJ: Prentice Hall, 2002).]

69. Substitution of $v = dy/dx$ in the differential equation for $y = y(x)$ gives

$$a\frac{dv}{dx} = \sqrt{1+v^2},$$

and separation of variables then yields

$$\int \frac{dv}{\sqrt{1+v^2}} = \int \frac{dx}{a}; \quad \sinh^{-1} v = \frac{x}{a} + C_1; \quad \frac{dy}{dx} = \sinh\left(\frac{x}{a} + C_1\right).$$

The fact that $y'(0) = 0$ implies that $C_1 = 0$, so it follows that

$$\frac{dy}{dx} = \sinh\left(\frac{x}{a}\right); \qquad y(x) = a\cosh\left(\frac{x}{a}\right) + C.$$

Of course the (vertical) position of the x-axis can be adjusted so that $C = 0$, and the units in which T and ρ are measured may be adjusted so that $a = 1$. In essence, then the shape of the hanging cable is the hyperbolic cosine graph $y = \cosh x$.

SECTION 1.5

LINEAR FIRST-ORDER EQUATIONS

1. $\rho = \exp\left(\int 1\,dx\right) = e^x; \quad D_x\left(y \cdot e^x\right) = 2e^x; \quad y \cdot e^x = 2e^x + C; \quad y(x) = 2 + Ce^{-x}$

$y(0) = 0$ implies $C = -2$ so $y(x) = 2 - 2e^{-x}$

2. $\rho = \exp\left(\int (-2)\,dx\right) = e^{-2x}; \quad D_x\left(y \cdot e^{-2x}\right) = 3; \quad y \cdot e^{-2x} = 3x + C; \quad y(x) = (3x + C)e^{2x}$

$y(0) = 0$ implies $C = 0$ so $y(x) = 3x\,e^{2x}$

3. $\rho = \exp\left(\int 3\,dx\right) = e^{3x}; \quad D_x\left(y \cdot e^{3x}\right) = 2x; \quad y \cdot e^{3x} = x^2 + C; \quad y(x) = (x^2 + C)e^{-3x}$

4. $\rho = \exp\left(\int(-2x)\,dx\right) = e^{-x^2}; \quad D_x\left(y \cdot e^{-x^2}\right) = 1; \quad y \cdot e^{-x^2} = x + C; \quad y(x) \;=\; (x+C)e^{x^2}$

5. $\rho = \exp\left(\int(2/x)\,dx\right) = e^{2\ln x} = x^2; \quad D_x\left(y \cdot x^2\right) = 3x^2; \quad y \cdot x^2 = x^3 + C$

$y(x) \;=\; x + C/x^2; \quad y(1) = 5$ implies $C = 4$ so $y(x) \;=\; x + 4/x^2$

6. $\rho = \exp\left(\int(5/x)\,dx\right) = e^{5\ln x} = x^5; \quad D_x\left(y \cdot x^5\right) = 7x^6; \quad y \cdot x^5 = x^7 + C$

$y(x) \;=\; x^2 + C/x^5; \quad y(2) = 5$ implies $C = 32$ so $y(x) \;=\; x^2 + 32/x^5$

7. $\rho = \exp\left(\int(1/2x)\,dx\right) = e^{(\ln x)/2} = \sqrt{x}; \quad D_x\left(y \cdot \sqrt{x}\right) = 5; \quad y \cdot \sqrt{x} = 5x + C$

$y(x) \;=\; 5\sqrt{x} + C/\sqrt{x}$

8. $\rho = \exp\left(\int(1/3x)\,dx\right) = e^{(\ln x)/3} = \sqrt[3]{x}; \quad D_x\left(y \cdot \sqrt[3]{x}\right) = 4\sqrt[3]{x}; \quad y \cdot \sqrt[3]{x} = 3x^{4/3} + C$

$y(x) \;=\; 3x + Cx^{-1/3}$

9. $\rho = \exp\left(\int(-1/x)\,dx\right) = e^{-\ln x} = 1/x; \quad D_x\left(y \cdot 1/x\right) = 1/x; \quad y \cdot 1/x = \ln x + C$

$y(x) \;=\; x\ln x + Cx; \quad y(1) = 7$ implies $C = 7$ so $y(x) \;=\; x\ln x + 7x$

10. $\rho = \exp\left(\int(-3/2x)\,dx\right) = e^{(-3\ln x)/2} = x^{-3/2};$

$D_x\left(y \cdot x^{-3/2}\right) = 9x^{1/2}/2; \quad y \cdot x^{-3/2} = 3x^{3/2} + C; \quad y(x) \;=\; 3x^3 + Cx^{3/2}$

11. $\rho = \exp\left(\int(1/x - 3)\,dx\right) = e^{\ln x - 3x} = x\,e^{-3x}; \quad D_x\left(y \cdot x\,e^{-3x}\right) = 0; \quad y \cdot x\,e^{-3x} = C$

$y(x) \;=\; C\,x^{-1}e^{3x}; \quad y(1) = 0$ implies $C = 0$ so $y(x) \;\equiv\; 0$ (constant)

12. $\rho = \exp\left(\int(3/x)\,dx\right) = e^{3\ln x} = x^3; \quad D_x\left(y \cdot x^3\right) = 2x^7; \quad y \cdot x^3 = \tfrac{1}{4}x^8 + C$

$y(x) \;=\; \tfrac{1}{4}x^5 + C\,x^{-3}; \quad y(2) = 1$ implies $C = -56$ so $y(x) \;=\; \tfrac{1}{4}x^5 - 56\,x^{-3}$

13. $\rho = \exp\left(\int 1\,dx\right) = e^x; \quad D_x\left(y \cdot e^x\right) = e^{2x}; \quad y \cdot e^x = \tfrac{1}{2}e^{2x} + C$

$y(x) \;=\; \tfrac{1}{2}e^x + C\,e^{-x}; \quad y(0) = 1$ implies $C = \tfrac{1}{2}$ so $y(x) \;=\; \tfrac{1}{2}e^x + \tfrac{1}{2}e^{-x}$

14. $\rho = \exp\left(\int(-3/x)\,dx\right) = e^{-3\ln x} = x^{-3}; \quad D_x\left(y \cdot x^{-3}\right) = x^{-1}; \quad y \cdot x^{-3} = \ln x + C$

$y(x) = x^3 \ln x + C x^3$; $y(1) = 10$ implies $C = 10$ so $y(x) = x^3 \ln x + 10 x^3$

15. $\rho = \exp\left(\int 2x\,dx \right) = e^{x^2}$; $D_x\left(y \cdot e^{x^2} \right) = x e^{x^2}$; $y \cdot e^{x^2} = \frac{1}{2} e^{x^2} + C$

$y(x) = \frac{1}{2} + C e^{-x^2}$; $y(0) = -2$ implies $C = -\frac{5}{2}$ so $y(x) = \frac{1}{2} - \frac{5}{2} e^{-x^2}$

16. $\rho = \exp\left(\int \cos x\,dx \right) = e^{\sin x}$; $D_x\left(y \cdot e^{\sin x} \right) = e^{\sin x} \cos x$; $y \cdot e^{\sin x} = e^{\sin x} + C$

$y(x) = 1 + C e^{-\sin x}$; $y(\pi) = 2$ implies $C = 1$ so $y(x) = 1 + e^{-\sin x}$

17. $\rho = \exp\left(\int 1/(1+x)\,dx \right) = e^{\ln(1+x)} = 1 + x$; $D_x\left(y \cdot (1+x) \right) = \cos x$; $y \cdot (1+x) = \sin x + C$

$y(x) = \dfrac{C + \sin x}{1+x}$; $y(0) = 1$ implies $C = 1$ so $y(x) = \dfrac{1 + \sin x}{1+x}$

18. $\rho = \exp\left(\int (-2/x)\,dx \right) = e^{-2\ln x} = x^{-2}$; $D_x\left(y \cdot x^{-2} \right) = \cos x$; $y \cdot x^{-2} = \sin x + C$

$y(x) = x^2 \left(\sin x + C \right)$

19. $\rho = \exp\left(\int \cot x\,dx \right) = e^{\ln(\sin x)} = \sin x$; $D_x\left(y \cdot \sin x \right) = \sin x \cos x$

$y \cdot \sin x = \frac{1}{2} \sin^2 x + C$; $y(x) = \frac{1}{2} \sin x + C \csc x$

20. $\rho = \exp\left(\int (-1-x)\,dx \right) = e^{-x-x^2/2}$; $D_x\left(y \cdot e^{-x-x^2/2} \right) = (1+x) e^{-x-x^2/2}$

$y \cdot e^{-x-x^2/2} = -e^{-x-x^2/2} + C$; $y(x) = -1 + C e^{-x-x^2/2}$

$y(0) = 0$ implies $C = 1$ so $y(x) = -1 + e^{-x-x^2/2}$

21. $\rho = \exp\left(\int (-3/x)\,dx \right) = e^{-3\ln x} = x^{-3}$; $D_x\left(y \cdot x^{-3} \right) = \cos x$; $y \cdot x^{-3} = \sin x + C$

$y(x) = x^3 \sin x + C x^3$; $y(2\pi) = 0$ implies $C = 0$ so $y(x) = x^3 \sin x$

22. $\rho = \exp\left(\int (-2x)\,dx \right) = e^{-x^2}$; $D_x\left(y \cdot e^{-x^2} \right) = 3x^2$; $y \cdot e^{-x^2} = x^3 + C$

$y(x) = \left(x^3 + C \right) e^{+x^2}$; $y(0) = 5$ implies $C = 5$ so $y(x) = \left(x^3 + 5 \right) e^{+x^2}$

23. $\rho = \exp\left(\int (2 - 3/x)\,dx \right) = e^{2x - 3\ln x} = x^{-3} e^{2x}$; $D_x\left(y \cdot x^{-3} e^{2x} \right) = 4 e^{2x}$

$y \cdot x^{-3} e^{2x} = 2 e^{2x} + C$; $y(x) = 2x^3 + C x^3 e^{-2x}$

24. $\rho = \exp\left(\int 3x/(x^2+4)\,dx\right) = e^{3\ln(x^2+4)/2} = (x^2+4)^{3/2}$; $\quad D_x\left(y\cdot(x^2+4)^{3/2}\right) = x(x^2+4)^{1/2}$

$y\cdot(x^2+4)^{3/2} = \frac{1}{3}(x^2+4)^{3/2} + C$; $\quad y(x) = \frac{1}{3} + C(x^2+4)^{-3/2}$

$y(0) = 1$ implies $C = \frac{16}{3}$ so $y(x) = \frac{1}{3}\left[1 + 16(x^2+4)^{-3/2}\right]$

25. First we calculate

$$\int \frac{3x^3\,dx}{x^2+1} = \int\left[3x - \frac{3x}{x^2+1}\right]dx = \frac{3}{2}\left[x^2 - \ln(x^2+1)\right].$$

It follows that $\rho = (x^2+1)^{-3/2}\exp(3x^2/2)$ and thence that

$$\begin{aligned} D_x\left(y\cdot(x^2+1)^{-3/2}\exp(3x^2/2)\right) &= 6x(x^2+4)^{-5/2}, \\ y\cdot(x^2+1)^{-3/2}\exp(3x^2/2) &= -2(x^2+4)^{-3/2} + C, \\ y(x) &= -2\exp(3x^2/2) + C(x^2+1)^{3/2}\exp(-3x^2/2). \end{aligned}$$

Finally, $y(0) = 1$ implies that $C = 3$ so the desired particular solution is

$$y(x) = -2\exp(3x^2/2) + 3(x^2+1)^{3/2}\exp(-3x^2/2).$$

26. With $x' = dx/dy$, the differential equation is $y^3x' + 4y^2x = 1$. Then with y as the independent variable we calculate

$$\rho(y) = \exp\left(\int(4/y)\,dy\right) = e^{4\ln y} = y^4; \quad D_y\left(x\cdot y^4\right) = y$$

$$x\cdot y^4 = \frac{1}{2}y^2 + C; \quad x(y) = \frac{1}{2y^2} + \frac{C}{y^4}$$

27. With $x' = dx/dy$, the differential equation is $x' - x = ye^y$. Then with y as the independent variable we calculate

$$\rho(y) = \exp\left(\int(-1)\,dy\right) = e^{-y}; \quad D_y\left(x\cdot e^{-y}\right) = y$$

$$x\cdot e^{-y} = \tfrac{1}{2}y^2 + C; \quad x(y) = \left(\tfrac{1}{2}y^2 + C\right)e^y$$

28. With $x' = dx/dy$, the differential equation is $(1+y^2)x' - 2yx = 1$. Then with y as the independent variable we calculate

$$\rho(y) = \exp\left(\int(-2y/(1+y^2)\,dy\right) = e^{-\ln(y^2+1)} = (1+y^2)^{-1}$$

$$D_y\left(x\cdot(1+y^2)^{-1}\right) = (1+y^2)^{-2}$$

An integral table (or trigonometric substitution) now yields

$$\frac{x}{1+y^2} = \int\frac{dy}{\left(1+y^2\right)^2} = \frac{1}{2}\left(\frac{y}{1+y^2}+\tan^{-1}y+C\right)$$

$$x(y) = \tfrac{1}{2}\left[y+\left(1+y^2\right)\left(\tan^{-1}y+C\right)\right]$$

29. $\rho = \exp\left(\int(-2x)\,dx\right) = e^{-x^2};\quad D_x\left(y\cdot e^{-x^2}\right) = e^{-x^2};\quad y\cdot e^{-x^2} = C+\int_0^x e^{-t^2}\,dt$

$$y(x) = e^{x^2}\left(C+\tfrac{1}{2}\sqrt{\pi}\,\mathrm{erf}(x)\right)$$

30. After division of the given equation by $2x$, multiplication by the integrating factor $\rho = x^{-1/2}$ yields

$$x^{-1/2}y'-\tfrac{1}{2}x^{-3/2}y = x^{-1/2}\cos x,$$

$$D_x\left(x^{-1/2}y\right) = x^{-1/2}\cos x,$$

$$x^{-1/2}y = C+\int_1^x t^{-1/2}\cos t\,dt.$$

The initial condition $y(1) = 0$ implies that $C = 0$, so the desired particular solution is

$$y(x) = x^{1/2}\int_1^x t^{-1/2}\cos t\,dt\,.$$

31. **(a)** $y_c' = Ce^{-\int P\,dx}(-P) = -P\,y_c$, so $y_c'+P\,y_c = 0.$

(b) $y_p' = (-P)e^{-\int P\,dx}\cdot\left[\int\left(Q\,e^{\int P\,dx}\right)dx\right]+e^{-\int P\,dx}\cdot Q\,e^{\int P\,dx} = -Py_p+Q$

32. **(a)** If $y = A\cos x+B\sin x$ then

$$y'+y = (A+B)\cos x+(B-A)\sin x = 2\sin x$$

provided that $A=-1$ and $B=1$. These coefficient values give the particular solution $y_p(x) = \sin x-\cos x$.

(b) The general solution of the equation $y' + y = 0$ is $y(x) = Ce^{-x}$ so addition to the particular solution found in part (a) gives $y(x) = Ce^{-x} + \sin x - \cos x$.

(c) The initial condition $y(0) = 1$ implies that $C = 2$, so the desired particular solution is $y(x) = 2e^{-x} + \sin x - \cos x$.

33. The amount $x(t)$ of salt (in kg) after t seconds satisfies the differential equation $x' = -x/200$, so $x(t) = 100e^{-t/200}$. Hence we need only solve the equation $10 = 100e^{-t/200}$ for $t = 461$ sec = 7 min 41 sec (approximately).

34. Let $x(t)$ denote the amount of pollutants in the lake after t days, measured in millions of cubic feet (mft^3). The volume of the lake is 8000 mft^3, and the initial amount $x(0)$ of pollutants is $x_0 = (0.25\%)(8000) = 20$ mft^3. We want to know when $x(t) = (0.10\%)(8000) = 8$ mft^3. We set up the differential equation in infinitesimal form by writing

$$dx = [\text{in}] - [\text{out}] = (0.0005)(500)\,dt - \frac{x}{8000} \cdot 500\,dt,$$

which simplifies to

$$\frac{dx}{dt} = \frac{1}{4} - \frac{x}{16}, \qquad \text{or} \qquad \frac{dx}{dt} + \frac{1}{16}x = \frac{1}{4}.$$

Using the integrating factor $\rho = e^{t/16}$, we readily derive the solution $x(t) = 4 + 16e^{-t/16}$ for which $x(0) = 20$. Finally, we find that $x = 8$ when $t = 16 \ln 4 \approx 22.2$ days.

35. The only difference from the Example 4 solution in the textbook is that $V = 1640$ km^3 and $r = 410$ km^3/yr for Lake Ontario, so the time required is

$$t = \frac{V}{r} \ln 4 = 4 \ln 4 \approx 5.5452 \text{ years}.$$

36. **(a)** The volume of brine in the tank after t min is $V(t) = 60 - t$ gal, so the initial value problem is

$$\frac{dx}{dt} = 2 - \frac{3x}{60 - t}, \qquad x(0) = 0.$$

The solution is

$$x(t) = (60 - t) - \frac{(60 - t)^3}{3600}.$$

(b) The maximum amount ever in the tank is $40/\sqrt{3} \approx 23.09$ lb. This occurs after $t = 60 - 20\sqrt{3} \approx 25/36$ min.

37. The volume of brine in the tank after t min is $V(t) = 100 + 2t$ gal, so the initial value problem is

$$\frac{dx}{dt} = 5 - \frac{3x}{100+2t}, \qquad x(0) = 50.$$

The integrating factor $\rho(t) = (100 + 2t)^{3/2}$ leads to the solution

$$x(t) = (100+2t) - \frac{50000}{(100+2t)^{3/2}}.$$

such that $x(0) = 50$. The tank is full after $t = 150$ min, at which time $x(150) = 393.75$ lb.

38. **(a)** $dx/dt = -x/20$ and $x(0) = 50$ so $x(t) = 50e^{-t/20}$.

(b) The solution of the linear differential equation

$$\frac{dy}{dt} = \frac{5x}{100} - \frac{5y}{200} = \frac{5}{2}e^{-t/20} - \frac{1}{40}y$$

with $y(0) = 50$ is

$$y(t) = 150e^{-t/40} - 100e^{-t/20}.$$

(c) The maximum value of y occurs when

$$y'(t) = -\frac{15}{4}e^{-t/40} + 5e^{-t/20} = -\frac{5}{4}e^{-t/40}\left(3 - 4e^{-t/40}\right) = 0.$$

We find that $y_{\max} = 56.25$ lb when $t = 40\ln(4/3) \approx 11.51$ min.

39. **(a)** The initial value problem

$$\frac{dx}{dt} = -\frac{x}{10}, \qquad x(0) = 100$$

for Tank 1 has solution $x(t) = 100e^{-t/10}$. Then the initial value problem

$$\frac{dy}{dt} = \frac{x}{10} - \frac{y}{10} = 10e^{-t/10} - \frac{y}{10}, \quad y(0) = 0$$

for Tank 2 has solution $y(t) = 10t\,e^{-t/10}$.

(b) The maximum value of y occurs when

$$y'(t) = 10e^{-t/10} - te^{-t/10} = 0$$

and thus when t = 10. We find that $y_{\max} = y(10) = 100e^{-1} \approx 36.79$ gal.

40. **(b)** Assuming inductively that $x_n = t^n e^{-t/2}/\left(n!2^n\right)$, the equation for x_{n+1} is

$$\frac{dx_{n+1}}{dt} = \frac{1}{2}x_n - \frac{1}{2}x_{n+1} = \frac{t^n\, e^{-t/2}}{n!\, 2^{n+1}} - \frac{1}{2}x_{n+1}.$$

We easily solve this first–order equation with $x_{n+1}(0) = 0$ and find that

$$x_{n+1} = \frac{t^{n+1}\, e^{-t/2}}{(n+1)!\, 2^{n+1}},$$

thereby completing the proof by induction.

41. **(a)** $A'(t) = 0.06A + 0.12S = 0.06A + 3.6e^{0.05t}$

(b) The solution with $A(0) = 0$ is

$$A(t) = 360(e^{0.06\,t} - e^{0.05\,t}),$$

so $A(40) \approx 1308.283$ thousand dollars.

42. The mass of the hailstone at time t is $m = (4/3)\pi r^3 = (4/3)\pi k^3 t^3$. Then the equation $d(mv)/dt = mg$ simplifies to

$$tv' + 3v = gt.$$

The solution satisfying the initial condition $v(0) = 0$ is $v(t) = gt/4$, so $v'(t) = g/4$.

43. The solution of the initial value problem $y' = x - y,\quad y(-5) = y_0$ is

$$y(x) = x - 1 + (y_0 + 6)e^{-x-5}.$$

Substituting $x = 5$, we therefore solve the equation $4 + (y_0 + 6)e^{-10} = y_1$ with $y_1 = 3.998,\ 3.999,\ 4,\ 4.001,\ 4.002$ for the desired initial values $y_0 = -50.0529,\ -28.0265,\ -6.0000,\ 16.0265,\ 38.0529$, respectively.

44. The solution of the initial value problem $y' = x + y, \quad y(-5) = y_0$ is

$$y(x) \ = \ -x - 1 + (y_0 - 4)e^{x+5}.$$

Substituting $x = 5$, we therefore solve the equation $-6 + (y_0 - 4)e^{10} \ = \ y_1$ with $y_1 \ = \ -10, \ -5, \ 0, \ 5, \ 10$ for the desired initial values $y_0 \ = \ 3.99982, \ 4.00005, \ 4.00027, \ 4.00050, \ 4.00073$, respectively.

45. With the pollutant measured in millions of liters and the reservoir water in millions of cubic meters, the inflow-outflow rate is $r = \frac{1}{5}$, the pollutant concentration in the inflow is $c_o = 10$, and the volume of the reservoir is $V = 2$. Substituting these values in the equation $x' = rc_o - (r/V)x$, we get the equation

$$\frac{dx}{dt} \ = \ 2 - \frac{1}{10}x$$

for the amount $x(t)$ of pollutant in the lake after t months. With the aid of the integrating factor $\rho = e^{t/10}$, we readily find that the solution with $x(0) = 0$ is

$$x(t) \ = \ 20\left(1 - e^{-t/10}\right).$$

Then we find that $x = 10$ when $t = 10 \ln 2 \approx 6.93$ months, and observe finally that, as expected, $x(t) \to 20$ as $t \to \infty$.

46. With the pollutant measured in millions of liters and the reservoir water in millions of cubic meters, the inflow-outflow rate is $r = \frac{1}{5}$, the pollutant concentration in the inflow is $c_o = 10(1 + \cos t)$, and the volume of the reservoir is $V = 2$. Substituting these values in the equation $x' = rc_o - (r/V)x$, we get the equation

$$\frac{dx}{dt} \ = \ 2(1 + \cos t) - \frac{1}{10}x, \qquad \text{that is,} \qquad \frac{dx}{dt} + \frac{1}{10}x = \ 2(1 + \cos t)$$

for the amount $x(t)$ of pollutant in the lake after t months. With the aid of the integrating factor $\rho = e^{t/10}$, we get

$$\begin{aligned} x \cdot e^{t/10} \ &= \ \int (2e^{t/10} + 2e^{t/10} \cos t) \, dt \\ &= \ 20e^{t/10} + 2 \cdot \frac{e^{t/10}}{(\frac{1}{10})^2 + 1^2} \left(\frac{1}{10} \cos t + \sin t \right) + C. \end{aligned}$$

When we impose the condition $x(0) = 0$, we get the desired particular solution

$$x(t) = \frac{20}{101}\left(101 - 102e^{-t/10} + \cos t + 10\sin t\right).$$

In order to determine when $x = 10$, we need to solve numerically. For instance, we can use the *Mathematica* commands

```
x = (20/101)(101 - 102 Exp[-t/10] + Cos[t] + 10 Sin[t]);
FindRoot[ x == 10, {t,7} ]
{t -> 6.474591767017537}
```

and find that this occurs after about 6.47 months. Finally, as $t \to \infty$ we observe that $x(t)$ approaches the function $20 + \frac{20}{101}(\cos t + 10\sin t)$ that does, indeed, oscillate about the equilibrium solution $x(t) \equiv 20$.

SECTION 1.6

SUBSTITUTION METHODS AND EXACT EQUATIONS

It is traditional for every elementary differential equations text to include the particular types of equations that are found in this section. However, no one of them is vitally important solely in its own right. Their main purpose (at this point in the course) is to familiarize students with the technique of transforming a differential equation by substitution. The subsection on airplane flight trajectories (together with Problems 56–59) is included as an application, but is optional material and may be omitted if the instructor desires.

The differential equations in Problems 1–15 are homogeneous, so we make the substitutions

$$v = \frac{y}{x}, \qquad y = vx, \qquad \frac{dy}{dx} = v + x\frac{dv}{dx}.$$

For each problem we give the differential equation in x, $v(x)$, and $v' = dv/dx$ that results, together with the principal steps in its solution.

1. $x(v+1)v' = -\left(v^2 + 2v - 1\right)$; $\displaystyle\int \frac{2(v+1)\,dv}{v^2 + 2v - 1} = -\int 2x\,dx$; $\ln\left(v^2 + 2v - 1\right) = -2\ln x + \ln C$

$x^2\left(v^2 + 2v - 1\right) = C$; $y^2 + 2xy - x^2 = C$

2. $2xvv' = 1$; $\displaystyle\int 2v\,dv = \int \frac{dx}{x}$; $v^2 = \ln x + C$; $y^2 = x^2\left(\ln x + C\right)$

3. $xv' = 2\sqrt{v}$; $\displaystyle\int \frac{dv}{2\sqrt{v}} = \int \frac{dx}{x}$; $\sqrt{v} = \ln x + C$; $y = x\left(\ln x + C\right)^2$

4. $x(v-1)v' = -(v^2+1); \quad \int \frac{2(1-v)\,dv}{v^2+1} = \int \frac{2\,dx}{x}; \quad 2\tan^{-1} v - \ln(v^2+1) = 2\ln x + C$

$2\tan^{-1}(y/x) - \ln(y^2/x^2+1) \;=\; 2\ln x + C$

5. $x(v+1)v' = -2v^2; \quad \int \left(\frac{1}{v} + \frac{1}{v^2}\right) dv = -\int \frac{2\,dx}{x}; \quad \ln v - \frac{1}{v} \;=\; -2\ln x + C$

$\ln y - \ln x - \frac{x}{y} \;=\; -2\ln x + C; \quad \ln(xy) \;=\; C + \frac{x}{y}$

6. $x(2v+1)v' = -2v^2; \quad \int \left(\frac{2}{v} + \frac{1}{v^2}\right) dv = -\int \frac{2\,dx}{x}; \quad \ln v^2 - \frac{1}{v} \;=\; -2\ln x + C$

$2\ln y - 2\ln x - \frac{x}{y} \;=\; -2\ln x + C; \quad 2y\ln y \;=\; x + Cy$

7. $x v^2 v' = 1; \quad \int 3v^2\,dv = \int \frac{3\,dx}{x}; \quad v^3 \;=\; 3\ln x + C; \quad y^3 \;=\; x^3(3\ln x + C)$

8. $x v' = e^v; \quad -\int e^{-v}\,dv = -\int \frac{dx}{x}; \quad e^{-v} \;=\; -\ln x + C; \quad -v \;=\; \ln(C - \ln x)$

$y \;=\; -x\ln(C - \ln x)$

9. $x v' = v^2; \quad -\int \frac{dv}{v^2} = -\int \frac{dx}{x}; \quad \frac{1}{v} \;=\; -\ln x + C; \quad x \;=\; y(C - \ln x)$

10. $x v v' = 2v^2 + 1; \quad \int \frac{4v\,dv}{2v^2+1} = \int \frac{4\,dx}{x}; \quad \ln(2v^2+1) \;=\; 4\ln x + \ln C$

$2y^2/x^2 + 1 \;=\; Cx^4; \quad x^2 + 2y^2 \;=\; Cx^6$

11. $x(1-v^2)v' = v + v^3; \quad \int \frac{1-v^2}{v^3+v}\,dv = \int \frac{dx}{x}; \quad \int \left(\frac{1}{v} - \frac{2v}{v^2+1}\right) dv = \int \frac{dx}{x}$

$\ln v - \ln(v^2+1) \;=\; \ln x + \ln C; \quad v \;=\; Cx(v^2+1); \quad y \;=\; C(x^2+y^2)$

12. $x v v' = \sqrt{v^2+4}; \quad \int \frac{v\,dv}{\sqrt{v^2+4}} = \int \frac{dx}{x}; \quad \sqrt{v^2+4} \;=\; \ln x + C$

$v^2 + 4 \;=\; (\ln x + C)^2; \quad 4x^2 + y^2 \;=\; x^2(\ln x + C)^2$

13. $x v' = \sqrt{v^2+1}; \quad \int \frac{dv}{\sqrt{v^2+1}} = \int \frac{dx}{x}; \quad \ln\left(v+\sqrt{v^2+1}\right) = \ln x + \ln C$

$v + \sqrt{v^2+1} = C x; \quad y + \sqrt{x^2+y^2} = C x^2$

14. $x v v' = \sqrt{1+v^2} - \left(1+v^2\right)$

$$\begin{aligned} \ln x &= \int \frac{v\,dv}{\sqrt{1+v^2}-(1+v^2)} \\ &= \frac{1}{2}\int \frac{du}{\sqrt{u}\left(1-\sqrt{u}\right)} \qquad (u = 1+v^2) \\ &= -\int \frac{dw}{w} = -\ln w + \ln C \end{aligned}$$

with $w = 1-\sqrt{u}$. Back-substitution and simplification finally yields the implicit solution $x - \sqrt{x^2+y^2} = C$.

15. $x(v+1)v' = -2\left(v^2+2v\right); \quad \int \frac{2(v+1)\,dv}{v^2+2v} = -\int \frac{4\,dx}{x}; \quad \ln\left(v^2+2v\right) = -4\ln x + \ln C$

$v^2+2v = C/x^4; \quad x^2y^2+2x^3y = C$

16. The substitution $v = x + y + 1$ leads to

$$\begin{aligned} x &= \int \frac{dv}{1+\sqrt{v}} = \int \frac{2u\,du}{1+u} \qquad (v = u^2) \\ &= 2u - 2\ln(1+u) + C \\ x &= 2\sqrt{x+y+1} - 2\ln(1+\sqrt{x+y+1}) + C \end{aligned}$$

17. $v = 4x + y; \quad v' = v^2+4; \quad x = \int \frac{dv}{v^2+4} = \frac{1}{2}\tan^{-1}\frac{v}{2} + \frac{C}{2}$

$v = 2\tan(2x-C); \quad y = 2\tan(2x-C) - 4x$

18. $v = x + y; \quad v v' = v+1; \quad x = \int \frac{v\,dv}{v+1} = \int \left(1 - \frac{1}{v+1}\right)dv = v - \ln(v+1) - C$

$y = \ln(x + y + 1) + C.$

Problems 19–25 are Bernoulli equations. For each, we indicate the appropriate substitution as specified in Equation (10) of this section, the resulting linear differential equation in v, its integrating factor ρ, and finally the resulting solution of the original Bernoulli equation.

19. $v = y^{-2}; \quad v' - 4v/x = -10/x^2; \quad \rho = 1/x^4; \quad y^2 = x/\left(Cx^5 + 2\right)$

20. $v = y^3; \quad v' + 6xv = 18x; \quad \rho = e^{3x^2}; \quad y^3 = 3 + Ce^{-3x^2}$

21. $v = y^{-2}; \quad v' + 2v = -2; \quad \rho = e^{2x}; \quad y^2 = 1/\left(Ce^{-2x} - 1\right)$

22. $v = y^{-3}; \quad v' - 6v/x = -15/x^2; \quad \rho = x^{-6}; \quad y^3 = 7x/\left(7Cx^7 + 15\right)$

23. $v = y^{-1/3}; \quad v' - 2v/x = -1; \quad \rho = x^{-2}; \quad y = \left(x + Cx^2\right)^{-3}$

24. $v = y^{-2}; \quad v' + 2v = e^{-2x}/x; \quad \rho = e^{2x}; \quad y^2 = e^{2x}/(C + \ln x)$

25. $v = y^3; \quad v' + 3v/x = 3/\sqrt{1+x^4}; \quad \rho = x^3; \quad y^3 = \left(C + 3\sqrt{1+x^4}\right)/\left(2x^3\right)$

26. The substitution $v = y^3$ yields the linear equation $v' + v = e^{-x}$ with integrating factor $\rho = e^x$. Solution: $y^3 = e^{-x}(x + C)$

27. The substitution $v = y^3$ yields the linear equation $xv' - v = 3x^4$ with integrating factor $\rho = 1/x$. Solution: $y = (x^4 + Cx)^{1/3}$

28. The substitution $v = e^y$ yields the linear equation $xv' - 2v = 2x^3e^{2x}$ with integrating factor $\rho = 1/x^2$. Solution: $y = \ln(Cx^2 + x^2e^{2x})$

29. The substitution $v = \sin y$ yields the homogeneous equation $2xvv' = 4x^2 + v^2$. Solution: $\sin^2 y = 4x^2 - Cx$

30. First we multiply each side of the given equation by e^y. Then the substitution $v = e^y$ gives the homogeneous equation $(x + v)v' = x - v$ of Problem 1 above. Solution: $x^2 - 2xe^y - e^{2y} = C$

Each of the differential equations in Problems 31–42 is of the form $M\,dx + N\,dy = 0$, and the exactness condition $\partial M/\partial y = \partial N/\partial x$ is routine to verify. For each problem we give the principal steps in the calculation corresponding to the method of Example 9 in this section.

31. $F = \int(2x + 3y)\,dx = x^2 + 3xy + g(y); \quad F_y = 3x + g'(y) = 3x + 2y = N$

$g'(y) = 2y; \quad g(y) = y^2; \qquad x^2 + 3xy + y^2 = C$

32. $F = \int (4x - y)\,dx = 2x^2 - xy + g(y); \quad F_y = -x + g'(y) = 6y - x = N$

$g'(y) = 6y; \quad g(y) = 3y^2; \qquad x^2 - xy + 3y^2 = C$

33. $F = \int (3x^2 + 2y^2)\,dx = x^3 + xy^2 + g(y); \quad F_y = 4xy + g'(y) = 4xy + 6y^2 = N$

$g'(y) = 6y^2; \quad g(y) = 2y^3; \qquad x^3 + 2xy^2 + 2y^3 = C$

34. $F = \int (2xy^2 + 3x^2)\,dx = x^3 + x^2y^2 + g(y); \quad F_y = 2x^2y + g'(y) = 2x^2y + 4y^3 = N$

$g'(y) = 4y^3; \quad g(y) = y^4; \qquad x^3 + x^2y^2 + y^4 = C$

35. $F = \int (x^3 + y/x)\,dx = \tfrac{1}{4}x^4 + y\ln x + g(y); \quad F_y = \ln x + g'(y) = y^2 + \ln x = N$

$g'(y) = y^2; \quad g(y) = \tfrac{1}{3}y^3; \qquad \tfrac{1}{4}x^3 + \tfrac{1}{3}y^2 + y\ln x = C$

36. $F = \int (1 + ye^{xy})\,dx = x + e^{xy} + g(y); \quad F_y = xe^{xy} + g'(y) = 2y + xe^{xy} = N$

$g'(y) = 2y; \quad g(y) = y^2; \qquad x + e^{xy} + y^2 = C$

37. $F = \int (\cos x + \ln y)\,dx = \sin x + x\ln y + g(y); \quad F_y = x/y + g'(y) = x/y + e^y = N$

$g'(y) = e^y; \quad g(y) = e^y; \qquad \sin x + x\ln y + e^y = C$

38. $F = \int (x + \tan^{-1} y)\,dx = \tfrac{1}{2}x^2 + x\tan^{-1} y + g(y); \quad F_y = \dfrac{x}{1+y^2} + g'(y) = \dfrac{x+y}{1+y^2} = N$

$g'(y) = \dfrac{y}{1+y^2}; \quad g(y) = \tfrac{1}{2}\ln(1+y^2); \qquad \tfrac{1}{2}x^2 + x\tan^{-1} y + \tfrac{1}{2}\ln(1+y^2) = C$

39. $F = \int (3x^2y^3 + y^4)\,dx = x^3y^3 + xy^4 + g(y);$

$F_y = 3x^3y^2 + 4xy^3 + g'(y) = 3x^3y^2 + y^4 + 4xy^3 = N$

$g'(y) = y^4; \quad g(y) = \tfrac{1}{5}y^5; \qquad x^3y^3 + xy^4 + \tfrac{1}{5}y^5 = C$

40. $F = \int (e^x \sin y + \tan y)\,dx = e^x \sin y + x\tan y + g(y);$

$F_y = e^x \cos y + x\sec^2 y + g'(y) = e^x \cos y + x\sec^2 y = N$

$g'(y) = 0; \quad g(y) = 0; \qquad e^x \sin y + x\tan y = C$

41. $$F = \int \left(\frac{2x}{y} - \frac{3y^2}{x^4} \right) dx = \frac{x^2}{y} + \frac{y^2}{x^3} + g(y);$$

$$F_y = -\frac{x^2}{y^2} + \frac{2y}{x^3} + g'(y) = -\frac{x^2}{y^2} + \frac{2y}{x^3} + \frac{1}{\sqrt{y}} = N$$

$$g'(y) = \frac{1}{\sqrt{y}}; \quad g(y) = 2\sqrt{y}; \qquad \frac{x^2}{y} + \frac{y^2}{x^3} + 2\sqrt{y} = C$$

42. $$F = \int \left(y^{-2/3} - \frac{3}{2} x^{-5/2} y \right) dx = x\,y^{-2/3} + x^{-3/2} y + g(y);$$

$$F_y = -\frac{2}{3} x\,y^{-5/3} + x^{-3/2} + g'(y) = x^{-3/2} - \frac{2}{3} x\,y^{-5/3} = N$$

$$g'(y) = 0; \quad g(y) = 0; \qquad x\,y^{-2/3} + x^{-3/2} y = C$$

43. The substitution $y' = p,\ y'' = p'$ in $xy'' = y'$ yields

$$xp' = p, \qquad \text{(separable)}$$
$$\int \frac{dp}{p} = \int \frac{dx}{x} \quad \Rightarrow \quad \ln p = \ln x + \ln C,$$
$$y' = p = Cx,$$
$$y(x) = \tfrac{1}{2} Cx^2 + B = Ax^2 + B.$$

44. The substitution $y' = p,\ y'' = p\,p' = p(dp/dy)$ in $yy'' + (y')^2 = 0$ yields

$$ypp' + p^2 = 0 \quad \Rightarrow \quad yp' = -p, \quad \text{(separable)}$$
$$\int \frac{dp}{p} = -\int \frac{dy}{y} \quad \Rightarrow \quad \ln p = -\ln y + \ln C,$$
$$p = C/y \quad \Rightarrow \quad x = \int \frac{1}{p} dy = \int \frac{y}{C} dy$$
$$x(y) = \frac{y^2}{2C} + B = Ay^2 + B.$$

45. The substitution $y' = p,\ y'' = p\,p' = p(dp/dy)$ in $y'' + 4y = 0$ yields

$$pp' + 4y = 0, \qquad \text{(separable)}$$
$$\int p\,dp = -\int 4y\,dy \quad \Rightarrow \quad \tfrac{1}{2} p^2 = -2y^2 + C,$$
$$p^2 = -4y^2 + 2C = 4\left(\tfrac{1}{2} C - y^2\right),$$

$$
\begin{aligned}
x &= \int \frac{1}{p}\,dy = \int \frac{dy}{2\sqrt{k^2-y^2}} = \frac{1}{2}\sin^{-1}\frac{y}{k} + D, \\
y(x) &= k\sin[2x-2D] = k(\sin 2x\cos 2D - \cos 2x\sin 2D), \\
y(x) &= A\cos 2x + B\sin 2x.
\end{aligned}
$$

46. The substitution $y' = p,\ y'' = p'$ in $xy'' + y' = 4x$ yields

$$
\begin{aligned}
xp' + p &= 4x, \qquad \text{(linear in } p) \\
D_x[x\cdot p] &= 4x \quad \Rightarrow \quad x\cdot p = 2x^2 + A, \\
p &= \frac{dy}{dx} = 2x + \frac{A}{x}, \\
y(x) &= x^2 + A\ln x + B.
\end{aligned}
$$

47. The substitution $y' = p,\ y'' = p'$ in $y'' = (y')^2$ yields

$$
\begin{aligned}
p' &= p^2, \qquad \text{(separable)} \\
\int \frac{dp}{p^2} &= \int x\,dx \quad \Rightarrow \quad -\frac{1}{p} = x + B, \\
\frac{dy}{dx} &= -\frac{1}{x+B}, \\
y(x) &= A - \ln|x+A|.
\end{aligned}
$$

48. The substitution $y' = p,\ y'' = p'$ in $x^2y'' + 3xy' = 2$ yields

$$
\begin{aligned}
x^2p' + 3xp &= 2 \quad \Rightarrow \quad p' + \frac{3}{p}p = \frac{2}{x^2}, \qquad \text{(linear in } p) \\
D_x[x^3\cdot p] &= 2x \quad \Rightarrow \quad x^3\cdot p = x^2 + C, \\
\frac{dy}{dx} &= \frac{1}{x} + \frac{C}{x^3}, \\
y(x) &= \ln x + \frac{A}{x^2} + B.
\end{aligned}
$$

49. The substitution $y' = p,\ y'' = p\,p' = p(dp/dy)$ in $yy'' + (y')^2 = yy'$ yields

$$
\begin{aligned}
yp\,p' + p^2 &= yp \quad \Rightarrow \quad y\,p' + p = y \qquad \text{(linear in } p), \\
D_y[y\cdot p] &= y, \\
yp &= \frac{1}{2}y^2 + \frac{1}{2}C \quad \Rightarrow \quad p = \frac{y^2+C}{2y},
\end{aligned}
$$

$$x = \int \frac{1}{p}dy = \int \frac{2y\,dy}{y^2+C} = \ln\left(y^2+C\right) - \ln B,$$

$$y^2 + C = Be^x \quad \Rightarrow \quad y(x) = \pm\left(A + Be^x\right)^{1/2}.$$

50. The substitution $y' = p,\ y'' = p'$ in $y'' = \left(x + y'\right)^2$ gives $p' = (x+p)^2$, and then the substitution $v = x + p,\ p' = v' - 1$ yields

$$v' - 1 = v^2 \quad \Rightarrow \quad \frac{dv}{dx} = 1 + v^2,$$

$$\int \frac{dv}{1+v^2} = \int dx \quad \Rightarrow \tan^{-1} v = x + A,$$

$$v = x + y' = \tan(x+A) \quad \Rightarrow \quad \frac{dy}{dx} = \tan(x+A) - x,$$

$$y(x) = \ln\left|\sec(x+A)\right| - \tfrac{1}{2}x^2 + B.$$

51. The substitution $y' = p,\ y'' = p\,p' = p(dp/dy)$ in $y'' = 2y\left(y'\right)^3$ yields

$$p\,p' = 2yp^3 \quad \Rightarrow \quad \int \frac{dp}{p^2} = \int 2y\,dy \quad \Rightarrow -\frac{1}{p} = y^2 + C,$$

$$x = \int \frac{1}{p}dy = -\frac{1}{3}y^3 - Cx + D,$$

$$y^3 + 3x + Ay + B = 0$$

52. The substitution $y' = p,\ y'' = p\,p' = p(dp/dy)$ in $y^3 y'' = 1$ yields

$$y^3 p\,p' = 1 \quad \Rightarrow \quad \int p\,dp = \int \frac{dy}{y^3} \quad \Rightarrow \quad \frac{1}{2}p^2 = -\frac{1}{2y^2} + \frac{A}{2},$$

$$p^2 = \frac{Ay^2 - 1}{y^2} \quad \Rightarrow \quad x = \int \frac{1}{p}dy = \int \frac{y\,dy}{\sqrt{Ay^2-1}},$$

$$x = \frac{1}{A}\sqrt{Ay^2-1} + C \quad \Rightarrow \quad Ax + B = \sqrt{Ay^2-1},$$

$$Ay^2 - (Ax+B)^2 = 1.$$

53. The substitution $y' = p,\ y'' = p\,p' = p(dp/dy)$ in $y'' = 2yy'$ yields

$$p\,p' = 2yp \quad \Rightarrow \quad \int dp = \int 2y\,dy \quad \Rightarrow \quad p = y^2 + A^2,$$

$$x = \int \frac{1}{p}dy = \int \frac{dy}{y^2 + A^2} = \frac{1}{A}\tan^{-1}\frac{y}{A} + C,$$

$$\tan^{-1}\frac{y}{A} = A(x-C) \quad \Rightarrow \quad \frac{y}{A} = \tan(Ax - AC),$$
$$y(x) = A\tan(Ax+B).$$

54. The substitution $y' = p,\ y'' = p\,p' = p(dp/dy)$ in $yy'' = 3(y')^2$ yields

$$yp\,p' = 3p^2 \quad \Rightarrow \quad \int \frac{dp}{p} = \int \frac{3\,dy}{y}$$
$$\ln p = 3\ln y + \ln C \quad \Rightarrow \quad p = Cy^3,$$
$$x = \int \frac{1}{p}\,dy = \int \frac{dy}{Cy^3} = -\frac{1}{2Cy^2} + B,$$
$$Ay^2(B-x) = 1.$$

55. The substitution $v = ax + by + c,\ y = (v - ax - c)/b$ in $y' = F(ax+by+c)$ yields the separable differential equation $(dv/dx - a)/b = F(v)$, that is, $dv/dx = a + bF(v)$.

56. If $v = y^{1-n}$ then $y = v^{1/(1-n)}$ so $y' = v^{n/(1-n)}v'/(1-n)$. Hence the given Bernoulli equation transforms to

$$\frac{v^{n/(1-n)}}{1-n}\frac{dv}{dx} + P(x)\,v^{1/(1-n)} = Q(x)\,v^{n/(1-n).}$$

Multiplication by $(1-n)/v^{n/(1-n)}$ then yields the linear differential equation $v' + (1-n)Pv = (1-n)Q\,v.$

57. If $v = \ln y$ then $y = e^v$ so $y' = e^v v'$. Hence the given equation transforms to $e^v v' + P(x)\,e^v = Q(x)\,v\,e^v$. Cancellation of the factor e^v then yields the linear differential equation $v' - Q(x)v = P(x)$.

58. The substitution $v = \ln y,\ y = e^v,\ y' = e^v\,v'$ yields the linear equation $x\,v' + 2\,v = 4x^2$ with integrating factor $\rho = x^2$. Solution: $y = \exp(x^2 + C/x^2)$

59. The substitution $x = u - 1,\ y = v - 2$ yields the homogeneous equation

$$\frac{dv}{du} = \frac{u-v}{u+v}.$$

The substitution $v = pu$ leads to

$$\ln u = -\int \frac{(p+1)\,dp}{(p^2+2p-1)} = -\frac{1}{2}\left[\ln\left(p^2+2p-1\right)-\ln C\right].$$

We thus obtain the implicit solution

$$\begin{aligned}
u^2\left(p^2+2p-1\right) &= C\\
u^2\left(\frac{v^2}{u^2}+2\frac{v}{u}-1\right) &= v^2+2uv-u^2 = C\\
(y+2)^2+2(x+1)(y+2)-(x+1)^2 &= C\\
y^2+2xy-x^2+2x+6y &= C.
\end{aligned}$$

60. The substitution $x = u-3,\;\; y = v-2$ yields the homogeneous equation

$$\frac{dv}{du} = \frac{-u+2v}{4u-3v}.$$

The substitution $v = pu$ leads to

$$\begin{aligned}
\ln u &= \int \frac{(4-3p)\,dp}{(3p+1)(p-1)} = \frac{1}{4}\int\left(\frac{1}{p-1}-\frac{15}{3p+1}\right)dp\\
&= \frac{1}{4}\left[\ln(p-1)-5\ln(3p+1)+\ln C\right].
\end{aligned}$$

We thus obtain the implicit solution

$$\begin{aligned}
u^4 &= \frac{C(p-1)}{(3p+1)^5} = \frac{C(v/u-1)}{(3v/u+1)^5} = \frac{Cu^4(v-u)}{(3v+u)^5}\\
(3v+u)^5 &= C(v-u)\\
(x+3y+3)^5 &= C(y-x-5).
\end{aligned}$$

61. The substitution $v = x-y$ yields the separable equation $v' = 1-\sin v$. With the aid of the identity

$$\frac{1}{1-\sin v} = \frac{1+\sin v}{\cos^2 v} = \sec^2 v+\sec v\tan v$$

we obtain the solution

$$x = \tan(x-y)+\sec(x-y)+C.$$

62. The substitution $y = vx$ in the given homogeneous differential equation yields the separable equation $x\left(2v^3-1\right)v' = -\left(v^4+v\right)$ that we solve as follows:

$$\int \frac{2v^3-1}{v^4+v}\,dv = -\int \frac{dx}{x}$$
$$\int \left(\frac{2v-1}{v^2-v+1} - \frac{1}{v} + \frac{1}{v+1}\right)dv = -\int \frac{dx}{x} \qquad \text{(partial fractions)}$$
$$\ln(v^2-v+1) - \ln v + \ln(v+1) = -\ln x + \ln C$$
$$x\,(v^2-v+1)(v+1) = C\,v$$
$$(y^2-xy+x^2)(x+y) = C\,xy$$
$$x^3+y^3 = C\,xy$$

63. If we substitute $y = y_1 + 1/v,\ y' = y_1' - v'/v^2$ (primes denoting differentiation with respect to x) into the Riccati equation $y' = Ay^2 + By + C$ and use the fact that $y_1' = Ay_1^2 + By_1 + C$, then we immediately get the linear differential equation $v' + (B + 2A\,y_1)\,v = -A$.

In Problems 64 and 65 we outline the application of the method of Problem 63 to the given Riccati equation.

64. The substitution $y = x + 1/v$ yields the linear equation $v' - 2x\,v = 1$ with integrating factor $\rho = e^{-x^2}$. In Problem 29 of Section 1.5 we saw that the general solution of this linear equation is $v(x) = e^{x^2}\left[C + \frac{\sqrt{\pi}}{2}\operatorname{erf}(x)\right]$ in terms of the *error function* erf(x) introduced there Hence the general solution of our Riccati equation is given by $y(x) = x + e^{-x^2}\left[C + \frac{\sqrt{\pi}}{2}\operatorname{erf}(x)\right]^{-1}$.

65. The substitution $y = x + 1/v$ yields the trivial linear equation $v' = -1$ with immediate solution $v(x) = C - x$. Hence the general solution of our Riccati equation is given by $y(x) = x + 1/(C - x)$.

66. The substitution $y' = C$ in the Clairaut equation immediately yields the general solution $y = Cx + g(C)$.

67. Clearly the line $y = Cx - C^2/4$ and the tangent line at $(C/2, C^2/4)$ to the parabola $y = x^2$ both have slope C.

68. $\ln\left(v + \sqrt{1+v^2}\right) = -k\ln x + k\ln a = \ln\left(x/a\right)^{-k}$

$$v + \sqrt{1+v^2} = \left(x/a\right)^{-k}$$

$$\left[(x/a)^{-k} - v\right]^2 = 1 + v^2$$

$$(x/a)^{-2k} - 2v(x/a)^{-k} + v^2 = 1 + v^2$$

$$v = \frac{1}{2}\left[\left(\frac{x}{a}\right)^{-2k} - 1\right] / \left(\frac{x}{a}\right)^{-k} = \frac{1}{2}\left[\left(\frac{x}{a}\right)^{-k} - \left(\frac{x}{a}\right)^{k}\right]$$

69. With $a = 100$ and $k = 1/10$, Equation (19) in the text is

$$y = 50[(x/100)^{9/10} - (x/100)^{11/10}].$$

The equation $y'(x) = 0$ then yields

$$(x/100)^{1/10} = (9/11)^{1/2},$$

so it follows that

$$y_{\max} = 50[(9/11)^{9/2} - (9/11)^{11/2}] \approx 3.68 \text{ mi.}$$

70. With $k = w/v_0 = 10/500 = 1/10$, Eq. (16) in the text gives

$$\ln\left(v + \sqrt{1+v^2}\right) = -\frac{1}{10}\ln x + C$$

where $v = y/x$. Substitution of $x = 200, y = 150, v = 3/4$ yields $C = \ln\left(2 \cdot 200^{1/10}\right)$, thence

$$\ln\left(\frac{y}{x} + \sqrt{1 + \frac{y^2}{x^2}}\right) = -\frac{1}{10}\ln x + \ln\left(2 \cdot 200^{1/10}\right),$$

which — after exponentiation and then multiplication of the resulting equation by x — simplifies as desired to $y + \sqrt{x^2 + y^2} = 2\left(200x^9\right)^{1/10}$. If $x = 0$ then this equation yields $y = 0$, thereby verifying that the airplane reaches the airport at the origin.

71. **(a)** With $a = 100$ and $k = w/v_0 = 2/4 = 1/2$, the solution given by equation (19) in the textbook is $y(x) = 50[(x/100)^{1/2} - (x/100)^{3/2}]$. The fact that $y(0) = 0$ means that this trajectory goes through the origin where the tree is located.

(b) With $k = 4/4 = 1$ the solution is $y(x) = 50[1 - (x/100)^2]$ and we see that the swimmer hits the bank at a distance $y(0) = 50$ north of the tree.

(c) With $k = 6/4 = 1$ the solution is $y(x) = 50[(x/100)^{-1/2} - (x/100)^{5/2}]$. This trajectory is asymptotic to the positive x-axis, so we see that the swimmer never reaches the west bank of the river.

72. The substitution $y' = p,\ y'' = p'$ in $ry'' = [1+(y')^2]^{3/2}$ yields

$$rp' = (1+p^2)^{3/2} \quad \Rightarrow \quad \int \frac{rp\,dp}{(1+p^2)^{3/2}} = \int dx.$$

Now integral formula #52 in the back of our favorite calculus textbook gives

$$\frac{rp}{\sqrt{1+p^2}} = x-a \quad \Rightarrow \quad r^2p^2 = (1+p^2)(x-a)^2,$$

and we solve readily for

$$p^2 = \frac{(x-a)^2}{r^2-(x-a)^2} \quad \Rightarrow \quad \frac{dy}{dx} = p = \frac{x-a}{\sqrt{r^2-(x-a)^2}},$$

whence

$$y = \int \frac{(x-a)\,dx}{\sqrt{r^2-(x-a)^2}} = -\sqrt{r^2-(x-a)^2}+b,$$

which finally gives $(x-a)^2+(y-b)^2 = r^2$ as desired.

CHAPTER 1 Review Problems

The main objective of this set of review problems is practice in the identification of the different types of first-order differential equations discussed in this chapter. In each of Problems 1–36 we identify the type of the given equation and indicate an appropriate method of solution.

1. If we write the equation in the form $y' - (3/x)y = x^2$ we see that it is *linear* with integrating factor $\rho = x^{-3}$. The method of Section 1.5 then yields the general solution $y = x^3(C + \ln x)$.

2. We write this equation in the *separable* form $y'/y^2 = (x+3)/x^2$. Then separation of variables and integration as in Section 1.4 yields the general solution $y = x/(3 - Cx - x\ln x)$.

3. This equation is *homogeneous*. The substitution $y = vx$ of Equation (8) in Section 1.6 leads to the general solution $y = x/(C - \ln x)$.

4. We note that $D_y\left(2xy^3+e^x\right) = D_x\left(3x^2y^2+\sin y\right) = 6xy^2$, so the given equation is *exact*. The method of Example 9 in Section 1.6 yields the implicit general solution $x^2y^3 + e^x - \cos y = C$.

5. We write this equation in the *separable* form $y'/y^2 = (2x-3)/x^4$. Then separation of variables and integration as in Section 1.4 yields the general solution $y = C\exp[(1-x)/x^3]$.

6. We write this equation in the *separable* form $y'/y^2 = (1-2x)/x^2$. Then separation of variables and integration as in Section 1.4 yields the general solution $y = x/(1 + Cx + 2x\ln x)$.

7. If we write the equation in the form $y' + (2/x)y = 1/x^3$ we see that it is *linear* with integrating factor $\rho = x^2$. The method of Section 1.5 then yields the general solution $y = x^{-2}(C + \ln x)$.

8. This equation is *homogeneous*. The substitution $y = vx$ of Equation (8) in Section 1.6 leads to the general solution $y = 3Cx/(C - x^3)$.

9. If we write the equation in the form $y' + (2/x)y = 6x\sqrt{y}$ we see that it is a *Bernoulli equation* with $n = 1/2$. The substitution $v = y^{-1/2}$ of Eq. (10) in Section 1.6 then yields the general solution $y = (x^2 + C/x)^2$.

10. We write this equation in the *separable* form $y'/\left(1+y^2\right) = 1+x^2$. Then separation of variables and integration as in Section 1.4 yields the general solution $y = \tan(C + x + x^3/3)$.

11. This equation is *homogeneous*. The substitution $y = vx$ of Equation (8) in Section 1.6 leads to the general solution $y = x/(C - 3\ln x)$.

12. We note that $D_y\left(6xy^3 + 2y^4\right) = D_x\left(9x^2y^2 + 8xy^3\right) = 18xy^2 + 8y^3$, so the given equation is *exact*. The method of Example 9 in Section 1.6 yields the implicit general solution $3x^2y^3 + 2xy^4 = C$.

13. We write this equation in the *separable* form $y'/y^2 = 5x^4 - 4x$. Then separation of variables and integration as in Section 1.4 yields the general solution $y = 1/(C + 2x^2 - x^5)$.

14. This equation is *homogeneous*. The substitution $y = vx$ of Equation (8) in Section 1.6 leads to the implicit general solution $y^2 = x^2/(C + 2\ln x)$.

15. This is a *linear* differential equation with integrating factor $\rho = e^{3x}$. The method of Section 1.5 yields the general solution $y = (x^3 + C)e^{-3x}$.

16. The substitution $v = y - x,\ y = v + x,\ y' = v' + 1$ gives the separable equation $v' + 1 = (y-x)^2 = v^2$ in the new dependent variable v. The resulting implicit general solution of the original equation is $y - x - 1 = C\,e^{2x}(y - x + 1)$.

17. We note that $D_y\left(e^x + y\,e^{xy}\right) = D_x\left(e^y + x\,e^{xy}\right) = e^{xy} + xy\,e^{xy}$, so the given equation is *exact*. The method of Example 9 in Section 1.6 yields the implicit general solution $e^x + e^y + e^{xy} = C$.

18. This equation is *homogeneous*. The substitution $y = vx$ of Equation (8) in Section 1.6 leads to the implicit general solution $y^2 = Cx^2(x^2 - y^2)$.

19. We write this equation in the *separable* form $y'/y^2 = \left(2 - 3x^5\right)/x^3$. Then separation of variables and integration as in Section 1.4 yields the general solution $y = x^2 / (x^5 + Cx^2 + 1)$.

20. If we write the equation in the form $y' + (3/x)y = 3x^{-5/2}$ we see that it is *linear* with integrating factor $\rho = x^3$. The method of Section 1.5 then yields the general solution $y = 2x^{-3/2} + Cx^{-3}$.

21. If we write the equation in the form $y' + \left(1/(x+1)\right)y = 1/(x^2-1)$ we see that it is *linear* with integrating factor $\rho = x + 1$. The method of Section then 1.5 yields the general solution $y = [C + \ln(x - 1)] / (x + 1)$.

22. If we write the equation in the form $y' - (6/x)y = 12x^3y^{2/3}$ we see that it is a *Bernoulli equation* with $n = 1/3$. The substitution $v = y^{-2/3}$ of Eq. (10) in Section 1.6 then yields the general solution $y = (2x^4 + Cx^2)^3$.

23. We note that $D_y\left(e^y + y\cos x\right) = D_x\left(x\,e^y + \sin x\right) = e^y + \cos x$, so the given equation is *exact*. The method of Example 9 in Section 1.6 yields the implicit general solution $x\,e^y + y\sin x = C$

24. We write this equation in the *separable* form $y'/y^2 = \left(1 - 9x^2\right)/x^{3/2}$. Then separation of variables and integration as in Section 1.4 yields the general solution $y = x^{1/2} / (6x^2 + Cx^{1/2} + 2)$.

25. If we write the equation in the form $y' + \left(2/(x+1)\right)y = 3$ we see that it is *linear* with integrating factor $\rho = (x+1)^2$. The method of Section 1.5 then yields the general solution $y = x + 1 + \mathrm{C}\,(x + 1)^{-2}$.

26. We note that $D_y\left(9x^{1/2}y^{4/3} - 12x^{1/5}y^{3/2}\right) = D_x\left(8x^{3/2}y^{1/3} - 15x^{6/5}y^{1/2}\right) =$

$12x^{1/2}y^{1/3} - 18x^{1/5}y^{1/2}$, so the given equation is *exact*. The method of Example 9 in Section 1.6 yields the implicit general solution $6x^{3/2}y^{4/3} - 10x^{6/5}y^{3/2} = C$.

27. If we write the equation in the form $y' + (1/x)y = -x^2y^4/3$ we see that it is a *Bernoulli equation* with $n = 4$. The substitution $v = y^{-3}$ of Eq. (10) in Section 1.6 then yields the general solution $y = x^{-1}(C + \ln x)^{-1/3}$.

28. If we write the equation in the form $y' + (1/x)y = 2e^{2x}/x$ we see that it is *linear* with integrating factor $\rho = x$. The method of Section 1.5 then yields the general solution $y = x^{-1}(C + e^{2x})$.

29. If we write the equation in the form $y' + \left(1/(2x+1)\right)y = (2x+1)^{1/2}$ we see that it is *linear* with integrating factor $\rho = \left(2x+1\right)^{1/2}$. The method of Section 1.5 then yields the general solution $y = (x^2 + x + C)(2x + 1)^{-1/2}$.

30. The substitution $v = x + y,\ y = v - x,\ y' = v' - 1$ gives the separable equation $v' - 1 = \sqrt{v}$ in the new dependent variable v. The resulting implicit general solution of the original equation is $x = 2(x + y)^{1/2} - 2\ln[1 + (x + y)^{1/2}] + C$.

31. $dy/(y+7) = 3x^2dx$ is separable; $y' + 3x^2y = 21x^2$ is linear.

32. $dy/(y^2-1) = x\,dx$ is separable; $y' + x\,y = x\,y^3$ is a Bernoulli equation with $n = 3$.

33. $(3x^2 + 2y^2)\,dx + 4xy\,dy = 0$ is exact; $y' = -\tfrac{1}{4}\left(3x/y + 2y/x\right)$ is homogeneous.

34. $(x + 3y)\,dx + (3x - y)\,dy = 0$ is exact; $y' = \dfrac{1 + 3y/x}{y/x - 3}$ is homogeneous.

35. $dy/(y+1) = 2x\,dx/\left(x^2+1\right)$ is separable; $y' - \left(2x/(x^2+1)\right)y = 2x/(x^2+1)$ is linear.

36. $dy/\left(\sqrt{y} - y\right) = \cot x\,dx$ is separable; $y' + (\cot x)y = (\cot x)\sqrt{y}$ is a Bernoulli equation with $n = 1/2$.

CHAPTER 2

MATHEMATICAL MODELS AND NUMERICAL METHODS

SECTION 2.1

POPULATION MODELS

Section 2.1 introduces the first of the two major classes of mathematical models studied in the textbook, and is a prerequisite to the discussion of equilibrium solutions and stability in Section 2.2.

In Problems 1–8 we outline the derivation of the desired particular solution, and then sketch some typical solution curves.

1. Noting that $x > 1$ because $x(0) = 2$, we write

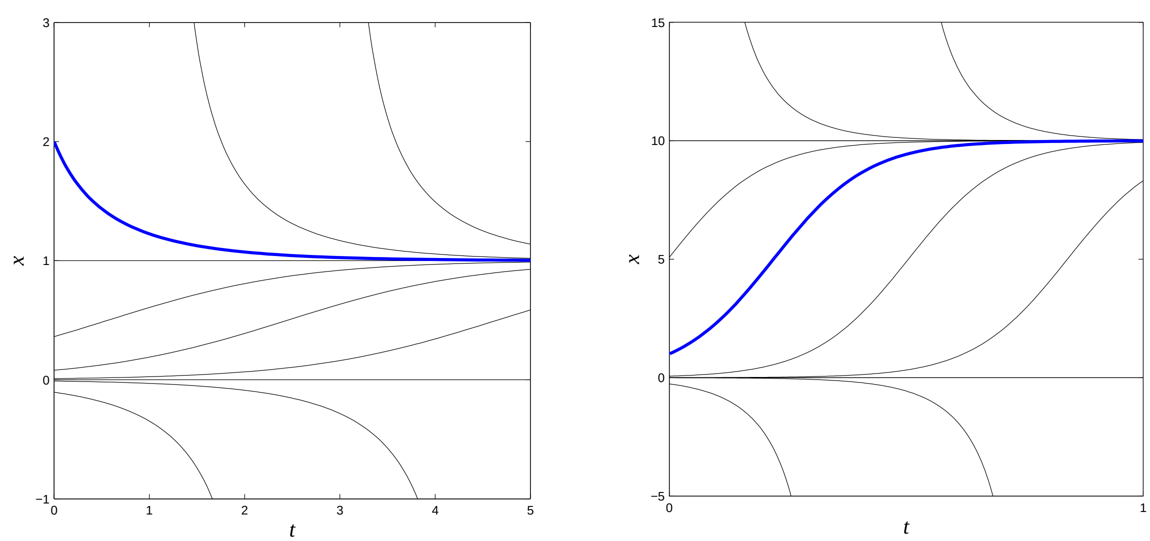
$$\int \frac{dx}{x(1-x)} = \int 1\,dt; \qquad \int \left(\frac{1}{x} - \frac{1}{x-1}\right) dx = \int 1\,dt$$

$$\ln x - \ln(x-1) = t + \ln C; \qquad \frac{x}{x-1} = C e^t$$

$$x(0) = 2 \quad \text{implies} \quad C = 2; \qquad x = 2(x-1)e^t$$

$$x(t) = \frac{2e^t}{2e^t - 1} = \frac{2}{2 - e^{-t}}.$$

Typical solution curves are shown in the figure on the left below.



2. Noting that $x < 10$ because $x(0) = 1$, we write

$$\int \frac{dx}{x(10-x)} = \int 1\,dt; \qquad \int \left(\frac{1}{x} + \frac{1}{10-x} \right) dx = \int 10\,dt$$

$$\ln x - \ln(10-x) = 10t + \ln C; \qquad \frac{x}{10-x} = C\,e^{10t}$$

$$x(0) = 1 \quad \text{implies} \quad C = \frac{1}{9}; \qquad 9x = (10-x)\,e^{10t}$$

$$x(t) = \frac{10e^{10t}}{9+e^{10t}} = \frac{10}{1+9e^{-10t}}.$$

Typical solution curves are shown in the figure on the right at the bottom of the preceding page.

3. Noting that $x > 1$ because $x(0) = 3$, we write

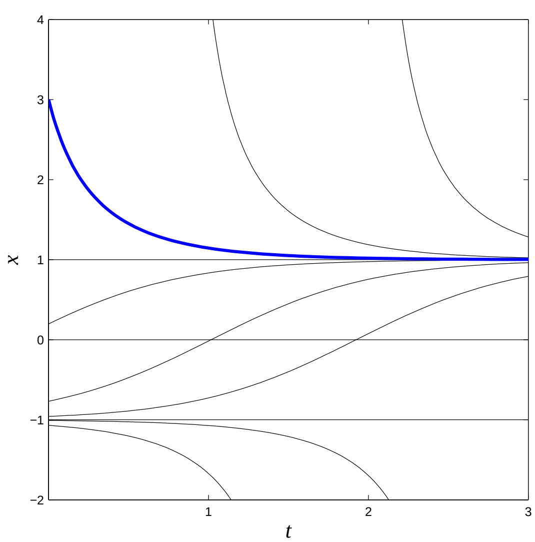
$$\int \frac{dx}{(1+x)(1-x)} = \int 1\,dt; \qquad \int \left(\frac{1}{x-1} - \frac{1}{x+1} \right) dx = \int (-2)\,dt$$

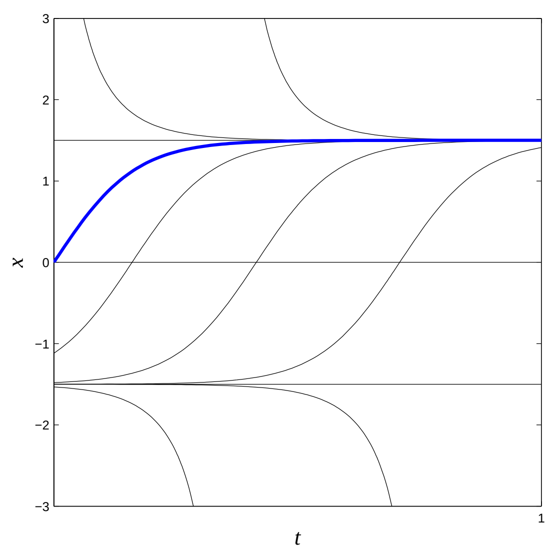
$$\ln(x-1) - \ln(x+1) = -2t + \ln C; \qquad \frac{x-1}{x+1} = C\,e^{-2t}$$

$$x(0) = 3 \quad \text{implies} \quad C = \frac{1}{2}; \qquad 2(x-1) = (x+1)\,e^{-2t}$$

$$x(t) = \frac{2+e^{-2t}}{2-e^{-2t}} = \frac{2e^{2t}+1}{2e^{2t}-1}.$$

Typical solution curves are shown in the figure on the left below.





4. Noting that $|x| < \frac{3}{2}$ because $x(0) = 0$, we write

$$\int \frac{dx}{(3+2x)(3-2x)} = \int 1\, dt; \qquad \int \left(\frac{1}{3+2x} + \frac{1}{3-2x} \right) dx = \int 6\, dt$$

$$\frac{1}{2}\ln(3+2x) - \frac{1}{2}\ln(3-2x) = 6t + \frac{1}{2}\ln C; \qquad \frac{3+2x}{3-2x} = C\,e^{12t}$$

$$x(0) = 0 \quad \text{implies} \quad C = 1; \qquad 3+2x = (3-2x)\,e^{12t}$$

$$x(t) = \frac{3e^{12t} - 3}{2e^{12t} + 2} = \frac{3\left(e^{12t} - 1\right)}{2\left(e^{12t} + 1\right)}.$$

Typical solution curves are shown in the figure on the right at the bottom of the preceding page.

5. Noting that $x > 5$ because $x(0) = 8$, we write

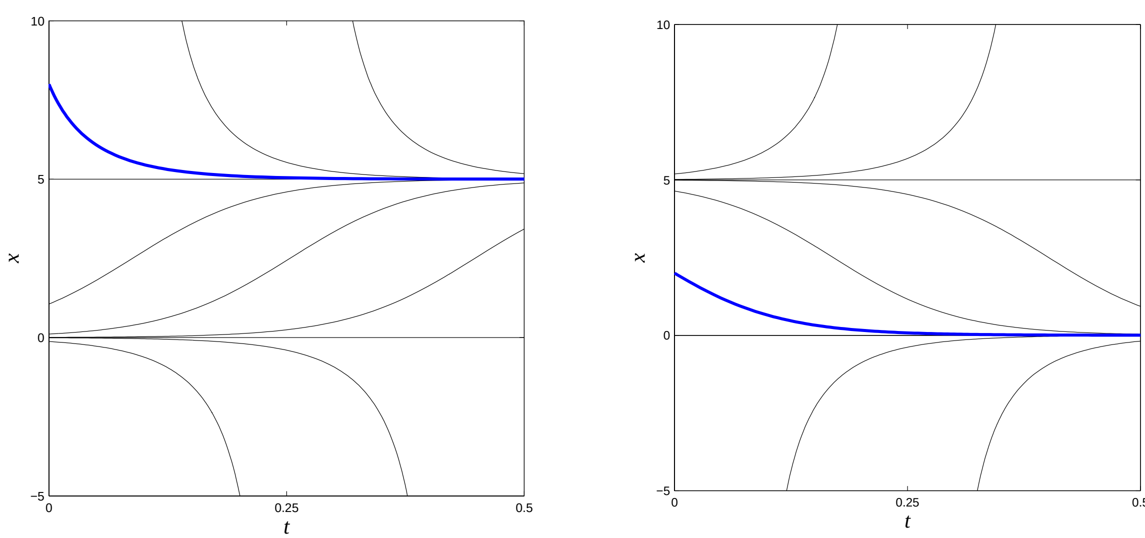
$$\int \frac{dx}{x(x-5)} = \int (-3)\, dt; \qquad \int \left(\frac{1}{x} - \frac{1}{x-5} \right) dx = \int 15\, dt$$

$$\ln x - \ln(x-5) = 15t + \ln C; \qquad \frac{x}{x-5} = C\,e^{15t}$$

$$x(0) = 8 \quad \text{implies} \quad C = 8/3; \qquad 3x = 8(x-5)\,e^{15t}$$

$$x(t) = \frac{-40\,e^{15t}}{3 - 8\,e^{15t}} = \frac{40}{8 - 3\,e^{-15t}}.$$

Typical solution curves are shown in the figure on the left below.



6. Noting that $x < 5$ because $x(0) = 2$, we write

$$\int \frac{dx}{x(5-x)} = \int (-3)\,dt; \qquad \int \left(\frac{1}{x} + \frac{1}{5-x}\right) dx = \int (-15)\,dt$$

$$\ln x - \ln(5-x) = -15t + \ln C; \qquad \frac{x}{5-x} = C\,e^{-15t}$$

$$x(0) = 2 \quad \text{implies} \quad C = 2/3; \qquad 3x = 2(5-x)\,e^{-15t}$$

$$x(t) = \frac{10\,e^{-15t}}{3 + 2\,e^{-15t}} = \frac{10}{2 + 3\,e^{15t}}.$$

Typical solution curves are shown in the figure on the right at the bottom of the preceding page.

7. Noting that $x > 7$ because $x(0) = 11$, we write

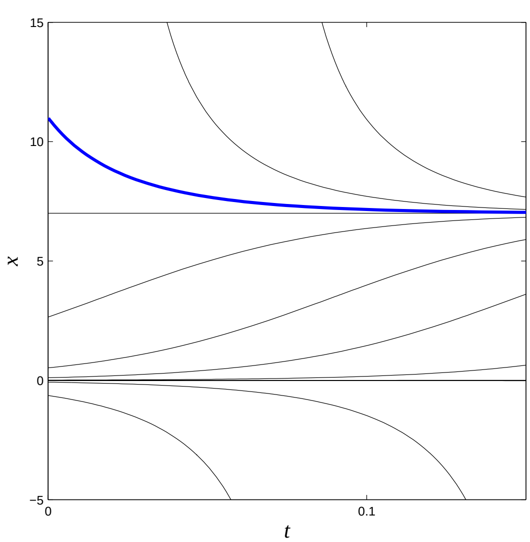
$$\int \frac{dx}{x(x-7)} = \int (-4)\,dt; \qquad \int \left(\frac{1}{x} - \frac{1}{x-7}\right) dx = \int 28\,dt$$

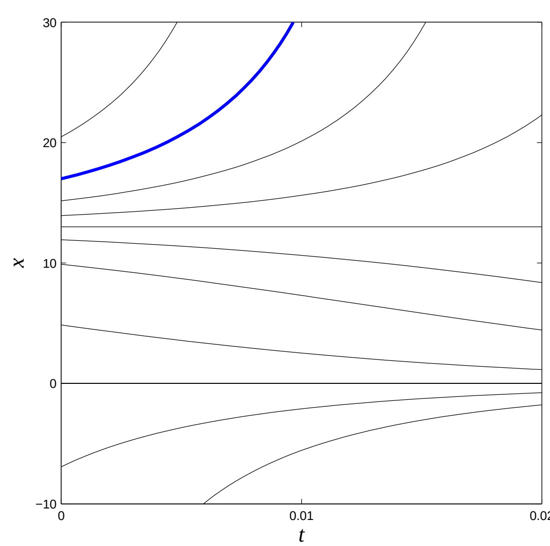
$$\ln x - \ln(x-7) = 28t + \ln C; \qquad \frac{x}{x-7} = C\,e^{28t}$$

$$x(0) = 11 \quad \text{implies} \quad C = 11/4; \qquad 4x = 11(x-17)\,e^{28t}$$

$$x(t) = \frac{-77\,e^{28t}}{4 - 11\,e^{28t}} = \frac{77}{11 - 4\,e^{-28t}}.$$

Typical solution curves are shown in the figure on the left below.





8. Noting that $x > 13$ because $x(0) = 17$, we write

$$\int \frac{dx}{x(x-13)} = \int 7\,dt; \qquad \int \left(\frac{1}{x} - \frac{1}{x-13}\right) dx = \int (-91)\,dt$$

$$\ln x - \ln(x-13) = -91t + \ln C; \qquad \frac{x}{x-13} = C\,e^{-91t}$$

$$x(0) = 17 \quad \text{implies} \quad C = 17/4; \qquad 4x = 17(x-13)\,e^{-91t}$$

$$x(t) = \frac{-221\,e^{-91t}}{4 - 17\,e^{-91t}} = \frac{221}{17 - 4\,e^{91t}}.$$

Typical solution curves are shown in the figure on the right at the bottom of the preceding page.

9. Substitution of $P(0) = 100$ and $P'(0) = 20$ into $P' = k\sqrt{P}$ yields $k = 2$, so the differential equation is $P' = 2\sqrt{P}$. Separation of variables and integration, $\int dP/2\sqrt{P} = \int dt$, gives $\sqrt{P} = t + C$. Then $P(0) = 100$ implies $C = 10$, so $P(t) = (t + 10)^2$. Hence the number of rabbits after one year is $P(12) = 484$.

10. Given $P' = -\delta P = -(k/\sqrt{P})\,P = -k\sqrt{P}$, separation of variables and integration as in Problem 9 yields $2\sqrt{P} = -kt + C$. The initial condition $P(0) = 900$ gives $C = 60$, and then the condition $P(6) = 441$ implies that $k = 3$. Therefore $2\sqrt{P} = -3t + 60$, so $P = 0$ after $t = 20$ weeks.

11. **(a)** Starting with $dP/dt = k\sqrt{P}$, $dP/dt = k\sqrt{P}$, we separate the variables and integrate to get $P(t) = (kt/2 + C)^2$. Clearly $P(0) = P_0$ implies $C = \sqrt{P_0}$.

(b) If $P(t) = (kt/2 + 10)^2$, then $P(6) = 169$ implies that $k = 1$. Hence $P(t) = (t/2 + 10)^2$, so there are 256 fish after 12 months.

12. Solution of the equation $P' = k\,P^2$ by separation of variables and integration,

$$\int \frac{dP}{P^2} = \int k\,dt; \qquad -\frac{1}{P} = kt - C,$$

gives $P(t) = 1/(C - kt)$. Now $P(0) = 12$ implies that $C = 1/12$, so now $P(t) = 12/(1 - 12kt)$. Then $P(10) = 24$ implies that $k = 1/240$, so finally $P(t) = 240/(20 - t)$. Hence $P = 48$ when $t = 15$, that is, in the year 2003. And obviously $P \to \infty$ as $t \to 20$.

13. **(a)** If the birth and death rates both are proportional to P^2 and $\beta > \delta$, then Eq. (1) in this section gives $P' = kP^2$ with k positive. Separating variables and integrating as in Problem 12, we find that $P(t) = 1/(C - kt)$. The initial condition $P(0) = P_0$ then gives $C = 1/P_0$, so $P(t) = 1/(1/P_0 - kt) = P_0/(1 - kP_0 t)$.

(b) If $P_0 = 6$ then $P(t) = 6/(1 - 6kt)$. Now the fact that $P(10) = 9$ implies that $k = 180$, so $P(t) = 6/(1 - t/30) = 180/(30 - t)$. Hence it is clear that $P \to \infty$ as $t \to 30$ (doomsday).

14. Now $dP/dt = -kP^2$ with $k > 0$, and separation of variables yields $P(t) = 1/(kt + C)$. Clearly $C = 1/P_0$ as in Problem 13, so $P(t) = P_0/(1 + kP_0 t)$. Therefore it is clear that $P(t) \to 0$ as $t \to \infty$, so the population dies out in the long run.

15. If we write $P' = bP(a/b - P)$ we see that $M = a/b$. Hence

$$\frac{B_0 P_0}{D_0} = \frac{(aP_0)P_0}{bP_0^2} = \frac{a}{b} = M.$$

Note also (for Problems 16 and 17) that $a = B_0/P_0$ and $b = D_0/P_0^2 = k$.

16. The relations in Problem 15 give $k = 1/2400$ and $M = 160$. The solution is $P(t) = 19200/(120 + 40e^{-t/15})$. We find that $P = 0.95M$ after about 27.69 months.

17. The relations in Problem 15 give $k = 1/2400$ and $M = 180$. The solution is $P(t) = 43200/(240 - 60e^{-3t/80})$. We find that $P = 1.05M$ after about 44.22 months.

18. If we write $P' = aP(P - b/a)$ we see that $M = b/a$. Hence

$$\frac{D_0 P_0}{B_0} = \frac{(bP_0)P_0}{aP_0^2} = \frac{b}{a} = M.$$

Note also (for Problems 19 and 20) that $b = D_0/P_0$ and $a = B_0/P_0^2 = k$.

19. The relations in Problem 18 give $k = 1/1000$ and $M = 90$. The solution is $P(t) = 9000/(100 - 10e^{9t/100})$. We find that $P = 10M$ after about 24.41 months.

20. The relations in Problem 18 give $k = 1/1100$ and $M = 120$. The solution is $P(t) = 13200/(110 + 10e^{6t/55})$. We find that $P = 0.1M$ after about 42.12 months.

21. Starting with the differential equation $dP/dt = kP(200 - P)$, we separate variables and integrate, noting that $P < 200$ because $P_0 = 100$:

$$\int \frac{dP}{P(200-P)} = \int k\,dt \quad \Rightarrow \quad \int\left(\frac{1}{P}+\frac{1}{200-P}\right)dP = \int 200k\,dt;$$

$$\ln\frac{P}{200-P} = 200kt + \ln C \quad \Rightarrow \quad \frac{P}{200-P} = Ce^{200kt}.$$

Now $P(0)=100$ gives $C=1$, and $P'(0)=1$ implies that $1=k\cdot 100(200-100)$, so we find that $k=1/10000$. Substitution of these numerical values gives

$$\frac{P}{200-P} = e^{200t/10000} = e^{t/50},$$

and we solve readily for $P(t)=200/\left(1+e^{-t/50}\right)$. Finally, $P(60)=200/\left(1+e^{-6/5}\right)\approx 153.7$ million.

22. We work in thousands of persons, so $M = 100$ for the total fixed population. We substitute $M = 100$, $P'(0) = 1$, and $P_0 = 50$ in the logistic equation, and thereby obtain

$$1 = k(50)(100-50), \qquad \text{so} \qquad k = 0.0004.$$

If t denotes the number of days until 80 thousand people have heard the rumor, then Eq. (7) in the text gives

$$80 = \frac{50\times 100}{50+(100-50)e^{-0.04t}},$$

and we solve this equation for $t\approx 34.66$. Thus the rumor will have spread to 80% of the population in a little less than 35 days.

23. **(a)** $x' = 0.8x - 0.004x^2 = 0.004x(200-x)$, so the maximum amount that will dissolve is $M = 200$ g.

(b) With $M = 200$, $P_0 = 50$, and $k = 0.004$, Equation (4) in the text yields the solution

$$x(t) = \frac{10000}{50+150\,e^{-0.08t}}.$$

Substituting $x = 100$ on the left, we solve for $t = 1.25 \ln 3 \approx 1.37$ sec.

24. The differential equation for $N(t)$ is $N'(t) = kN(15-N)$. When we substitute $N(0) = 5$ (thousands) and $N'(0) = 0.5$ (thousands/day) we find that $k = 0.01$. With N in place of P, this is the logistic equation in Eq. (3) of the text, so its solution is given by Equation (7):

$$N(t) = \frac{15\times 5}{5+10\exp[-(0.01)(15)t]} = \frac{15}{1+2\,e^{-0.15\,t}}.$$

Upon substituting $N = 10$ on the left, we solve for $t = (\ln 4)/(0.15) \approx 9.24$ days.

25. Proceeding as in Example 3 in the text, we solve the equations

$$25.00k(M-25.00) = 3/8, \qquad 47.54k(M-47.54) = 1/2$$

for $M = 100$ and $k = 0.0002$. Then Equation (4) gives the population function

$$P(t) = \frac{2500}{25+75e^{-0.02t}}.$$

We find that $P = 75$ when $t = 50 \ln 9 \approx 110$, that is, in 2035 A. D.

26. The differential equation for $P(t)$ is

$$P'(t) = 0.001P^2 - \delta P.$$

When we substitute $P(0) = 100$ and $P'(0) = 8$ we find that $\delta = 0.02$, so

$$\frac{dP}{dt} = 0.001P^2 - 0.02P = 0.001P(P-20).$$

We separate variables and integrate, noting that $P > 20$ because $P_0 = 100$:

$$\int \frac{dP}{P(P-20)} = \int 0.001\,dt \quad\Rightarrow\quad \int\left(\frac{1}{P-20}-\frac{1}{P}\right)dP = \int 0.02\,dt;$$

$$\ln\frac{P-20}{P} = \frac{1}{50}t + \ln C \quad\Rightarrow\quad \frac{P-20}{P} = Ce^{t/50}.$$

Now $P(0) = 100$ gives $C = 4/5$, hence

$$5(P-20) = 4P\,e^{t/50} \quad\Rightarrow\quad P(t) = \frac{100}{5-4e^{t/50}}.$$

It follows readily that $P = 200$ when $t = 50 \ln(9/8) \approx 5.89$ months.

27. We are given that

$$P' = kP^2 - 0.01P,$$

When we substitute $P(0) = 200$ and $P'(0) = 2$ we find that $k = 0.0001$, so

$$\frac{dP}{dt} = 0.0001P^2 - 0.01P = 0.0001P(P-100).$$

We separate variables and integrate, noting that $P > 100$ because $P_0 = 200$:

$$\int \frac{dP}{P(P-100)} = \int 0.0001\, dt \quad \Rightarrow \quad \int\left(\frac{1}{P-100} - \frac{1}{P}\right) dP = \int 0.01\, dt;$$

$$\ln \frac{P-100}{P} = \frac{1}{100}t + \ln C \quad \Rightarrow \quad \frac{P-100}{P} = Ce^{t/100}.$$

Now $P(0) = 100$ gives $C = 1/2$, hence

$$2(P-100) = P\,e^{t/100} \quad \Rightarrow \quad P(t) = \frac{200}{2-e^{t/100}}.$$

(a) $P = 1000$ when $t = 100 \ln(9/5) \approx 58.78$.

(b) $P \to \infty$ as $t \to 100 \ln 2 \approx 69.31$.

28. Our alligator population satisfies the equation

$$\frac{dP}{dt} = 0.0001x^2 - 0.01x = 0.0001x(x-100).$$

With x in place of P, this is the same differential equation as in Problem 27, but now we use absolute values to allow both possibilities $x < 100$ and $x > 100$:

$$\int \frac{dx}{x(x-100)} = \int 0.0001\, dt \quad \Rightarrow \quad \int\left(\frac{1}{x-100} - \frac{1}{x}\right) dP = \int 0.01\, dt;$$

$$\ln \frac{|x-100|}{x} = \frac{1}{100}t + \ln C \quad \Rightarrow \quad \frac{|x-100|}{x} = Ce^{t/100}. \qquad (*)$$

(a) If $x(0) = 25$ then $x < 100$ and $|x-100| = 100 - x$, so (*) gives $C = 3$ and hence

$$100 - x = 3x\,e^{t/100} \quad \Rightarrow \quad x(t) = \frac{100}{1+3e^{t/100}}.$$

We therefore see that $x(t) \to 0$ as $t \to \infty$.

(b) But if $x(0)=150$ then $x>100$ and $|x-100|=x-100$, so (*) gives $C=1/3$ and hence

$$3(x-100) = x\,e^{t/100} \quad\Rightarrow\quad x(t) = \frac{300}{3-e^{t/100}}.$$

Now $x(t)\to+\infty$ as $t\to(100\ln 3)^{-}$, so doomsday occurs after about 109.86 months.

29. Here we have the logistic equation

$$\frac{dP}{dt} = 0.03135P-0.0001489P^2 = 0.0001489P(210.544-P)$$

where $k=0.0001489$ and $P=210.544$. With $P_0=3.9$ also, Eq. (7) in the text gives

$$P(t) = \frac{(210.544)(3.9)}{(3.9)+(210.544-3.9)e^{-(0.0001489)(210.544)t}} = \frac{821.122}{3.9+206.644e^{-0.03135t}}.$$

(a) This solution gives $P(140)\approx 127.008$, fairly close to the actual 1930 U.S. census population of 123.2 million.

(b) The limiting population as $t\to\infty$ is $821.122/3.9=210.544$ million.

(c) Since the actual U.S. population in 200 was about 281 million — already exceeding the maximum population predicted by the logistic equation — we see that that this model did *not* continue to hold throughout the 20th century.

30. The equation is separable, so we have

$$\int\frac{dP}{P} = \int\beta_0 e^{-\alpha t}dt, \qquad\text{so}\qquad \ln P = -\frac{\beta_0}{\alpha}e^{-\alpha t}+C.$$

The initial condition $P(0)=P_0$ gives $C=\ln P_0+\beta_0/\alpha$, so

$$P(t) = P_0\exp\left[\frac{\beta_0}{\alpha}\left(1-e^{-\alpha t}\right)\right].$$

31. If we substitute $P(0) = 10^6$ and $P'(0) = 3\times10^5$ into the differential equation

$$P'(t) = \beta_0 e^{-\alpha t}P,$$

we find that $\beta_0 = 0.3$. Hence the solution given in Problem 30 is

$$P(t) = P_0\exp[(0.3/\alpha)(1-e^{-\alpha t})].$$

The fact that $P(6) = 2P_0$ now yields the equation

$$f(\alpha) = (0.3)(1 - e^{-6\alpha}) - \alpha \ln 2 = 0$$

for α. We apply Newton's iterative formula

$$\alpha_{n+1} = \alpha_n - \frac{f(\alpha_n)}{f'(\alpha_n)}$$

with $f'(\alpha) = 1.8\,e^{-6\alpha} - \ln 2$ and initial guess $\alpha_0 = 1$, and find that $\alpha \approx 0.3915$. Therefore the limiting cell population as $t \to \infty$ is

$$P_0 \exp(\beta_0 / \alpha) = 10^6 \exp(0.3/0.3915) \approx 2.15 \times 10^6.$$

Thus the tumor does not grow much further after 6 months.

32. We separate the variables in the logistic equation and use absolute values to allow for both possibilities $P_0 < M$ and $P_0 > M$:

$$\int \frac{dP}{P(M-P)} = \int k\,dt \quad \Rightarrow \quad \int \left(\frac{1}{P} + \frac{1}{M-P} \right) dP = \int kM\,dt;$$

$$\ln \frac{P}{|M-P|} = kMt + \ln C \quad \Rightarrow \quad \frac{P}{|M-P|} = Ce^{kMt}. \qquad (*)$$

If $P_0 < M$ then $P < M$ and $|M - P| = M - P$, so substitution of $t = 0$, $P = P_0$ in (*) gives $C = P_0/(M - P_0)$. It follows that

$$\frac{P}{M-P} = \frac{P_0}{M-P_0} e^{kMt}.$$

But if $P_0 > M$ then $P > M$ and $|M - P| = P - M$, so substitution of $t = 0$, $P = P_0$ in (*) gives $C = P_0/(P_0 - M)$, and it follows that

$$\frac{P}{P-M} = \frac{P_0}{P_0-M} e^{kMt}.$$

We see that the preceding two equations are equivalent, and either yields

$$(M - P_0)P = (M - P)P_0\,e^{kMt} \quad \Rightarrow \quad P(t) = \frac{MP_0 e^{kMt}}{(M - P_0) + P_0 e^{kMt}},$$

which gives the desired result upon division of numerator and denominator by e^{kMt}.

33. (a) We separate the variables in the extinction-explosion equation and use absolute values to allow for both possibilities $P_0 < M$ and $P_0 > M$:

$$\int \frac{dP}{P(P-M)} = \int k\,dt \quad \Rightarrow \quad \int \left(\frac{1}{P-M} - \frac{1}{P} \right) dP = \int kM\,dt;$$

$$\ln \frac{|P-M|}{P} = kMt + \ln C \quad \Rightarrow \quad \frac{|P-M|}{P} = Ce^{kMt}. \qquad (*)$$

If $P_0 < M$ then $P < M$ and $|P-M| = M - P$, so substitution of $t = 0$, $P = P_0$ in (*) gives $C = (M - P_0)/P_0$. It follows that

$$\frac{M-P}{P} = \frac{M-P_0}{P_0} e^{kMt}.$$

But if $P_0 > M$ then $P > M$ and $|P-M| = P - M$, so substitution of $t = 0$, $P = P_0$ in (*) gives $C = (P_0 - M)/P_0$, and it follows that

$$\frac{P-M}{P} = \frac{P_0 - M}{P_0} e^{kMt}.$$

We see that the preceding two equations are equivalent, and either yields

$$(P-M)P_0 = (P_0 - M)P e^{kMt} \quad \Rightarrow \quad P(t) = \frac{MP_0}{P_0 + (M - P_0)e^{kMt}}.$$

(b) If $P_0 < M$ then the coefficient $M - P_0$ is positive and the denominator increases without bound, so $P(t) \to 0$ as $t \to \infty$. But if $P_0 > M$, then the denominator $P_0 - (P_0 - M)e^{kMt}$ approaches zero — so $P(t) \to +\infty$ — as t approaches the value $(1/kM)\ln([P_0/(P_0 - M)] > 0$ from the left.

34. Differentiation of both sides of the logistic equation $P' = kP \cdot (M - P)$ yields

$$\begin{aligned} P'' &= \frac{dP'}{dP} \cdot \frac{dP}{dt} \\ &= [k \cdot (M-P) + kP \cdot (-1)] \cdot kP(M-P) \\ &= k[M - 2P] \cdot kP(M-P) = 2k^2 P(M - \tfrac{1}{2}P)(M-P) \end{aligned}$$

as desired. The conclusions that $P'' > 0$ if $0 < P < \frac{1}{2}M$, that $P'' = 0$ if $P = \frac{1}{2}M$, and that $P'' < 0$ if $\frac{1}{2}M < P < M$ are then immediate. Thus it follows that each of the curves for which $P_0 < M$ has an inflection point where it crosses the horizontal line $P = \frac{1}{2}M$.

35. Any way you look at it, you should see that, the larger the parameter $k > 0$ is, the faster the logistic population $P(t)$ approaches its limiting population M.

36. With $x = e^{-50kM}$, $P_0 = 5.308$, $P_1 = 23.192$, and $P_2 = 76.212$, Eqs. (7) in the text take the form

$$\frac{P_0 M}{P_0 + (M - P_0)x} = P_1, \quad \frac{P_0 M}{P_0 + (M - P_0)x^2} = P_2$$

from which we get

$$P_0 + (M - P_0)x = P_0 M / P_1, \quad P_0 + (M - P_0)x^2 = P_0 M / P_2$$

$$x = \frac{P_0(M - P_1)}{P_1(M - P_0)}, \quad x^2 = \frac{P_0(M - P_2)}{P_2(M - P_0)} \tag{i}$$

$$\frac{P_0^2(M - P_1)^2}{P_1^2(M - P_0)^2} = \frac{P_0(M - P_2)}{P_2(M - P_0)}$$

$$P_0 P_2 (M - P_1)^2 = P_1^2 (M - P_0)(M - P_2)$$

$$P_0 P_2 M^2 - 2P_0 P_1 P_2 M + P_0 P_1^2 P_2 = P_1^2 M^2 - P_1^2(P_0 + P_2)M + P_0 P_1^2 P_2$$

We cancel the final terms on the two sides of this last equation and solve for

$$M = \frac{P_1(2P_0P_2 - P_0P_1 - P_1P_2)}{P_0P_2 - P_1^2}. \tag{ii}$$

Substitution of the given values $P_0 = 5.308$, $P_1 = 23.192$, and $P_2 = 76.212$ now gives $M = 188.121$. The first equation in (i) and $x = \exp(-kMt_1)$ yield

$$k = -\frac{1}{Mt_1} \ln \frac{P_0(M - P_1)}{P_1(M - P_0)}. \tag{iii}$$

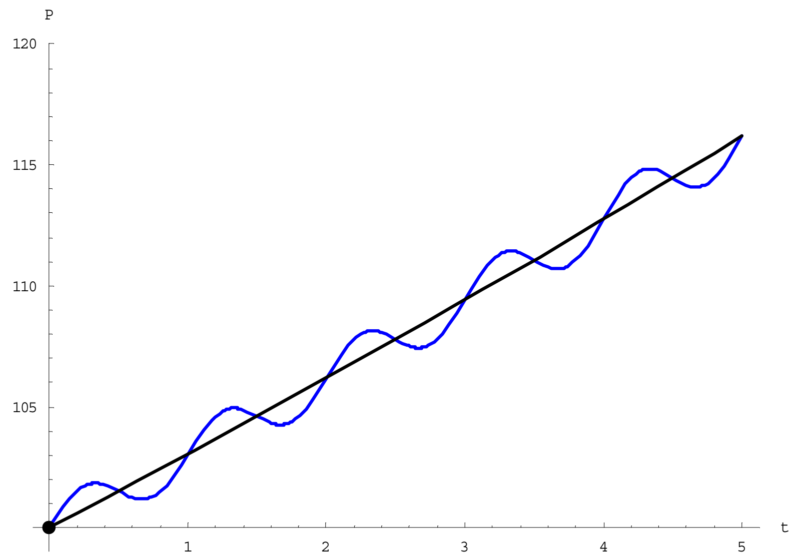
Now substitution of $t_1 = 50$ and our numerical values of M, P_0, P_1, P_2 gives $k = 0.000167716$. Finally, substitution of these values of k and M (and P_0) in the logistic solution (4) gives the logistic model of Eq. (8) in the text.

In Problems 37 and 38 we give just the values of k and M calculated using Eqs. (ii) and (iii) in Problem 36 above, the resulting logistic solution, and the predicted year 2000 population.

37. $k = 0.0000668717$ and $M = 338.027$, so $P(t) = \dfrac{25761.7}{76.212 + 261.815 e^{-0.0226045t}}$,

predicting $P = 192.525$ in the year 2000.

38. $k = 0.000146679$ and $M = 208.250$, so $P(t) = \dfrac{4829.73}{23.192 + 185.058\, e^{-0.0305458\, t}}$, predicting $P = 248.856$ in the year 2000.



39. We readily separate the variables and integrate:

$$\int \frac{dP}{P} = \int (k + b\cos 2\pi t)\, dt \quad \Rightarrow \quad \ln P = kt + \frac{b}{2\pi}\sin 2\pi t + \ln C.$$

Clearly $C = P_0$, so we find that $P(t) = P_0 \exp\left(kt + \dfrac{b}{2\pi}\sin 2\pi t\right)$. The colored curve in the figure above shows the graph that results with the typical numerical values $P_0 = 100$, $k = 0.03$, and $b = 0.06$. It oscillates about the black curve which represents natural growth with P_0 and $k = 0.03$. We see that the two agree at the end of each full year.

SECTION 2.2

EQUILIBRIUM SOLUTIONS AND STABILITY

In Problems 1–12 we identify the stable and unstable critical points as well as the funnels and spouts along the equilibrium solutions. In each problem the indicated solution satisfying $x(0) = x_0$ is derived fairly routinely by separation of variables. In some cases, various signs in the solution depend on the initial value, and we give a typical solution. For each problem we show typical solution curves corresponding to different values of x_0.

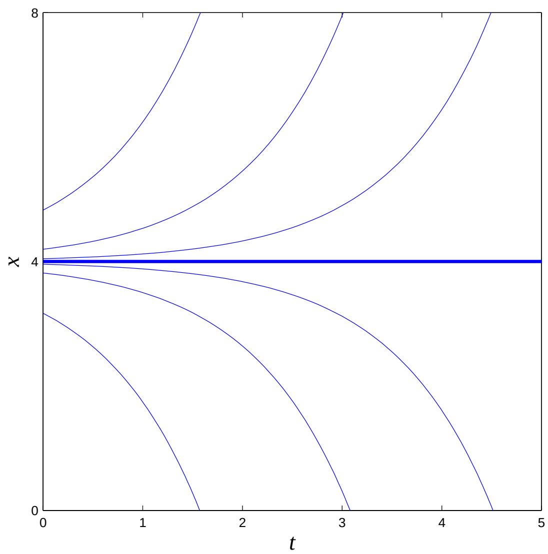
1. Unstable critical point: $x = 4$
Spout: Along the equilibrium solution $x(t) = 4$

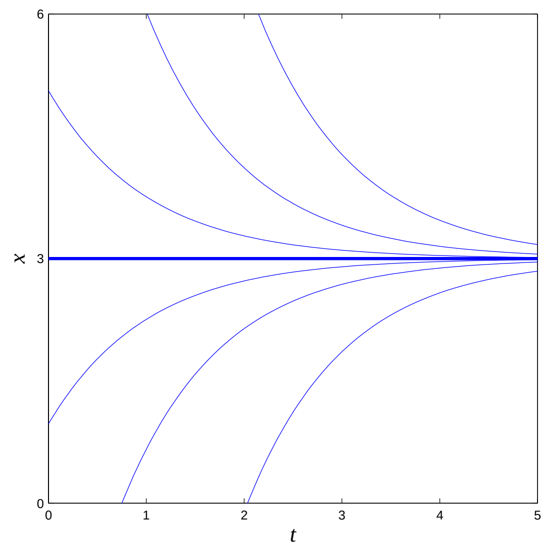
Solution: If $x_0 > 4$ then

$$\int \frac{dx}{x-4} = \int dt; \quad \ln(x-4) = t + C; \quad C = \ln(x_0 - 4)$$

$$x - 4 = (x_0 - 4)e^t; \quad x(t) = 4 + (x_0 - 4)e^t .$$

Typical solution curves are shown in the figure on the left below.





2. Stable critical point: $x = 3$
Funnel: Along the equilibrium solution $x(t) = 3$

Solution: If $x_0 > 3$ then

$$\int \frac{dx}{x-3} = \int (-1)\,dt; \quad \ln(x-3) = -t + C; \quad C = \ln(x_0 - 3)$$

$$x - 3 = (x_0 - 3)e^{-t}; \quad x(t) = 3 + (x_0 - 3)e^{-t} .$$

Typical solution curves are shown in the figure on the right above.

3. Stable critical point: $x = 0$
Unstable critical point: $x = 4$
Funnel: Along the equilibrium solution $x(t) = 0$
Spout: Along the equilibrium solution $x(t) = 4$

Solution: If $x_0 > 4$ then

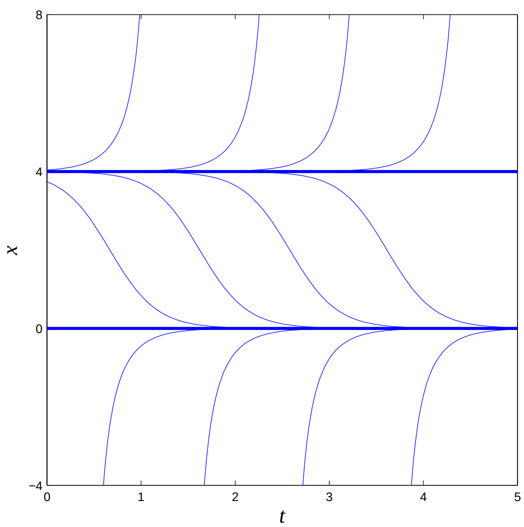
$$\int 4\,dt = \int \frac{4\,dx}{x(x-4)} = \int \left(\frac{1}{x-4} - \frac{1}{x} \right) dx$$

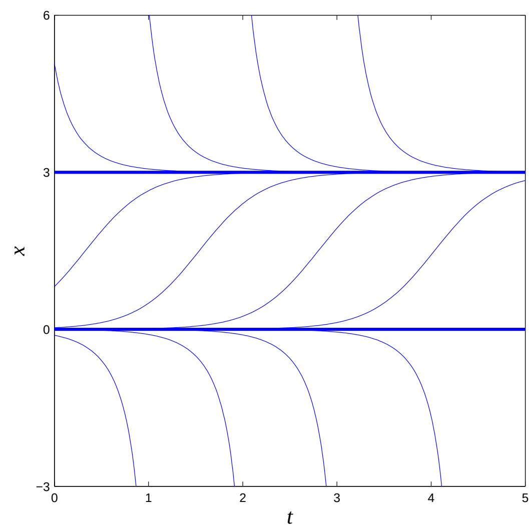
$$4t + C = \ln\frac{x-4}{x}; \qquad C = \ln\frac{x_0 - 4}{x_0}$$

$$4t = \ln\frac{x_0(x-4)}{x(x_0-4)}; \qquad e^{4t} = \frac{x_0(x-4)}{x(x_0-4)}$$

$$x(t) = \frac{4x_0}{x_0 + (4-x_0)e^{4t}}.$$

Typical solution curves are shown in the figure on the left below.





4. Stable critical point: $x = 3$
Unstable critical point: $x = 0$
Funnel: Along the equilibrium solution $x(t) = 3$
Spout: Along the equilibrium solution $x(t) = 0$

Solution: If $x_0 > 3$ then

$$\int(-3)\,dt = \int\frac{3\,dx}{x(x-3)} = \int\left(\frac{1}{x-3} - \frac{1}{x}\right)dx$$

$$-3t + C = \ln\frac{x-3}{x}; \qquad C = \ln\frac{x_0-3}{x_0}$$

$$-3t = \ln\frac{x_0(x-3)}{x(x_0-3)}; \qquad e^{-3t} = \frac{x_0(x-3)}{x(x_0-3)}$$

$$x(t) = \frac{3x_0}{x_0 + (3-x_0)e^{-3t}}.$$

Typical solution curves are shown in the figure on the right above.

5. Stable critical point: $x = -2$
Unstable critical point: $x = 2$
Funnel: Along the equilibrium solution $x(t) = -2$
Spout: Along the equilibrium solution $x(t) = 2$

Solution: If $x_0 > 2$ then

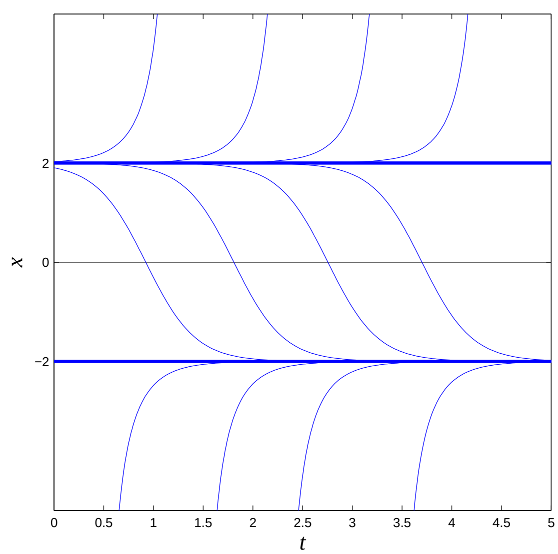
$$\int 4\,dt = \int \frac{4\,dx}{x^2-4} = \int \left(\frac{1}{x-2} - \frac{1}{x+2}\right)dx$$

$$4t + C = \ln\frac{x-2}{x+2}; \qquad C = \ln\frac{x_0-2}{x_0+2}$$

$$4t = \ln\frac{(x-2)(x_0+2)}{(x+2)(x_0-2)}; \qquad e^{4t} = \frac{(x-2)(x_0+2)}{(x+2)(x_0-2)}$$

$$x(t) = \frac{2\left[(x_0+2)+(x_0-2)e^{4t}\right]}{(x_0+2)-(x_0-2)e^{4t}}.$$

Typical solution curves are shown in the figure below.



6. Stable critical point: $x = 3$
Unstable critical point: $x = -3$
Funnel: Along the equilibrium solution $x(t) = 3$
Spout: Along the equilibrium solution $x(t) = -3$

Solution: If $x_0 > 3$ then

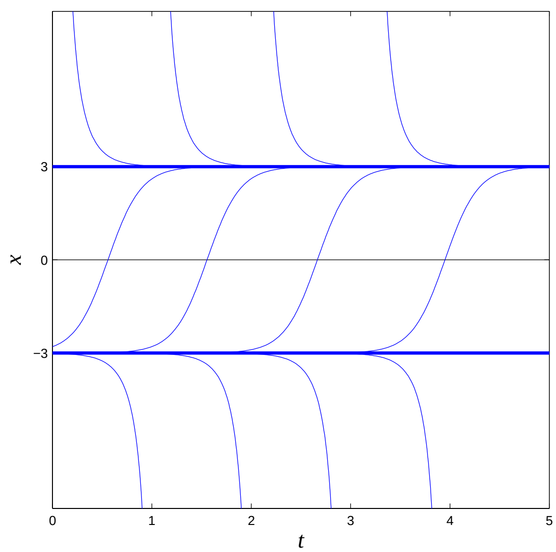
$$\int 6\,dt = \int \frac{6\,dx}{9-x^2} = \int\left(\frac{1}{3+x}+\frac{1}{3-x}\right)dx$$

$$6t + C = \ln\frac{x+3}{x-3}; \qquad C = \ln\frac{x_0+3}{x_0-3}$$

$$6t = \ln\frac{(x+3)(x_0-3)}{(x_0+3)(x-3)}; \qquad e^{6t} = \frac{(x+3)(x_0-3)}{(x_0+3)(x-3)}$$

$$x(t) = \frac{3\left[(x_0-3)+(x_0+3)e^{6t}\right]}{(3-x_0)+(x_0+3)e^{6t}}.$$

Typical solution curves are shown in the figure below.



7. Critical point: $x = 2$

This single critical point is *semi-stable*, meaning that solutions with $x_0 > 2$ go to infinity as t increases, while solutions with $x_0 < 2$ approach 2.

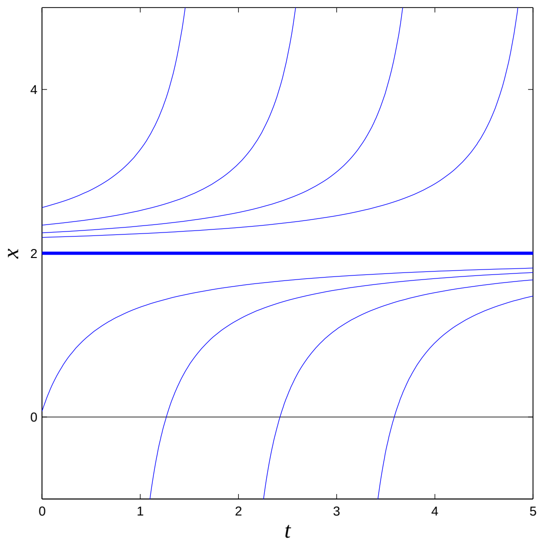
Solution: If $x_0 > 2$ then

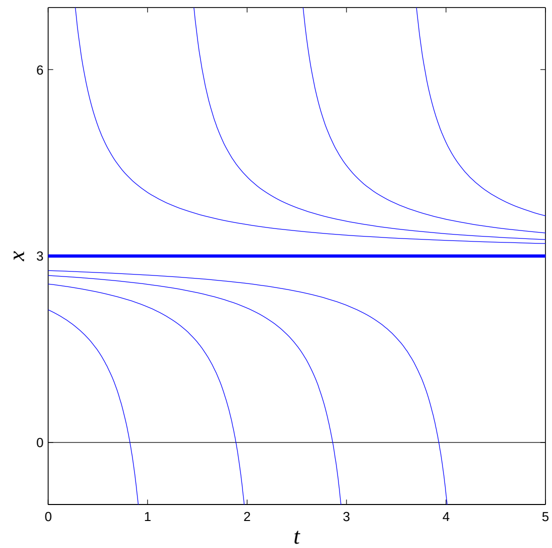
$$\int \frac{-dx}{(x-2)^2} = \int(-1)\,dt; \qquad \frac{1}{x-2} = -t + C; \qquad C = \frac{1}{x_0-2}$$

$$\frac{1}{x-2} = -t + \frac{1}{x_0-2} = \frac{1-t(x_0-2)}{x_0-2}$$

$$x(t) \;=\; 2+\frac{x_0-2}{1-t(x_0-2)} \;=\; \frac{x_0(2t-1)-4t}{t\,x_0-2t-1}.$$

Typical solution curves are shown in the figure on the left below.





8. Critical point: $x \;=\; 3$

This single critical point is *semi-stable*, meaning that solutions with $x_0 < 3$ go to minus infinity as t increases, while solutions with $x_0 > 3$ approach 3.

Solution: If $x_0 > 3$ then

$$\int \frac{-dx}{(x-3)^2} \;=\; \int dt; \qquad \frac{1}{x-3} \;=\; t+C; \qquad C \;=\frac{1}{x_0-3}$$

$$\frac{1}{x-3} \;=\; t+\frac{1}{x_0-3} \;=\; \frac{1+t(x_0-3)}{x_0-3}$$

$$x(t) \;=\; 3+\frac{x_0-3}{1+t(x_0-3)} \;=\; \frac{x_0(3t+1)-9t}{t\,x_0-3t+1}.$$

Typical solution curves are shown in the figure on the right above.

9. Stable critical point: $x=1$
Unstable critical point: $x=4$
Funnel: Along the equilibrium solution $x(t) \;=\; 1$
Spout: Along the equilibrium solution $x(t) \;=\; 4$

Solution: If $x_0 > 4$ then

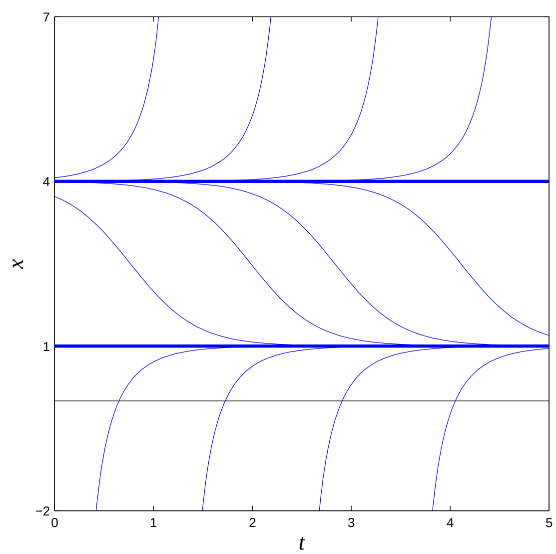
$$\int 3\,dt = \int \frac{3\,dx}{(x-4)(x-1)} = \int \left(\frac{1}{x-4} - \frac{1}{x-1} \right) dx$$

$$3t + C = \ln \frac{x-4}{x-1}; \quad C = \ln \frac{x_0 - 4}{x_0 - 1}$$

$$3t = \ln \frac{(x-4)(x_0-1)}{(x-1)(x_0-4)}; \quad e^{3t} = \frac{(x-4)(x_0-1)}{(x-1)(x_0-4)}$$

$$x(t) = \frac{4(1-x_0) + (x_0-4)e^{3t}}{(1-x_0) + (x_0-4)e^{3t}}.$$

Typical solution curves are shown in the figure on the left below.



10. Stable critical point: $x = 5$
Unstable critical point: $x = 2$
Funnel: Along the equilibrium solution $x(t) = 5$
Spout: Along the equilibrium solution $x(t) = 2$

Solution: If $x_0 > 5$ then

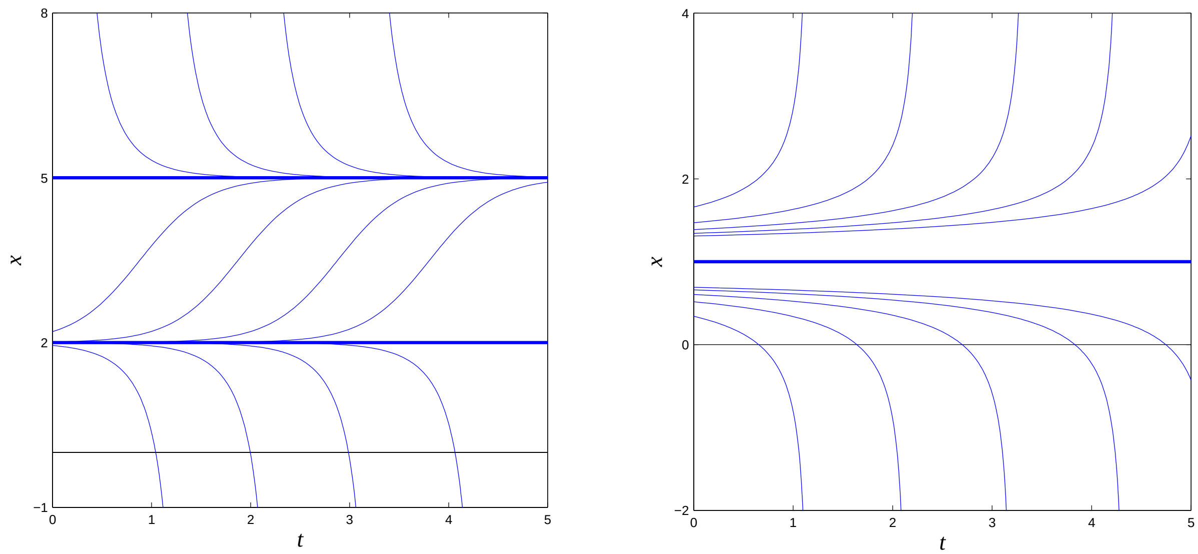
$$\int 3\,dt = \int \frac{(-3)\,dx}{(x-5)(x-2)} = \int \left(\frac{1}{x-2} - \frac{1}{x-5} \right) dx$$

$$3t + C = \ln \frac{x-2}{x-5}; \quad C = \ln \frac{x_0 - 2}{x_0 - 5}$$

$$3t = \ln \frac{(x-2)(x_0-5)}{(x-5)(x_0-2)}; \quad e^{3t} = \frac{(x-2)(x_0-5)}{(x-5)(x_0-2)}$$

$$x(t) = \frac{2(5-x_0)+5(x_0-2)e^{3t}}{(5-x_0)+(x_0-2)e^{3t}}.$$

Typical solution curves are shown in the figure on the left below.

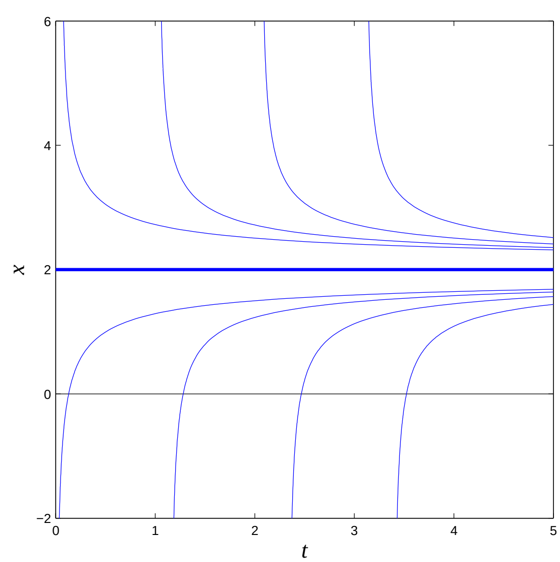


11. Unstable critical point: $x = 1$

Spout: Along the equilibrium solution $x(t) = 1$

Solution: $\displaystyle\int \frac{-2\,dx}{(x-1)^3} = \int(-2)\,dt; \qquad \frac{1}{(x-1)^2} = -2t + \frac{1}{(x_0-1)^2}.$

Typical solution curves are shown in the figure on the right above.



12. Stable critical point: $x = 2$

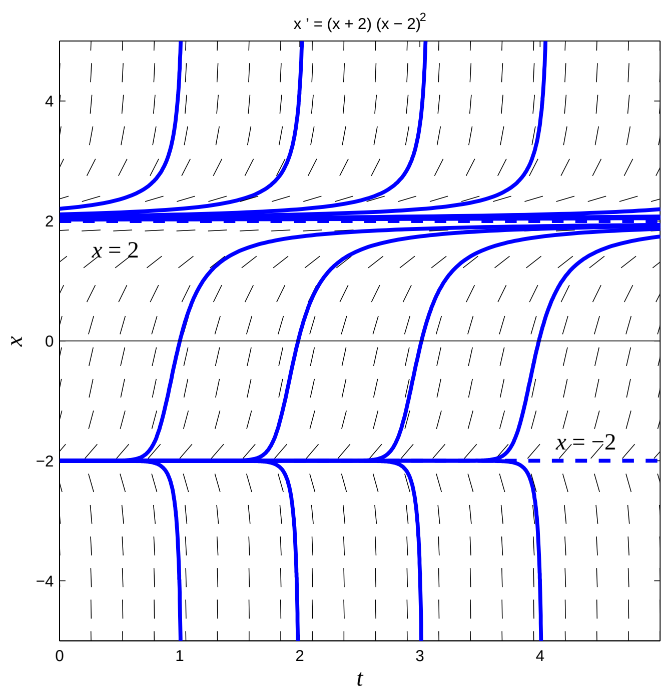
Funnel: Along the equilibrium solution $x(t) = 2$

Solution: $\displaystyle\int \frac{2\,dx}{(2-x)^3} = \int 2\,dt; \quad \frac{1}{(2-x)^2} = 2t + \frac{1}{(2-x_0)^2}.$

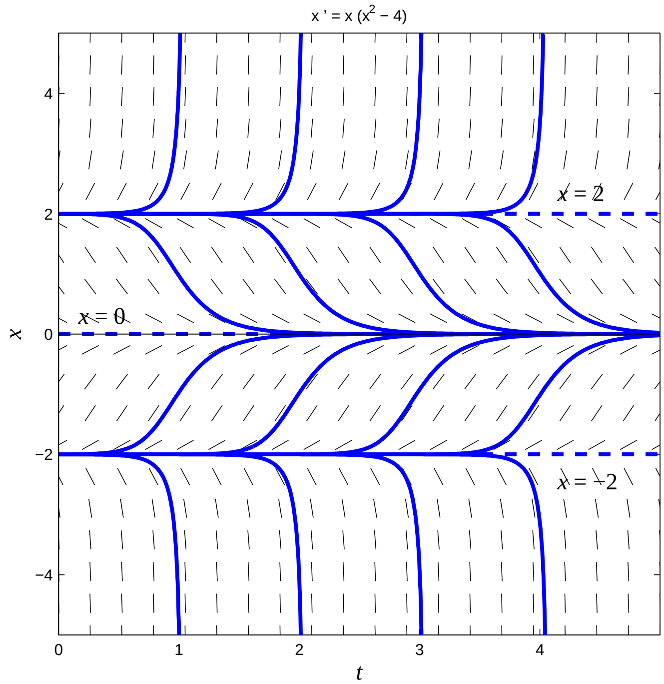
Typical solution curves are shown in the figure at the bottom of the preceding page.

In each of Problems 13 through 18 we present the figure showing slope field and typical solution curves, and then record the visually apparent classification of critical points for the given differential equation.

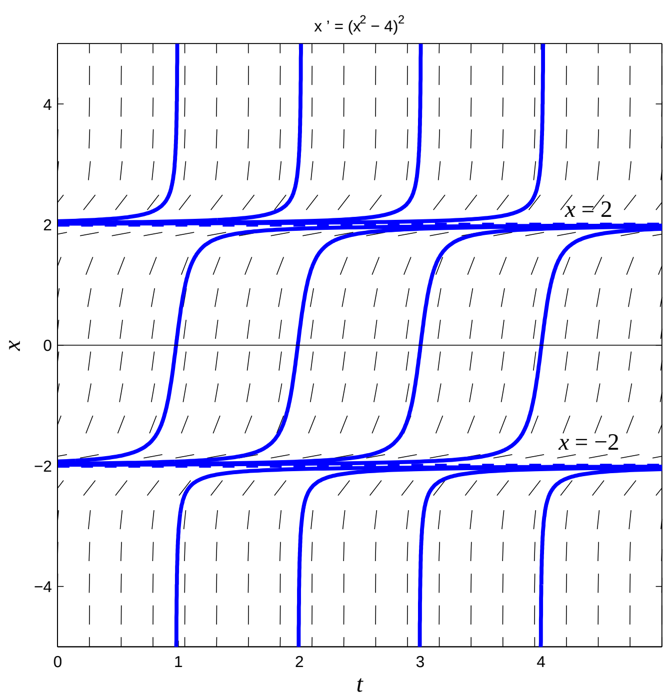
13. The critical points $x = 2$ and $x = -2$ are both unstable. A slope field and typical solution curves of the differential equation are shown below.



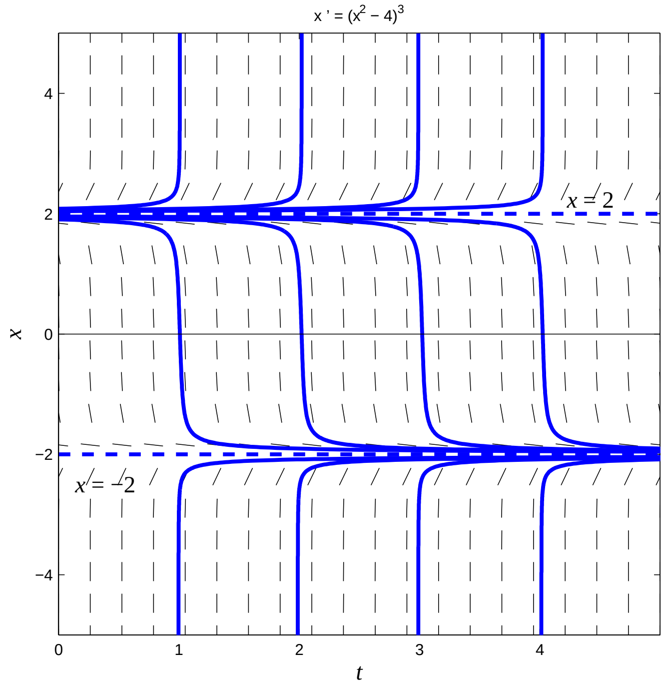
14. The critical points $x = \pm 2$ are both unstable, while the critical point $x = 0$ is stable. A slope field and typical solution curves of the differential equation are shown below.



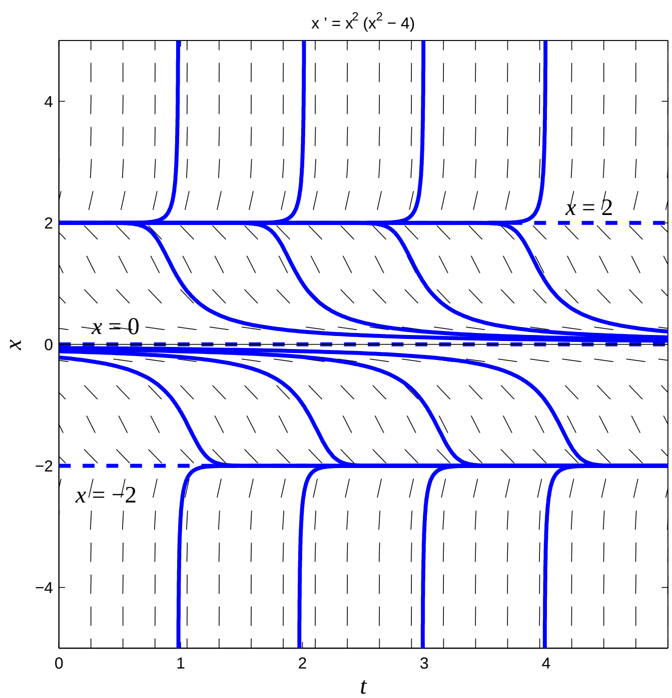
15. The critical points $x = 2$ and $x = -2$ are both unstable. A slope field and typical solution curves of the differential equation are shown below.



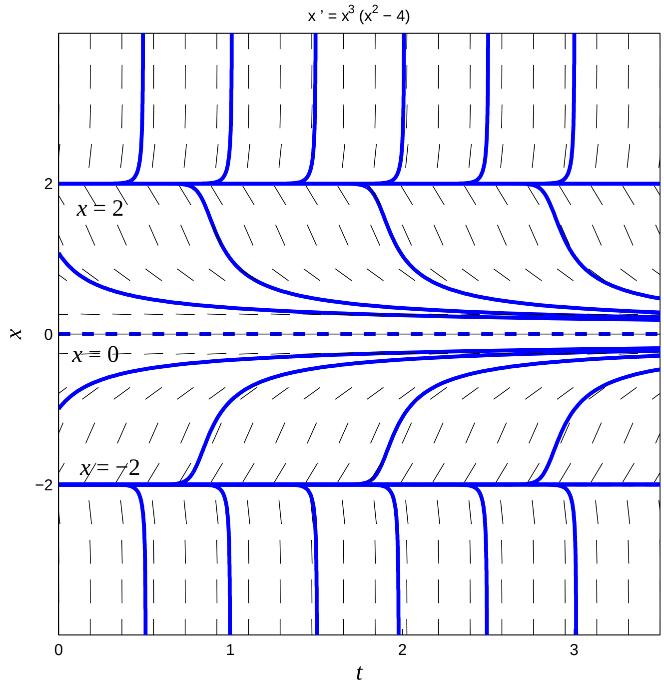
16. The critical point $x = 2$ is unstable, while the critical point $x = -2$ is stable. A slope field and typical solution curves of the differential equation are shown below.



17. The critical points $x = 2$ and $x = 0$ are unstable, while the critical point $x = -2$ is stable. A slope field and typical solution curves of the differential equation are shown below.



18. The critical points $x = 2$ and $x = -2$ are unstable, while the critical point $x = 0$ is stable. A slope field and typical solution curves of the differential equation are shown below.



19. The critical points of the given differential equation are the roots of the quadratic equation

$$\tfrac{1}{10}x(10-x)-h \;=\; 0, \qquad \text{that is,} \qquad x^2-10x+10h \;=\; 0.$$

Thus a critical point c is given in terms of h by

$$c \;=\; \frac{10\pm\sqrt{100-40h}}{2} \;=\; 5\pm\sqrt{25-10h}.$$

It follows that there is no critical point if $h > 2\tfrac{1}{2}$, only the single critical point $c = 0$ if $h = 2\tfrac{1}{2}$, and two distinct critical points if $h < 2\tfrac{1}{2}$ (so $10-25h > 0$). Hence the bifurcation diagram in the hc-plane is the parabola with the $(c-5)^2 = 25-10h$ that is obtained upon squaring to eliminate the square root above.

20. The critical points of the given differential equation are the roots of the quadratic equation

$$\tfrac{1}{100}x(x-5)+s \;=\; 0, \qquad \text{that is,} \qquad x^2-5x+100s \;=\; 0.$$

Thus a critical point c is given in terms of s by

$$c = \frac{5 \pm \sqrt{25-400s}}{2} = \frac{5}{2} \pm \frac{5}{2}\sqrt{1-16s}.$$

It follows that there is no critical point if $s > \frac{1}{16}$, only the single critical point $c = 0$ if $s = \frac{1}{16}$, and two distinct critical points if $s < \frac{1}{16}$ (so $1-16s > 0$). Hence the bifurcation diagram in the *sc*-plane is the parabola $(2c-5)^2 = 25(1-16s)$ that is obtained upon elimination of the radical above.

21. **(a)** If $k = -a^2$ where $a \geq 0$, then $kx - x^3 = -a^2x - x^3 = -x(a^2+x^2)$ is 0 only if $x = 0$, so the only critical point is $c = 0$. If $a > 0$ then we can solve the differential equation by writing

$$\int \frac{a^2 dx}{x(a^2+x^2)} = \int \left(\frac{1}{x} - \frac{x}{a^2+x^2} \right) dx = -\int a^2\, dt,$$

$$\ln x - \frac{1}{2}\ln(a^2+x^2) = -a^2t + \frac{1}{2}\ln C,$$

$$\frac{x^2}{a^2+x^2} = Ce^{-2a^2t} \quad \Rightarrow \quad x^2 = \frac{a^2Ce^{-2a^2t}}{1-Ce^{-2a^2t}}.$$

It follows that $x \to 0$ as $t \to 0$, so the critical point $c = 0$ is *stable*.

(b) If $k = +a^2$ where $a > 0$ then $kx - x^3 = +a^2x - x^3 = -x(x+a)(x-a)$ is 0 if either $x = 0$ or $x = \pm a = \pm\sqrt{k}$. Thus we have the three critical points $c = 0, \pm\sqrt{k}$, and this observation together with part (a) yields the pitchfork bifurcation diagram shown in Fig. 2.2.13 of the textbook. If $x \neq 0$ then we can solve the differential equation by writing

$$\int \frac{2a^2 dx}{x(x-a)(x+a)} = \int \left(-\frac{2}{x} + \frac{1}{x-a} + \frac{1}{x+a} \right) dx = -\int 2a^2\, dt,$$

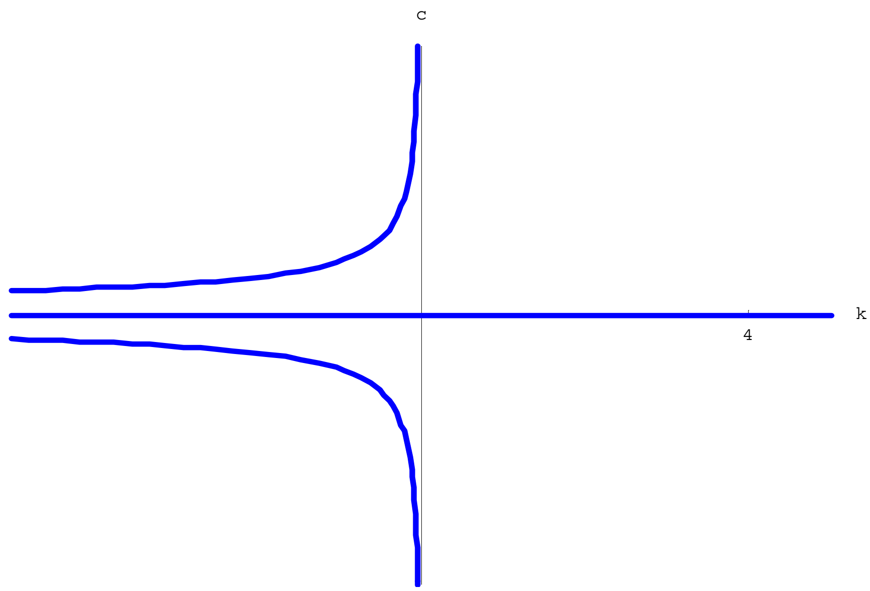
$$-2\ln x + \ln(x-a) + \ln(x-a) = -2a^2t + \ln C,$$

$$\frac{x^2-a^2}{x^2} = Ce^{-2a^2t} \quad \Rightarrow \quad x^2 = \frac{a^2}{1-Ce^{-2a^2t}} \quad \Rightarrow \quad x = \frac{\pm\sqrt{k}}{\sqrt{1-Ce^{-2a^2t}}}.$$

It follows that if $x(0) \neq 0$ then $x \to \sqrt{k}$ if $x > 0$, $x \to -\sqrt{k}$ if $x < 0$. This implies that the critical point $c = 0$ is *unstable*, while the critical points $c = \pm\sqrt{k}$ are *stable*.

22. If $k = 0$ then the only critical point $c = 0$ of the equation $x' = x$ is unstable, because the solutions $x(t) = x_0e^t$ diverge to infinity if $x_0 \neq 0$. If $k = +a^2 > 0$, then $x + a^2x^3 = x(1+a^2x^2) = 0$ only if $x = 0$, so again $c = 0$ is the only critical point. If

$k = -a^2 < 0$, then $x - a^2x^3 = x(1-a^2x^2) = x(1-ax)(1+ax) = 0$ if either $x = 0$ or $x = \pm 1/a = \pm\sqrt{-1/k}$. Hence the bifurcation diagram of the differential equation $x' = x + kx^3$ looks as pictured below:



23. **(a)** If $h < kM$ then the differential equation is $x' = k\,x\big((M - h/k) - x\big)$, which is a logistic equation with the *reduced* limiting population M - h / k.

(b) If $h > kM$ then the differential equation can be rewritten in the form $x' = -ax - bx^2$ with a and b both positive. The solution of this equation is

$$x(t) = \frac{a\,x_0}{(a + b\,x_0)e^{at} - bx_0}$$

so it is obvious that $x(t) \to 0$ as $t \to \infty$.

24. If $x_0 > N$ then

$$\int -k(N-H)\,dt = \int \frac{(N-H)\,dx}{(x-N)(x-H)} = \int \left(\frac{1}{x-N} - \frac{1}{x-H}\right)dx$$

$$-k(N-H)t + C = \ln\frac{x-N}{x-H}; \qquad C = \ln\frac{x_0 - N}{x_0 - H}$$

$$-k(N-H)t = \ln\frac{(x-N)(x_0-H)}{(x-H)(x_0-M)}; \qquad e^{-k(N-H)t} = \frac{(x-N)(x_0-H)}{(x-H)(x_0-M)}$$

$$x(t) = \frac{N(x_0 - H) - H(x_0 - N)e^{-k(N-H)t}}{(x_0 - H) - (x_0 - N)e^{-k(N-H)t}}$$

25. **(i)** In the first alternative form that is given, all of the coefficients within parentheses are positive if $H < x_0 < N$. Hence it is obvious that $x(t) \to N$ as $t \to \infty$.

(ii) In the second alternative form that is given, all of the coefficients within parentheses are positive if $x_0 < H$. Hence the denominator is initially equal to $N - H > 0$, but decreases as t increases, and reaches the value 0 when

$$t = \frac{1}{N-H} \ln \frac{N - x_0}{H - x_0} > 0.$$

26. If $4h = kM^2$ then Eqs. (13) and (14)in the text show that the differential equation takes the form $x' = -k(M/2 - x)^2$ with the single critical point $x = M/2$. This equation is readily solved by separation of variables, but clearly x' is negative whether x is less than or greater than $M/2$.

27. Separation of variables in the differential equation $x' = -k\left((x-a)^2 + b^2\right)$ yields

$$x(t) = a - b\tan\left(bk\,t + \tan^{-1}\frac{a - x_0}{b}\right).$$

It therefore follows that $x(t)$ goes to minus infinity in a finite period of time.

28. Aside from a change in sign, this calculation is the same as that indicated in Eqs. (13) and (14) in the text.

29. This is simply a matter of analyzing the signs of x' in the cases $x < a$, $a < x < b$, $b < x < c$, and $c > x$. Alternatively, plot slope fields and typical solution curves for the two differential equations using typical numerical values such as $a = -1, b = 1, c = 2$.

SECTION 2.3

ACCELERATION-VELOCITY MODELS

This section consists of three essentially independent subsections that can be studied separately: resistance proportional to velocity, resistance proportional to velocity-squared, and inverse-square gravitational acceleration.

1. Equation: $v' = k(250 - v), \quad v(0) = 0, \quad v(10) = 100$

Solution: $\int \frac{(-1)\,dv}{250 - v} = -\int k\,dt; \quad \ln(250 - v) = -kt + \ln C,$

$v(0) = 0$ implies $C = 250; \quad v(t) = 250(1 - e^{-kt})$

$v(10) = 100$ implies $k = \frac{1}{10}\ln(250/150) \approx 0.0511;$

Answer: $v = 200$ when $t = -(\ln 50/250)/k \approx 31.5\text{ sec}$

2. Equation: $v' = -kv, \quad v(0) = v_0; \quad x' = v, \quad x(0) = x_0$

Solution: $x'(t) = v(t) = v_0 e^{-kt}; \quad x(t) = -(v_0/k)e^{-kt} + C$

$C = x_0 + (v_0/k)e^{-kt}; \quad x(t) = x_0 + (v_0/k)\left(1 - e^{-kt}\right)$

Answer: $\lim_{t\to\infty} x(t) = \lim_{t\to\infty}\left[x_0 + (v_0/k)\left(1 - e^{-kt}\right)\right] = x_0 + (v_0/k)$

3. Equation: $v' = -kv, \quad v(0) = 40; \quad v(10) = 20 \qquad x' = v, \quad x(0) = 0$

Solution: $v(t) = 40\,e^{-kt}$ with $k = (1/10)\ln 2$

$x(t) = (40/k)(1 - e^{-kt})$

Answer: $x(\infty) = \lim_{t\to\infty}(40/k)(1 - e^{-kt}) = 40/k = 400/\ln 2 \approx 577$ ft

4. Equation: $v' = -kv^2, \quad v(0) = v_0; \quad x' = v, \quad x(0) = x_0$

Solution: $-\int \frac{dv}{v^2} = \int k\,dt; \quad \frac{1}{v} = kt + C; \quad C = \frac{1}{v_0}$

$x'(t) = v(t) = \frac{v_0}{1 + v_0 kt}; \quad x(t) = \frac{1}{k}\ln\left(1 + v_0 kt\right) + x_0$

$x(t) \to \infty$ as $x(t) \to \infty$

5. Equation: $v' = -kv, \quad v(0) = 40; \quad v(10) = 20 \qquad x' = v, \quad x(0) = 0$

Solution: $v = \frac{40}{1 + 40kt}$ (as in Problem 3)

$v(10) = 20$ implies $40k = 1/10$, so $v(t) = \frac{400}{10 + t}$

$x(t) = 400\ln[(10 + t)/10]$

Answer: $x(60) = 400\ln 7 \approx 778$ ft

6. Equation: $v' = -kv^{3/2}, \quad v(0) = v_0; \qquad x' = v, \quad x(0) = x_0$

Solution: $-\int \frac{dv}{2v^{3/2}} = \int \frac{k\,dt}{2}; \qquad \frac{1}{\sqrt{v}} = \frac{kt}{2} + C; \qquad C = \frac{1}{\sqrt{v_0}}$

$$x'(t) = v(t) = \frac{4v_0}{\left(2 + kt\sqrt{v_0}\right)^2}; \qquad x(t) = -\frac{4\sqrt{v_0}}{k\left(2 + kt\sqrt{v_0}\right)} + C$$

$$C = x_0 + \frac{2\sqrt{v_0}}{k}; \qquad x(t) = x_0 + \frac{2\sqrt{v_0}}{k}\left(1 - \frac{2}{2 + kt\sqrt{v_0}}\right)$$

$$x(\infty) = x_0 + 2\sqrt{v_0}/k$$

7. Equation: $v' = 10 - 0.1v, \qquad x(0) = v(0) = 0$

(a) $\int \frac{-0.1\,dv}{10 - 0.1v} = \int (-0.1)\,dt; \qquad \ln(10 - 0.1v) = -t/10 + \ln C$

$$v(0) = 0 \text{ implies } C = 10; \qquad \ln\left[(10 - 0.1v)/10\right] = -t/10$$

$$v(t) = 100(1 - e^{-t/10}); \qquad v(\infty) = 100 \text{ ft/sec} \quad \text{(limiting velocity)}$$

(b) $x(t) = 100t - 1000(1 - e^{-t/10})$

$v = 90$ ft/sec when $t = 23.0259$ sec and $x = 1402.59$ ft

8. Equation: $v' = 10 - 0.001v^2, \qquad x(0) = v(0) = 0$

(a) $\int \frac{0.01\,dv}{1 - 0.0001v^2} = \int \frac{dt}{10}; \qquad \tanh^{-1}\frac{v}{100} = \frac{t}{10} + C$

$$v(0) = 0 \text{ implies } C = 0 \text{ so } v(t) = 100\tanh(t/10)$$

$$v(\infty) = \lim_{t\to\infty} 100\tanh(t/10) = 100 \lim_{t\to\infty} \frac{e^{t/10} - e^{-t/10}}{e^{t/10} + e^{-t/10}} = 100 \text{ ft/sec}$$

(b) $x(t) = 1000\ln(\cosh t/10)$

$v = 90$ ft/sec when $t = 14.7222$ sec and $x = 830.366$ ft

9. The solution of the initial value problem

$$1000\,v' = 5000 - 100\,v, \qquad v(0) = 0$$

is

$$v(\mathrm{t}) = 50(1 - e^{-t/10}).$$

Hence, as $t \to \infty$, we see that $v(t)$ approaches $v_{\max} = 50$ ft/sec $\approx$ 34 mph.

10. Before opening parachute:

$$\begin{aligned}
v' &= -32 - 0.15v, \quad v(0) = 0, \quad y(0) = 10000 \\
v(t) &= 213.333(e^{-0.15t} - 1), \qquad v(20) = -202.712 \text{ ft/sec} \\
y(t) &= 11422.2 - 1422.22e^{-0.15ty} - 213.333t, \quad y(20) = 7084.75 \text{ ft}
\end{aligned}$$

After opening parachute:

$$\begin{aligned}
v' &= -32 - 1.5v, \quad v(0) = -202.712, \quad y(0) = 7084.75 \\
v(t) &= -21.3333 - 181.379e^{-1.5t} \\
y(t) &= 6964.83 + 120.919e^{-1.5t} - 21.3333t, \\
y &= 0 \text{ when } \quad t = 326.476
\end{aligned}$$

Thus she opens her parachute after 20 sec at a height of 7085 feet, and the total time of descent is $20 + 326.476 = 346.476$ sec, about 5 minutes and 46.5 seconds. Her impact speed is 21.33 ft/sec, about 15 mph.

11. If the paratrooper′s terminal velocity was 100 mph = 440/3 ft/sec, then Equation (7) in the text yields $\rho = 12/55$. Then we find by solving Equation (9) numerically with $y_0 = 1200$ and $v_0 = 0$ that $y = 0$ when $t \approx 12.5$ sec. Thus the newspaper account is inaccurate.

12. With $m = 640/32 = 20$ slugs, $W = 640$ lb, $B = (8)(62.5) = 500$ lb, and $F_R = -v$ lb (F_R is upward when $v < 0$), the differential equation is

$$20\,v'(t) = -640 + 500 - v = -140 - v\,.$$

Its solution with $v(0) = 0$ is

$$v(t) = 140\left(e^{-0.05t} - 1\right),$$

and integration with $y(0) = 0$ yields

$$y(t) = 2800\left(e^{-0.05t} - 1\right) - 140t\,.$$

Using these equations we find that $t = 20\ln(28/13) \approx 15.35$ sec when $v = -75$ ft/sec, and that $y(15.35) \approx -648.31$ ft. Thus the maximum safe depth is just under 650 ft.

Given the hints and integrals provided in the text, Problems 13–16 are fairly straightforward (and fairly tedious) integration problems.

17. To solve the initial value problem $v' = -9.8 - 0.0011v^2, \quad v(0) = 49$ we write

$$\int \frac{dv}{9.8 + 0.0011v^2} = -\int dt; \qquad \int \frac{0.010595\,dv}{1 + (0.010595\,v)^2} = -\int 0.103827\,dt$$

$$\tan^{-1}(0.010595\,v) = -0.103827\,t + C; \qquad v(0) = 49 \text{ implies } C = 0.478854$$

$$v(t) = 94.3841\tan(0.478854 - 0.103827\,t)$$

Integration with $y(0) = 0$ gives

$$y(t) = 108.468 + 909.052\ \ln(\cos(0.478854 - 0.103827\,t)).$$

We solve $v(0) = 0$ for $t = 4.612$, and then calculate $y(4.612) = 108.468$.

18. We solve the initial value problem $v' = -9.8 + 0.0011v^2, \quad v(0) = 0$ much as in Problem 17, except using hyperbolic rather than ordinary trigonometric functions. We first get

$$v(t) = -94.3841\tanh(0.103827\,t),$$

and then integration with $y(0) = 108.47$ gives

$$y(t) = 108.47 - 909.052\ \ln(\cosh(0.103827\,t)).$$

We solve $y(0) = 0$ for $t = \cosh^{-1}(\exp(108.47/909.052))/0.103.827 \approx 4.7992$, and then calculate $v(4.7992) = -43.489$.

19. Equation: $v' = 4 - (1/400)v^2, \quad v(0) = 0$

Solution: $$\int \frac{dv}{4 - (1/400)v^2} = \int dt; \qquad \int \frac{(1/40)\,dv}{1 - (v/40)^2} = \int \frac{1}{10}\,dt$$

$$\tanh^{-1}(v/40) = t/10 + C; \qquad C = 0; \qquad v(t) = 40\tanh(t/10)$$

Answer: $v(10) \approx 30.46$ ft/sec, $\quad v(\infty) = 40$ ft/sec

20. Equation: $v' = -32 - (1/800)v^2, \quad v(0) = 160, \quad y(0) = 0$

Solution: $$\int \frac{dv}{32 + (1/800)v^2} = -\int dt; \qquad \int \frac{(1/160)\,dv}{1 + (v/160)^2} = -\int \frac{1}{5}\,dt;$$

$$\tan^{-1}(v/160) = -t/5+C; \qquad v(0)=160 \text{ implies } C=\pi/4$$

$$v(t) = 160\tan\left(\frac{\pi}{4}-\frac{t}{5}\right)$$

$$y(t) = 800\ln\left(\cos\left(\frac{\pi}{4}-\frac{t}{5}\right)\right)+400\ln 2$$

We solve $v(t) = 0$ for $t=3.92699$ and then calculate $y(3.92699) = 277.26$ ft.

21. Equation: $\quad v' = -g-\rho v^2, \quad v(0) = v_0, \quad y(0) = 0$

Solution: $$\int\frac{dv}{g+\rho v^2} = -\int dt; \qquad \int\frac{\sqrt{\rho/g}\,dv}{1+\left(\sqrt{\rho/g}\,v\right)^2} = -\int\sqrt{g\rho}\,dt;$$

$$\tan^{-1}\left(\sqrt{\rho/g}\,v\right) = -\sqrt{g\rho}\,t+C; \qquad v(0)=v_0 \text{ implies } C=\tan^{-1}\left(\sqrt{\rho/g}\,v_0\right)$$

$$v(t) = -\sqrt{\frac{g}{\rho}}\tan\left(t\sqrt{g\rho}-\tan^{-1}\left(v_0\sqrt{\frac{\rho}{g}}\right)\right)$$

We solve $v(t) = 0$ for $t = \dfrac{1}{\sqrt{g\rho}}\tan^{-1}\left(v_0\sqrt{\dfrac{\rho}{g}}\right)$ and substitute in Eq. (17) for $y(t)$:

$$\begin{aligned}
y_{\max} &= \frac{1}{\rho}\ln\left|\frac{\cos\left(\tan^{-1}v_0\sqrt{\rho/g}-\tan^{-1}v_0\sqrt{\rho/g}\right)}{\cos\left(\tan^{-1}v_0\sqrt{\rho/g}\right)}\right| \\
&= \frac{1}{\rho}\ln\left(\sec\left(\tan^{-1}v_0\sqrt{\rho/g}\right)\right) = \frac{1}{\rho}\ln\sqrt{1+\frac{\rho v_0^2}{g}} \\
y_{\max} &= \frac{1}{2\rho}\ln\left(1+\frac{\rho v_0^2}{g}\right)
\end{aligned}$$

22. By an integration similar to the one in Problem 19, the solution of the initial value problem $v' = -32+0.075v^2, \quad v(0)=0$ is

$$v(t) = -20.666\tanh(1.54919\,t),$$

so the terminal speed is 20.666 ft/sec. Then a further integration with $y(0)=0$ gives

$$y(t) = 10000-13.333\ln(\cosh(1.54919\,t)).$$

We solve $y(0)=0$ for $t=484.57$. Thus the descent takes about 8 min 5 sec.

23. Before opening parachute:

$$v' = -32 + 0.00075v^2, \quad v(0) = 0, \quad y(0) = 10000$$
$$v(t) = -206.559\tanh(0.154919t) \qquad v(30) = -206.521 \text{ ft/sec}$$
$$y(t) = 10000 - 1333.33\ln(\cosh(0.154919t)), \quad y(30) = 4727.30 \text{ ft}$$

After opening parachute:

$$v' = -32 + 0.075v^2, \quad v(0) = -206.521, \quad y(0) = 4727.30$$
$$v(t) = -20.6559\tanh(1.54919t + 0.00519595)$$
$$y(t) = 4727.30 - 13.3333\ln(\cosh(1.54919t + 0.00519595))$$
$$y = 0 \text{ when } t = 229.304$$

Thus she opens her parachute after 30 sec at a height of 4727 feet, and the total time of descent is 30 + 229.304 = 259.304 sec, about 4 minutes and 19.3 seconds.

24. Let M denote the mass of the Earth. Then

(a) $\sqrt{2GM/R} = c$ implies $R = 0.884 \times 10^{-3}$ meters, about 0.88 cm;

(b) $\sqrt{2G(329320M)/R} = c$ implies $R = 2.91 \times 10^3$ meters, about 2.91 kilometers.

25. **(a)** The rocket's apex occurs when $v = 0$. We get the desired formula when we set $v = 0$ in Eq. (23),

$$v^2 = v_0^2 + 2GM\left(\frac{1}{r} - \frac{1}{R}\right),$$

and solve for r.

(b) We substitute $v = 0$, $r = R + 10^5$ (100 km = 10^5 m) and the mks values $G = 6.6726 \times 10^{-11}$, $M = 5.975 \times 10^{24}$, $R = 6.378 \times 10^6$ in Eq. (23) and solve for $v_0 = 1389.21$ m/s ≈ 1.389 km/s.

(c) When we substitute $v_0 = (9/10)\sqrt{2GM/R}$ in the formula derived in part (a), we find that $r_{\max} = 100R/19$.

26. By an elementary computation (as in Section 1.2) we find that an initial velocity of $v_0 = 16$ ft/sec is required to jump vertically 4 feet high on earth. We must determine whether this initial velocity is adequate for escape from the asteroid. Let r denote the ratio of the radius of the asteroid to the radius $R = 3960$ miles of the earth, so that

$$r = \frac{1.5}{3960} = \frac{1}{2640}.$$

Then the mass and radius of the asteroid are given by

$$M_a = r^3 M \quad \text{and} \quad R_a = rR$$

in terms of the mass M and radius R of the earth. Hence the escape velocity from the asteroid's surface is given by

$$v_a = \sqrt{\frac{2GM_a}{R_a}} = \sqrt{\frac{2G \cdot r^3 M}{rR_a}} = r\sqrt{\frac{2GM}{R}} = r v_0$$

in terms of the escape velocity v_0 from the earth's surface. Hence $v_a \approx 36680/2640$ ≈ 13.9 ft/sec. Since the escape velocity from this asteroid is thus less than the initial velocity of 16 ft/sec that your legs can provide, you can indeed jump right off this asteroid into space.

27. **(a)** Substitution of $v_0^2 = 2GM/R = k^2/R$ in Eq. (23) of the textbook gives

$$\frac{dr}{dt} = v = \sqrt{\frac{2GM}{r}} = \frac{k}{\sqrt{r}}.$$

We separate variables and proceed to integrate:

$$\int \sqrt{r}\, dr = \int k\, dt \quad \Rightarrow \quad \frac{2}{3} r^{3/2} = kt + \frac{2}{3} R^{3/2}$$

(using the fact that $r = R$ when $t = 0$). We solve for $r(t) = \left(\tfrac{2}{3}kt + R^{3/2}\right)^{2/3}$ and note that $r(t) \to \infty$ as $t \to \infty$.

(b) If $v_0 > 2GM/R$ then Eq. (23) gives

$$\frac{dr}{dt} = v = \sqrt{\frac{2GM}{r} + \left(v_0^2 - \frac{2GM}{R}\right)} = \sqrt{\frac{k^2}{r} + \alpha} > \frac{k}{\sqrt{r}}.$$

Therefore, at every instant in its ascent, the upward velocity of the projectile in this part is greater than the velocity at the same instant of the projectile of part (a). It's as though the projectile of part (a) is the fox, and the projectile of this part is a rabbit that runs faster. Since the fox goes to infinity, so does the faster rabbit.

28. **(a)** Integration of gives

$$\frac{1}{2} v^2 = GM\left(\frac{1}{r} - \frac{1}{r_0}\right)$$

and we solve for

$$\frac{dr}{dt} = v = -\sqrt{2GM\left(\frac{1}{r} - \frac{1}{r_0}\right)}$$

taking the negative square root because $v < 0$ in descent. Hence

$$\begin{aligned}
t &= -\sqrt{\frac{r_0}{2GM}}\int\sqrt{\frac{r}{r_0 - r}}\,dr \qquad (r = r_0\cos^2\theta) \\
&= \sqrt{r_0/2GM}\int 2r_0\cos^2\theta\,d\theta \\
&= \frac{r_0^{3/2}}{\sqrt{2GM}}(\theta + \sin\theta\cos\theta) \\
t &= \sqrt{\frac{r_0}{2GM}}\left(\sqrt{rr_0 - r^2} + r_0\cos^{-1}\sqrt{\frac{r}{r_0}}\right)
\end{aligned}$$

(b) Substitution of $G = 6.6726\times10^{-11}$, $M = 5.975\times10^{24}$ kg, $r = R = 6.378\times10^6$ m, and $r_0 = R + 10^6$ yields $t = 510.504$, that is, about $8\frac{1}{2}$ minutes for the descent to the surface of the earth. (Recall that we are ignoring air resistance.)

(c) Substitution of the same numeral values along with $v_0 = 0$ in the original differential equation of part (a) yields $v = -4116.42$ m/s ≈ -4.116 km/s for the velocity at impact with the earth's surface where $r = R$.

29. Integration of $v\dfrac{dv}{dy} = -\dfrac{GM}{(y+R)^2}$, $y(0) = 0$, $v(0) = v_0$ gives

$$\frac{1}{2}v^2 = \frac{GM}{y+R} - \frac{GM}{R} + \frac{1}{2}v_0^2$$

which simplifies to the desired formula for v^2. Then substitution of $G = 6.6726\times10^{-11}$, $M = 5.975\times10^{24}$ kg, $R = 6.378\times10^6$ m $v = 0$, and $v_0 = 1$ yields an equation that we easily solve for $y = 51427.3$, that is, about 51.427 km.

30. When we integrate

$$v\frac{dv}{dr} = -\frac{GM_e}{r^2} + \frac{GM_m}{(S-r)^2}, \quad r(0) = R, \quad r'(0) = v_0$$

in the usual way and solve for v, we get

$$v = \sqrt{\frac{2GM_e}{r} - \frac{2GM_e}{R} - \frac{2GM_m}{r-S} + \frac{2GM_m}{R-S} + v_0^2}.$$

The earth and moon attractions balance at the point where the right-hand side in the acceleration equation vanishes, which is when

$$r = \frac{\sqrt{M_e}\, S}{\sqrt{M_e} - \sqrt{M_m}}.$$

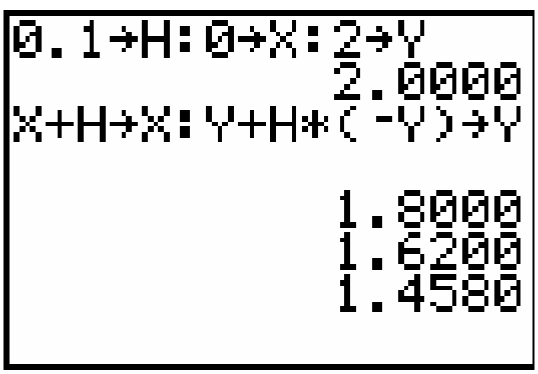
If we substitute this value of r, $M_m = 7.35\times 10^{22}$ kg, $S = 384.4\times 10^{6}$, and the usual values of the other constants involved, then set $v = 0$ (to just reach the balancing point), we can solve the resulting equation for $v_0 = 11{,}109$ m/s. Note that this is only 71 m/s less than the earth escape velocity of 11,180 m/s, so the moon really doesn't help much.

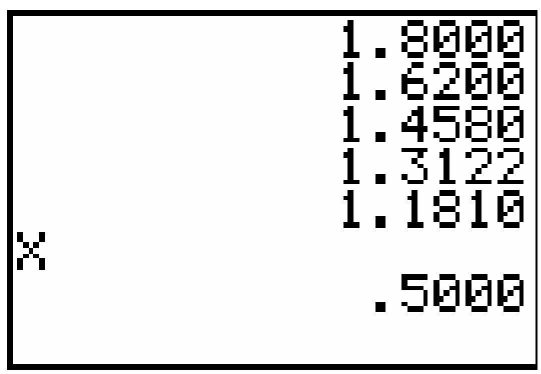
SECTION 2.4

NUMERICAL APPROXIMATION: EULER'S METHOD

In each of Problems 1–10 we also give first the explicit form of Euler's iterative formula for the given differential equation $y' = f(x,y)$. As we illustrate in Problem 1, the desired iterations are readily implemented, either manually or with a computer system or graphing calculator. Then we list the indicated values of $y(\frac{1}{2})$ rounded off accurate to 3 decimal places.

1. For the differential equation $y' = f(x,y)$ with $f(x,y) = -y$, the iterative formula of Euler's method is $y_{n+1} = y_n + h(-y_n)$. The TI-83 screen on the left shows a graphing calculator implementation of this iterative formula.





After the variables are initialized (in the first line), and the formula is entered, each press of the enter key carries out an additional step. The screen on the right shows the results of 5 steps from $x = 0$ to $x = 0.5$ with step size $h = 0.1$ — winding up with $y(0.5) \approx 1.181$.

Approximate values 1.125 and 1.181; true value $y(\frac{1}{2}) \approx 1.213$

The following *Mathematica* instructions produce precisely this line of data.

```
f[x_,y_] = -y;
g[x_] = 2 Exp[-x];
```

```
h = 0.25;   x = 0;   y1 = y0;
Do[  k = f[x,y1];           (* the left-hand slope     *)
     y1 = y1 + h*k;         (* Euler step to update y  *)
     x = x + h,             (* update x                *)
     {i,1,2} ]

h = 0.1;   x = 0;   y2 = y0;
Do[  k = f[x,y2];           (* the left-hand slope     *)
     y2 = y2 + h*k;         (* Euler step to update y  *)
     x = x + h,             (* update x                *)
     {i,1,5} ]

Print[x,"     ",y1,"     ",y2,"     ",g[0.5]]

0.5     1.125     1.18098     1.21306
```

2. Iterative formula: $y_{n+1} = y_n + h(2y_n)$

Approximate values 1.125 and 1.244; true value $y(\tfrac{1}{2}) \approx 1.359$

3. Iterative formula: $y_{n+1} = y_n + h(y_n + 1)$

Approximate values 2.125 and 2.221; true value $y(\tfrac{1}{2}) \approx 2.297$

4. Iterative formula: $y_{n+1} = y_n + h(x_n - y_n)$

Approximate values 0.625 and 0.681; true value $y(\tfrac{1}{2}) \approx 0.713$

5. Iterative formula: $y_{n+1} = y_n + h(y_n - x_n - 1)$

Approximate values 0.938 and 0.889; true value $y(\tfrac{1}{2}) \approx 0.851$

6. Iterative formula: $y_{n+1} = y_n + h(-2x_n y_n)$

Approximate values 1.750 and 1.627; true value $y(\tfrac{1}{2}) \approx 1.558$

7. Iterative formula: $y_{n+1} = y_n + h(-3x_n^2 y_n)$

Approximate values 2.859 and 2.737; true value $y(\tfrac{1}{2}) \approx 2.647$

8. Iterative formula: $y_{n+1} = y_n + h\exp(-y_n)$

Approximate values 0.445 and 0.420; true value $y(\tfrac{1}{2}) \approx 0.405$

9. Iterative formula: $y_{n+1} = y_n + h(1 + y_n^2)/4$

Approximate values 1.267 and 1.278; true value $y(\tfrac{1}{2}) \approx 1.287$

10. Iterative formula: $y_{n+1} = y_n + h(2x_n y_n^2)$

Approximate values 1.125 and 1.231; true value $y(\frac{1}{2}) \approx 1.333$

The tables of approximate and actual values called for in Problems 11–16 were produced using the following MATLAB script (appropriately altered for each problem).

```
% Section 2.4, Problems 11-16
x0 = 0;    y0 = 1;
% first run:
h = 0.01;
x = x0;  y = y0;  y1 = y0;
for  n = 1:100
   y = y + h*(y-2);
   y1 = [y1,y];
   x = x + h;
   end
% second run:
h = 0.005;
x = x0;  y = y0;  y2 = y0;
for  n = 1:200
   y = y + h*(y-2);
   y2 = [y2,y];
   x = x + h;
   end
% exact values
x = x0 : 0.2 : x0+1;
ye = 2 - exp(x);
% display table
ya = y2(1:40:201);
err = 100*(ye-ya)./ye;
[x; y1(1:20:101); ya; ye; err]
```

11. The iterative formula of Euler's method is $y_{n+1} = y_n + h(y_n - 2)$, and the exact solution is $y(x) = 2 - e^x$. The resulting table of approximate and actual values is

x	0.0	0.2	0.4	0.6	0.8	1.0
y (h=0.01)	1.0000	0.7798	0.5111	0.1833	–0.2167	–0.7048
y (h=0.005)	1.0000	0.7792	0.5097	0.1806	–0.2211	–0.7115
y actual	1.0000	0.7786	0.5082	0.1779	–0.2255	–0.7183
error	0%	–0.08%	–0.29%	–1.53%	1.97%	0.94%

12. Iterative formula: $y_{n+1} = y_n + h(y_n - 1)^2/2$

Exact solution: $y(x) = 1 + 2/(2 - x)$

x	0.0	0.2	0.4	0.6	0.8	1.0
y (h=0.01)	2.0000	2.1105	2.2483	2.4250	2.6597	2.9864
y (h=0.005)	2.0000	2.1108	2.2491	2.4268	2.6597	2.9931
y actual	2.0000	2.1111	2.2500	2.4286	2.6597	3.0000
error	0%	0.02%	0.04%	0.07%	0.13%	0.23%

13. Iterative formula: $y_{n+1} = y_n + 2hx_n^3/y_n$

Exact solution: $y(x) = (8 + x^4)^{1/2}$

x	1.0	1.2	1.4	1.6	1.8	2.0
y (h=0.01)	3.0000	3.1718	3.4368	3.8084	4.2924	4.8890
y (h=0.005)	3.0000	3.1729	3.4390	3.8117	4.2967	4.8940
y actual	3.0000	3.1739	3.4412	3.8149	4.3009	4.8990
error	0%	0.03%	0.06%	0.09%	0.10%	0.10%

14. Iterative formula: $y_{n+1} = y_n + hy_n^2/x_n$

Exact solution: $y(x) = 1/(1 - \ln x)$

x	1.0	1.2	1.4	1.6	1.8	2.0
y (h=0.01)	1.0000	1.2215	1.5026	1.8761	2.4020	3.2031
y (h=0.005)	1.0000	1.2222	1.5048	1.8814	2.4138	3.2304
y actual	1.0000	1.2230	1.5071	1.8868	2.4259	3.2589
error	0%	0.06%	0.15%	0.29%	0.50%	0.87%

15. Iterative formula: $y_{n+1} = y_n + h(3 - 2y_n/x_n)$

Exact solution: $y(x) = x + 4/x^2$

x	2.0	2.2	2.4	2.6	2.8	3.0
y (h=0.01)	3.0000	3.0253	3.0927	3.1897	3.3080	3.4422
y (h=0.005)	3.0000	3.0259	3.0936	3.1907	3.3091	3.4433
y actual	3.0000	3.0264	3.0944	3.1917	3.3102	3.4444
error	0%	0.019%	0.028%	0.032%	0.033%	0.032%

16. Iterative formula: $y_{n+1} = y_n + 2hx_n^5/y_n^2$

Exact solution: $y(x) = (x^6 - 37)^{1/3}$

x	2.0	2.2	2.4	2.6	2.8	3.0
y (h=0.01)	3.0000	4.2476	5.3650	6.4805	7.6343	8.8440
y (h=0.005)	3.0000	4.2452	5.3631	6.4795	7.6341	8.8445
y actual	3.0000	4.2429	5.3613	6.4786	7.6340	8.8451
error	0%	–0.056%	–0.034%	–0.015%	0.002%	0.006%

The tables of approximate values called for in Problems 17–24 were produced using a MATLAB script similar to the one listed preceding the Problem 11 solution above.

17.

x	0.0	0.2	0.4	0.6	0.8	1.0
y (h=0.1)	0.0000	0.0010	0.0140	0.0551	0.1413	0.2925
y (h=0.02)	0.0000	0.0023	0.0198	0.0688	0.1672	0.3379
y (h=0.004)	0.0000	0.0026	0.0210	0.0717	0.1727	0.3477
y (h=0.0008)	0.0000	0.0027	0.0213	0.0723	0.1738	0.3497

These data indicate that $y(1) \approx 0.35$, in contrast with Example 5 in the text, where the initial condition is $y(0) = 1$.

In Problems 18–24 we give only the final approximate values of y obtained using Euler's method with step sizes $h = 0.1$, $h = 0.02$, $h = 0.004$, and $h = 0.0008$.

18. With $x_0 = 0$ and $y_0 = 1$, the approximate values of $y(2)$ obtained are:

h	0.1	0.02	0.004	0.0008
y	1.6680	1.6771	1.6790	1.6794

19. With $x_0 = 0$ and $y_0 = 1$, the approximate values of $y(2)$ obtained are:

h	0.1	0.02	0.004	0.0008
y	6.1831	6.3653	6.4022	6.4096

20. With $x_0 = 0$ and $y_0 = -1$, the approximate values of $y(2)$ obtained are:

h	0.1	0.02	0.004	0.0008
y	−1.3792	−1.2843	−1.2649	−1.2610

21. With $x_0 = 1$ and $y_0 = 2$, the approximate values of $y(2)$ obtained are:

h	0.1	0.02	0.004	0.0008
y	2.8508	2.8681	2.8716	2.8723

22. With $x_0 = 0$ and $y_0 = 1$, the approximate values of $y(2)$ obtained are:

h	0.1	0.02	0.004	0.0008
y	6.9879	7.2601	7.3154	7.3264

23. With $x_0 = 0$ and $y_0 = 0$, the approximate values of $y(1)$ obtained are:

h	0.1	0.02	0.004	0.0008
y	1.2262	1.2300	1.2306	1.2307

24. With $x_0 = -1$ and $y_0 = 1$, the approximate values of $y(1)$ obtained are:

h	0.1	0.02	0.004	0.0008
y	0.9585	0.9918	0.9984	0.9997

25. Here $f(t,v) = 32 - 1.6v$ and $t_0 = 0$, $v_0 = 0$.

With $h = 0.01$, 100 iterations of $v_{n+1} = v_n + h\,f(t_n, v_n)$ yield $v(1) \approx 16.014$, and 200 iterations with $h = 0.005$ yield $v(1) \approx 15.998$. Thus we observe an approximate velocity of 16.0 ft/sec after 1 second — 80% of the limiting velocity of 20 ft/sec.

With $h = 0.01$, 200 iterations yield $v(2) \approx 19.2056$, and 400 iterations with $h = 0.005$ yield $v(2) \approx 19.1952$. Thus we observe an approximate velocity of 19.2 ft/sec after 2 seconds — 96% of the limiting velocity of 20 ft/sec.

26. Here $f(t,P) = 0.0225P - 0.003P^2$ and $t_0 = 0$, $P_0 = 25$.

With $h = 1$, 60 iterations of $P_{n+1} = P_n + h\,f(t_n, P_n)$ yield $P(60) \approx 49.3888$, and 120 iterations with $h = 0.5$ yield $P(60) \approx 49.3903$. Thus we observe a population of 49 deer after 5 years — 65% of the limiting population of 75 deer.

With $h = 1$, 120 iterations yield $P(120) \approx 66.1803$, and 240 iterations with $h = 0.5$ yield $P(60) \approx 66.1469$. Thus we observe a population of 66 deer after 10 years — 88% of the limiting population of 75 deer.

27. Here $f(x,y) = x^2 + y^2 - 1$ and $x_0 = 0$, $y_0 = 0$. The following table gives the approximate values for the successive step sizes h and corresponding numbers n of steps. It appears likely that $y(2) = 1.00$ rounded off accurate to 2 decimal places.

h	0.1	0.01	0.001	0.0001	0.00001
n	20	200	2000	20000	200000
$y(2)$	0.7772	0.9777	1.0017	1.0042	1.0044

28. Here $f(x,y) = x + \frac{1}{2}y^2$ and $x_0 = -2$, $y_0 = 0$. The following table gives the approximate values for the successive step sizes h and corresponding numbers n of steps. It appears likely that $y(2) = 1.46$ rounded off accurate to 2 decimal places.

h	0.1	0.01	0.001	0.0001	0.00001
n	40	400	4000	40000	400000
$y(2)$	1.2900	1.4435	1.4613	1.4631	1.4633

29. With step sizes $h = 0.15$, $h = 0.03$, and $h = 0.006$ we get the following results:

x	y with h=0.15	y with h=0.03	y with h=0.006
−1.0	1.0000	1.0000	1.0000
−0.7	1.0472	1.0512	1.0521
−0.4	1.1213	1.1358	1.1390
−0.1	1.2826	1.3612	1.3835
+0.2	0.8900	1.4711	0.8210
+0.5	0.7460	1.2808	0.7192

While the values for $h = 0.15$ alone are not conclusive, a comparison of the values of y for all three step sizes with $x > 0$ suggests some anomaly in the transition from negative to positive values of x.

30. With step sizes $h = 0.1$ and $h = 0.01$ we get the following results:

x	y with $h = 0.1$	y with $h = 0.01$
0.0	0.0000	0.0000
0.1	0.0000	0.0003
0.2	0.0010	0.0025
0.3	0.0050	0.0086
.	.	.
.	.	.
.	.	.
1.8	2.8200	4.3308
1.9	3.9393	7.9425
2.0	5.8521	28.3926

Clearly there is some difficulty near $x = 2$.

31. With step sizes $h = 0.1$ and $h = 0.01$ we get the following results:

x	y with $h = 0.1$	y with $h = 0.01$
0.0	1.0000	1.0000
0.1	1.2000	1.2200
0.2	1.4428	1.4967
.	.	.
.	.	.
.	.	.
0.7	4.3460	6.4643

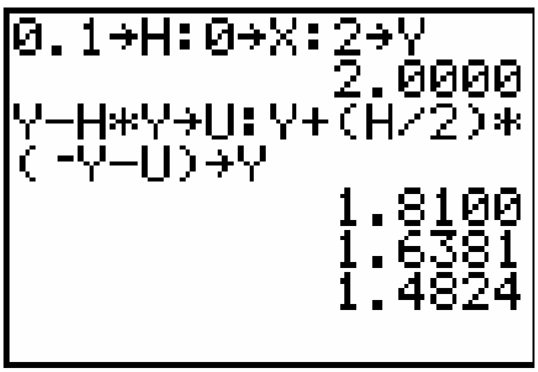
0.8	5.8670	11.8425
0.9	8.3349	39.5010

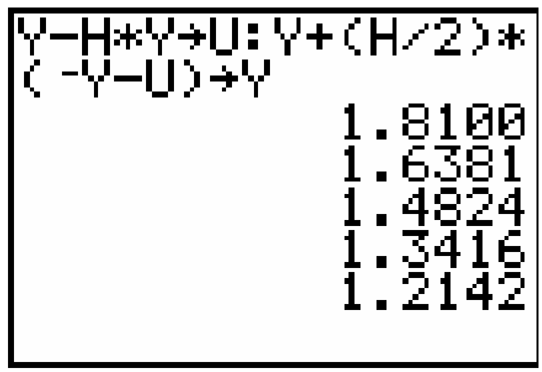
Clearly there is some difficulty near $x = 0.9$.

SECTION 2.5

A CLOSER LOOK AT THE EULER METHOD

In each of Problems 1–10 we give first the predictor formula for u_{n+1} and then the improved Euler corrector for y_{n+1}. These predictor-corrector iterations are readily implemented, either manually or with a computer system or graphing calculator (as we illustrate in Problem 1). We give in each problem a table showing the approximate values obtained, as well as the corresponding values of the exact solution.





1. $u_{n+1} = y_n + h(-y_n)$

$$y_{n+1} = y_n + (h/2)[-y_n - u_{n+1}]$$

The TI-83 screen on the left above shows a graphing calculator implementation of this iteration. After the variables are initialized (in the first line), and the formulas are entered, each press of the enter key carries out an additional step. The screen on the right shows the results of 5 steps from $x = 0$ to $x = 0.5$ with step size $h = 0.1$ — winding up with $y(0.5) \approx 1.2142$ — and we see the approximate values shown in the second row of the table below.

x	0.0	0.1	0.2	0.3	0.4	0.5
y with h=0.1	2.0000	1.8100	1.6381	1.4824	1.3416	1.2142
y actual	2.0000	1.8097	1.6375	1.4816	1.3406	1.2131

2. $u_{n+1} = y_n + 2hy_n$

$y_{n+1} = y_n + (h/2)[2y_n + 2u_{n+1}]$

x	0.0	0.1	0.2	0.3	0.4	0.5
y with h=0.1	0.5000	0.6100	0.7422	0.9079	1.1077	1.3514
y actual	0.5000	0.6107	0.7459	0.9111	1.1128	1.3591

3. $u_{n+1} = y_n + h(y_n + 1)$

$y_{n+1} = y_n + (h/2)[(y_n + 1) + (u_{n+1} + 1)]$

x	0.0	0.1	0.2	0.3	0.4	0.5
y with h=0.1	1.0000	1.2100	1.4421	1.6985	1.9818	2.2949
y actual	1.0000	1.2103	1.4428	1.6997	1.9837	2.2974

4. $u_{n+1} = y_n + h(x_n - y_n)$

$y_{n+1} = y_n + (h/2)[(x_n - y_n) + (x_n + h - u_{n+1})]$

x	0.0	0.1	0.2	0.3	0.4	0.5
y with h=0.1	1.0000	0.9100	0.8381	0.7824	0.7416	0.7142
y actual	1.0000	0.9097	0.8375	0.7816	0.7406	0.7131

5. $u_{n+1} = y_n + h(y_n - x_n - 1)$

$y_{n+1} = y_n + (h/2)[(y_n - x_n - 1) + (u_{n+1} - x_n - h - 1)]$

x	0.0	0.1	0.2	0.3	0.4	0.5
y with h=0.1	1.0000	0.9950	0.9790	0.9508	0.9091	0.8526
y actual	1.0000	0.9948	0.9786	0.9501	0.9082	0.8513

6. $u_{n+1} = y_n - 2x_ny_nh$

$y_{n+1} = y_n - (h/2)[2x_ny_n + 2(x_n + h)u_{n+1}]$

x	0.0	0.1	0.2	0.3	0.4	0.5
y with h=0.1	2.0000	1.9800	1.9214	1.8276	1.7041	1.5575
y actual	2.0000	1.9801	1.9216	1.8279	1.7043	1.5576

7. $u_{n+1} = y_n - 3x_n^2y_n h$

$y_{n+1} = y_n - (h/2)[3x_n^2y_n + 3(x_n + h)^2u_{n+1}]$

x	0.0	0.1	0.2	0.3	0.4	0.5
y with h=0.1	3.0000	2.9955	2.9731	2.9156	2.8082	2.6405
y actual	3.0000	2.9970	2.9761	2.9201	2.8140	2.6475

8. $$u_{n+1} = y_n + h\exp(-y_n)$$

$$y_{n+1} = y_n + (h/2)[\exp(-y_n) + \exp(-u_{n+1})]$$

x	0.0	0.1	0.2	0.3	0.4	0.5
y with h=0.1	0.0000	0.0952	0.1822	0.2622	0.3363	0.4053
y actual	0.0000	0.0953	0.1823	0.2624	0.3365	0.4055

9. $$u_{n+1} = y_n + h(1 + y_n^2)/4$$

$$y_{n+1} = y_n + h[1 + y_n^2 + 1 + (u_{n+1})^2]/8$$

x	0.0	0.1	0.2	0.3	0.4	0.5
y with h=0.1	1.0000	1.0513	1.1053	1.1625	1.2230	1.2873
y actual	1.0000	1.0513	1.1054	1.1625	1.2231	1.2874

10. $$u_{n+1} = y_n + 2x_n y_n^2 h$$

$$y_{n+1} = y_n + h[x_n y_n^2 + (x_n + h)(u_{n+1})^2]$$

x	0.0	0.1	0.2	0.3	0.4	0.5
y with h=0.1	1.0000	1.0100	1.0414	1.0984	1.1895	1.3309
y actual	1.0000	1.0101	1.0417	1.0989	1.1905	1.3333

The results given below for Problems 11–16 were computed using the following MATLAB script.

```
% Section 2.5, Problems 11-16
x0 = 0;  y0 = 1;
% first run:
h = 0.01;
x = x0;  y = y0;  y1 = y0;
for  n = 1:100
   u = y + h*f(x,y);                    %predictor
   y = y + (h/2)*(f(x,y)+f(x+h,u));    %corrector
   y1 = [y1,y];
   x = x + h;
   end
% second run:
h = 0.005;
x = x0;  y = y0;  y2 = y0;
```

```
for  n = 1:200
   u = y + h*f(x,y);                   %predictor
   y = y + (h/2)*(f(x,y)+f(x+h,u));    %corrector
   y2 = [y2,y];
   x = x + h;
end

% exact values
x = x0 : 0.2 : x0+1;
ye = g(x);

% display table
ya = y2(1:40:201);
err = 100*(ye-ya)./ye;
x = sprintf('%10.5f',x), sprintf('\n');
y1 = sprintf('%10.5f',y1(1:20:101)), sprintf('\n');
ya = sprintf('%10.5f',ya), sprintf('\n');
ye = sprintf('%10.5f',ye), sprintf('\n');
err = sprintf('%10.5f',err), sprintf('\n');
table = [x; y1; ya; ye; err]
```

For each problem the differential equation $y' = f(x, y)$ and the known exact solution $y = g(x)$ are stored in the files **f.m** and **g.m** — for instance, the files

```
function  yp = f(x,y)
yp = y-2;

function ye = g(x,y)
ye = 2-exp(x);
```

for Problem 11. (The exact solutions for Problems 11–16 here are given in the solutions for Problems 11–16 in Section 2.4.)

11.

x	0.0	0.2	0.4	0.6	0.8	1.0
y (h=0.01)	1.00000	0.77860	0.50819	0.17790	–0.22551	–0.71824
y (h=0.005)	1.00000	0.77860	0.50818	0.17789	–0.22553	–0.71827
y actual	1.00000	0.77860	0.50818	0.17788	–0.22554	–0.71828
error	0.000%	–0.000%	–0.001%	–0.003%	0.003%	0.002%

12.

x	0.0	0.2	0.4	0.6	0.8	1.0
y (h=0.01)	2.00000	2.11111	2.25000	2.42856	2.66664	2.99995
y (h=0.005)	2.00000	2.11111	2.25000	2.42857	2.66666	2.99999
y actual	2.00000	2.11111	2.25000	2.42857	2.66667	3.00000
error	0.0000%	0.0000%	0.0001%	0.0001%	0.0002%	0.0004%

13.

x	1.0	1.2	1.4	1.6	1.8	2.0
y (h=0.01)	3.00000	3.17390	3.44118	3.81494	4.30091	4.89901
y (h=0.005)	3.00000	3.17390	3.44117	3.81492	4.30089	4.89899
y actual	3.00000	3.17389	3.44116	3.81492	4.30088	4.89898
error	0.0000%	–0.0001%	–0.0001%	0.0001%	–0.0002%	–0.0002%

14.

x	1.0	1.2	1.4	1.6	1.8	2.0
y (h=0.01)	1.00000	1.22296	1.50707	1.88673	2.42576	3.25847
y (h=0.005)	1.00000	1.22297	1.50709	1.88679	2.42589	3.25878
y actual	1.00000	1.22297	1.50710	1.88681	2.42593	3.25889
error	0.0000%	0.0002%	0.0005%	0.0010%	0.0018%	0.0033%

15.

x	2.0	2.2	2.4	2.6	2.8	3.0
y (h=0.01)	3.000000	3.026448	3.094447	3.191719	3.310207	3.444448
y (h=0.005)	3.000000	3.026447	3.094445	3.191717	3.310205	3.444445
y actual	3.000000	3.026446	3.094444	3.191716	3.310204	3.444444
error	0.00000%	–0.00002%	–0.00002%	–0.00002%	–0.00002%	–0.00002%

16.

x	2.0	2.2	2.4	2.6	2.8	3.0
y (h=0.01)	3.000000	4.242859	5.361304	6.478567	7.633999	8.845112
y (h=0.005)	3.000000	4.242867	5.361303	6.478558	7.633984	8.845092
y actual	3.000000	4.242870	5.361303	6.478555	7.633979	8.845085
error	0.00000%	0.00006%	–0.00001%	–0.00005%	–0.00007%	–0.00007%

17. With $h = 0.1$: $y(1) \approx 0.35183$
With $h = 0.02$: $y(1) \approx 0.35030$
With $h = 0.004$: $y(1) \approx 0.35023$
With $h = 0.0008$: $y(1) \approx 0.35023$

The table of numerical results is

x	y with $h = 0.1$	y with $h = 0.02$	y with $h = 0.004$	y with $h = 0.0008$
0.0	0.00000	0.00000	0.00000	0.00000
0.2	0.00300	0.00268	0.00267	0.00267
0.4	0.02202	0.02139	0.02136	0.02136
0.6	0.07344	0.07249	0.07245	0.07245
0.8	0.17540	0.17413	0.17408	0.17408
1.0	0.35183	0.35030	0.35023	0.35023

In Problems 18–24 we give only the final approximate values of y obtained using the improved Euler method with step sizes $h = 0.1$, $h = 0.02$, $h = 0.004$, and $h = 0.0008$.

18. With $h = 0.1$: $y(2) \approx 1.68043$
With $h = 0.02$: $y(2) \approx 1.67949$
With $h = 0.004$: $y(2) \approx 1.67946$
With $h = 0.0008$: $y(2) \approx 1.67946$

19. With $h = 0.1$: $y(2) \approx 6.40834$
With $h = 0.02$: $y(2) \approx 6.41134$
With $h = 0.004$: $y(2) \approx 6.41147$
With $h = 0.0008$: $y(2) \approx 6.41147$

20. With $h = 0.1$: $y(2) \approx -1.26092$
With $h = 0.02$: $y(2) \approx -1.26003$
With $h = 0.004$: $y(2) \approx -1.25999$
With $h = 0.0008$: $y(2) \approx -1.25999$

21. With $h = 0.1$: $y(2) \approx 2.87204$
With $h = 0.02$: $y(2) \approx 2.87245$
With $h = 0.004$: $y(2) \approx 2.87247$
With $h = 0.0008$: $y(2) \approx 2.87247$

22. With $h = 0.1$: $y(2) \approx 7.31578$
With $h = 0.02$: $y(2) \approx 7.32841$
With $h = 0.004$: $y(2) \approx 7.32916$
With $h = 0.0008$: $y(2) \approx 7.32920$

23. With $h = 0.1$: $y(1) \approx 1.22967$
With $h = 0.02$: $y(1) \approx 1.23069$
With $h = 0.004$: $y(1) \approx 1.23073$
With $h = 0.0008$: $y(1) \approx 1.23073$

24. With $h = 0.1$: $y(1) \approx 1.00006$
With $h = 0.02$: $y(1) \approx 1.00000$
With $h = 0.004$: $y(1) \approx 1.00000$
With $h = 0.0008$: $y(1) \approx 1.00000$

25. Here $f(t,v) = 32 - 1.6v$ and $t_0 = 0$, $v_0 = 0$.

With $h = 0.01$, 100 iterations of

$$k_1 = f(t, v_n), \quad k_2 = f(t+h, v_n + hk_1), \quad v_{n+1} = v_n + \frac{h}{2}(k_1 + k_2)$$

yield $v(1) \approx 15.9618$, and 200 iterations with $h = 0.005$ yield $v(1) \approx 15.9620$. Thus we observe an approximate velocity of 15.962 ft/sec after 1 second — 80% of the limiting velocity of 20 ft/sec.

With $h = 0.01$, 200 iterations yield $v(2) \approx 19.1846$, and 400 iterations with $h = 0.005$ yield $v(2) \approx 19.1847$. Thus we observe an approximate velocity of 19.185 ft/sec after 2 seconds — 96% of the limiting velocity of 20 ft/sec.

26. Here $f(t,P) = 0.0225P - 0.003P^2$ and $t_0 = 0$, $P_0 = 25$.

With $h = 1$, 60 iterations of

$$k_1 = f(t, P_n), \qquad k_2 = f(t+h, P_n + hk_1), \qquad P_{n+1} = P_n + \frac{h}{2}(k_1 + k_2)$$

yield $P(60) \approx 49.3909$, and 120 iterations with $h = 0.5$ yield $P(60) \approx 49.3913$. Thus we observe an approximate population of 49.391 deer after 5 years — 65% of the limiting population of 75 deer.

With $h = 1$, 120 iterations yield $P(120) \approx 66.1129$, and 240 iterations with $h = 0.5$ yield $P(60) \approx 66.1134$. Thus we observe an approximate population of 66.113 deer after 10 years — 88% of the limiting population of 75 deer.

27. Here $f(x,y) = x^2 + y^2 - 1$ and $x_0 = 0$, $y_0 = 0$. The following table gives the approximate values for the successive step sizes h and corresponding numbers n of steps. It appears likely that $y(2) = 1.0045$ rounded off accurate to 4 decimal places.

h	0.1	0.01	0.001	0.0001
n	20	200	2000	20000
$y(2)$	1.01087	1.00452	1.00445	1.00445

28. Here $f(x,y) = x + \frac{1}{2}y^2$ and $x_0 = -2$, $y_0 = 0$. The following table gives the approximate values for the successive step sizes h and corresponding numbers n of steps. It appears likely that $y(2) = 1.4633$ rounded off accurate to 4 decimal places.

h	0.1	0.01	0.001	0.0001
n	40	400	4000	40000
$y(2)$	1.46620	1.46335	1.46332	1.46331

In the solutions for Problems 29 and 30 we illustrate the following general MATLAB ode solver.

```
function  [t,y] = ode(method, yp, t0,b, y0, n)
%  [t,y] = ode(method, yp, t0,b, y0, n)
%  calls the method described by 'method' for the
```

```
%  ODE 'yp' with function header
%
%             y' = yp(t,y)
%
%  on the interval  [t0,b]  with initial (column)
%  vector  y0.  Choices for method are 'euler',
%  'impeuler', 'rk' (Runge-Kutta), 'ode23', 'ode45'.
%  Results are saved at the endPoints of n subintervals,
%  that is, in steps of length  h = (b - t0)/n.  The
%  result  t  is an (n+1)-column vector from b to t1,
%  while  y  is a matrix with  n+1  rows (one for each
%  t-value) and one column for each dependent variable.

h = (b - t0)/n;             % step size
t = t0 : h : b;
t = t';                          % col. vector of t-values
y = y0';                    % 1st row of result matrix
for  i = 2 : n+1            % for i=2 to i=n+1
   t0 = t(i-1);             % old t
   t1 = t(i);                    % new t
   y0 = y(i-1,:)';               % old y-row-vector
   [T,Y] = feval(method, yp, t0,t1, y0);
   y = [y;Y'];                   % adjoin new y-row-vector
end
```

To use the improved Euler method, we call as **'method'** the following function.

```
function [t,y] = impeuler(yp, t0,t1, y0)
%
%  [t,y] = impeuler(yp, t0,t1, y0)
%  Takes one improved Euler step for
%
%       y' = yprime( t,y ),
%
%  from t0  to  t1  with initial value  the
%  column vector  y0.

h = t1 - t0;
k1 = feval( yp, t0, y0        );
k2 = feval( yp, t1, y0 + h*k1 );
k  = (k1 + k2)/2;
t = t1;
y = y0 + h*k;
```

29. Here our differential equation is described by the MATLAB function

```
function  vp = vpbolt1(t,v)
vp = -0.04*v - 9.8;
```

Then the commands

```
n = 50;
[t1,v1] = ode('impeuler','vpbolt1',0,10,49,n);
n = 100;
[t2,v2] = ode('impeuler','vpbolt1',0,10,49,n);
t = (0:10)';
ve = 294*exp(-t/25)-245;
[t, v1(1:5:51), v2(1:10:101), ve]
```

generate the table

t	with $n = 50$	with $n = 100$	actual v
0	49.0000	49.0000	49.0000
1	37.4722	37.4721	37.4721
2	26.3964	26.3963	26.3962
3	15.7549	15.7547	15.7546
4	5.5307	5.5304	5.5303
5	-4.2926	-4.2930	-4.2932
6	-13.7308	-13.7313	-13.7314
7	-22.7989	-22.7994	-22.7996
8	-31.5115	-31.5120	-31.5122
9	-39.8824	-39.8830	-39.8832
10	-47.9251	-47.9257	-47.9259

We notice first that the final two columns agree to 3 decimal places (each difference being than 0.0005). Scanning the $n = 100$ column for sign changes, we suspect that $v = 0$ (at the bolt's apex) occurs just after $t = 4.5$ sec. Then interpolation between $t = 4.5$ and $t = 4.6$ in the table

```
[t2(40:51),v2(40:51)]
```

3.9000	6.5345
4.0000	5.5304
4.1000	4.5303
4.2000	3.5341
4.3000	2.5420
4.4000	1.5538
4.5000	0.5696
4.6000	-0.4108
4.7000	-1.3872
4.8000	-2.3597
4.9000	-3.3283
5.0000	-4.2930

indicates that $t = 4.56$ at the bolt's apex. Finally, interpolation in

```
[t2(95:96),v2(95:96)]
```

9.4000	-43.1387
9.5000	-43.9445

gives the impact velocity $v(9.41) \approx -43.22$ m/s.

30. Now our differential equation is described by the MATLAB function

```
function  vp = vpbolt2(t,v)
vp = -0.0011*v.*abs(v) - 9.8;
```

Then the commands

```
n = 100;
[t1,v1] = ode('impeuler','vpbolt2',0,10,49,n);
n = 200;
[t2,v2] = ode('impeuler','vpbolt2',0,10,49,n);
t = (0:10)';
[t, v1(1:10:101), v2(1:20:201)]
```

generate the table

t	with $n = 100$	with $n = 200$
0	49.0000	49.0000
1	37.1547	37.1547
2	26.2428	26.2429
3.	15.9453	15.9455
4	6.0041	6.0044
5	-3.8020	-3.8016
6	-13.5105	-13.5102
7	-22.9356	-22.9355
8	-31.8984	-31.8985
9	-40.2557	-40.2559
10	-47.9066	-47.9070

We notice first that the final two columns agree to 2 decimal places (each difference being less than 0.005). Scanning the $n = 200$ column for sign changes, we suspect that $v = 0$ (at the bolt's apex) occurs just after $t = 4.6$ sec. Then interpolation between $t = 4.60$ and $t = 4.65$ in the table

```
[t2(91:101),v2(91:101)]

    4.5000    1.0964
    4.5500    0.6063
    4.6000    0.1163
    4.6500   -0.3737
    4.7000   -0.8636
    4.7500   -1.3536
    4.8000   -1.8434
    4.8500   -2.3332
    4.9000   -2.8228
```

```
4.9500   -3.3123
5.0000   -3.8016
```

indicates that $t = 4.61$ at the bolt's apex. Finally, interpolation in

```
[t2(189:190),v2(189:190)]

    9.4000  -43.4052
    9.4500  -43.7907
```

gives the impact velocity $v(9.41) \approx -43.48$ m/s.

SECTION 2.6

THE RUNGE-KUTTA METHOD

Each problem can be solved with a "template" of computations like those listed in Problem 1. We include a table showing the slope values $k_1,\ k_2,\ k_3,\ k_4$ and the *xy*-values at the ends of two successive steps of size $h = 0.25$.

1. To make the first step of size $h = 0.25$ we start with the function defined by

```
f[x_, y_] := -y
```

and the initial values

```
x = 0;     y = 2;    h = 0.25;
```

and then perform the calculations

```
k1 = f[x, y]
k2 = f[x + h/2, y + h*k1/2]
k3 = f[x + h/2, y + h*k2/2]
k4 = f[x + h, y + h*k3]
y  = y + h/6*(k1 + 2*k2 + 2*k3 + k4)
x  = x + h
```

in turn. Here we are using Mathematica notation that translates transparently to standard mathematical notation describing the corresponding manual computations. A repetition of this same block of calculations carries out a second step of size $h = 0.25$. The following table lists the intermediate and final results obtained in these two steps.

k_1	k_2	k_3	k_4	x	Approx. y	Actual y
–2	–1/75	–1.78125	–1.55469	0.25	1.55762	1.55760
–1.55762	–1.36292	–1.38725	–1.2108	0.5	1.21309	1.21306

2.

k_1	k_2	k_3	k_4	x	Approx. y	Actual y
1	1.25	1.3125	1.65625	0.25	0.82422	0.82436
1.64844	2.06055	2.16357	2.73022	0.5	1.35867	1.35914

3.

k_1	k_2	k_3	k_4	x	Approx. y	Actual y
2	2.25	2.28125	2.57031	0.25	1.56803	1.56805
2.56803	2.88904	2.92916	3.30032	0.5	2.29740	2.29744

4.

k_1	k_2	k_3	k_4	x	Approx. y	Actual y
–1	–0.75	–0.78128	–55469	0.25	0.80762	0.80760
–0.55762	–0.36292	–0.38725	–0.21080	0.5	0.71309	0.71306

5.

k_1	k_2	k_3	k_4	x	Approx. y	Actual y
0	–0.125	–0.14063	–0.28516	0.25	0.96598	0.96597
–28402	–0.44452	–0.46458	–0.65016	0.5	0.85130	0.85128

6.

k_1	k_2	k_3	k_4	x	Approx. y	Actual y
0	–0.5	–0.48438	–0.93945	0.25	1.87882	1.87883
–0.93941	–1.32105	–1.28527	–1.55751	0.5	1.55759	1.55760

7.

k_1	k_2	k_3	k_4	x	Approx. y	Actual y
0	–0.14063	–0.13980	–0.55595	0.25	2.95347	2.95349
–0.55378	–1.21679	–1.18183	–1.99351	0.5	2.6475	2.64749

8.

k_1	k_2	k_3	k_4	x	Approx. y	Actual y
1	0.88250	0.89556	0.79940	0.25	0.22315	0.22314
0.80000	0.72387	0.73079	0.66641	0.5	0.40547	0.40547

9.

k_1	k_2	k_3	k_4	x	Approx. y	Actual y
0.5	0.53223	0.53437	0.57126	0.25	1.13352	1.13352
0.57122	0.61296	0.61611	0.66444	0.5	1.28743	1.28743

10.

k_1	k_2	k_3	k_4	x	Approx. y	Actual y
0	0.25	0.26587	0.56868	0.25	1.06668	1.06667
0.56891	0.97094	1.05860	1.77245	0.5	1.33337	1.33333

The results given below for Problems 11–16 were computed using the following MATLAB script.

```
% Section 2.6, Problems 11-16
x0 = 0;  y0 = 1;

% first run:
h = 0.2;
x = x0;  y = y0;  y1 = y0;
for  n = 1:5
   k1 = f(x,y);
   k2 = f(x+h/2,y+h*k1/2);
   k3 = f(x+h/2,y+h*k2/2);
   k4 = f(x+h,y+h*k3);
   y = y +(h/6)*(k1+2*k2+2*k3+k4);
   y1 = [y1,y];
   x = x + h;
   end

% second run:
h = 0.1;
x = x0;  y = y0;  y2 = y0;
for  n = 1:10
   k1 = f(x,y);
   k2 = f(x+h/2,y+h*k1/2);
   k3 = f(x+h/2,y+h*k2/2);
   k4 = f(x+h,y+h*k3);
   y = y +(h/6)*(k1+2*k2+2*k3+k4);
   y2 = [y2,y];
   x = x + h;
end

% exact values
x = x0 : 0.2 : x0+1;
ye = g(x);

% display table
y2 = y2(1:2:11);
err = 100*(ye-y2)./ye;
x = sprintf('%10.6f',x), sprintf('\n');
y1 = sprintf('%10.6f',y1), sprintf('\n');
y2 = sprintf('%10.6f',y2), sprintf('\n');
ye = sprintf('%10.6f',ye), sprintf('\n');
err = sprintf('%10.6f',err), sprintf('\n');
table = [x;y1;y2;ye;err]
```

For each problem the differential equation $y' = f(x,y)$ and the known exact solution $y = g(x)$ are stored in the files `f.m` and `g.m` — for instance, the files

```
function  yp = f(x,y)
yp = y-2;
```

and

```
function ye = g(x,y)
ye = 2-exp(x);
```

for Problem 11.

11.

x	0.0	0.2	0.4	0.6	0.8	1.0
y (h=0.2)	1.000000	0.778600	0.508182	0.177894	–0.225521	–0.718251
y (h=0.1)	1.000000	0.778597	0.508176	0.177882	–0.225540	–0.718280
y actual	1.000000	0.778597	0.508175	0.177881	–0.225541	–0.718282
error	0.00000%	–0.00002%	–0.00009%	–0.00047%	–0.00061%	–0.00029%

12.

x	0.0	0.2	0.4	0.6	0.8	1.0
y (h=0.2)	2.000000	2.111110	2.249998	2.428566	2.666653	2.999963
y (h=0.1)	2.000000	2.111111	2.250000	2.428571	2.666666	2.999998
y actual	2.000000	2.111111	2.250000	2.428571	2.666667	3.000000
error	0.000000%	0.000002%	0.000006%	0.000014%	0.000032%	0.000080%

13.

x	1.0	1.2	1.4	1.6	1.8	2.0
y (h=0.2)	3.000000	3.173896	3.441170	3.814932	4.300904	4.899004
y (h=0.1)	3.000000	3.173894	3.441163	3.814919	4.300885	4.898981
y actual	3.000000	3.173894	3.441163	3.814918	4.300884	4.898979
error	0.00000%	–0.00001%	–0.00001%	–0.00002%	–0.00003%	–0.00003%

14.

x	1.0	1.2	1.4	1.6	1.8	2.0
y (h=0.2)	1.000000	1.222957	1.507040	1.886667	2.425586	3.257946
y (h=0.1)	1.000000	1.222973	1.507092	1.886795	2.425903	3.258821
y actual	1.000000	1.222975	1.507096	1.886805	2.425928	3.258891
error	0.0000%	0.0001%	0.0003%	0.0005%	0.0010%	0.0021%

15.

x	2.0	2.2	2.4	2.6	2.9	3.0
y (h=0.2)	3.000000	3.026448	3.094447	3.191719	3.310207	3.444447
y (h=0.1)	3.000000	3.026446	3.094445	3.191716	3.310204	3.444445
y actual	3.000000	3.026446	3.094444	3.191716	3.310204	3.444444
error	0.000000%	–0.000004%	–0.000005%	–0.000005%	–0.000005%	–0.000004%

16.

x	2.0	2.2	2.4	2.6	2.9	3.0
y (h=0.2)	3.000000	4.243067	5.361409	6.478634	7.634049	8.845150
y (h=0.1)	3.000000	4.242879	5.361308	6.478559	7.633983	8.845089
y actual	3.000000	4.242870	5.361303	6.478555	7.633979	8.845085
error	0.000000%	–0.000221%	–0.000094%	–0.000061%	–0.000047%	–0.000039%

17. With $h = 0.2$: $y(1) \approx 0.350258$
With $h = 0.1$: $y(1) \approx 0.350234$
With $h = 0.05$: $y(1) \approx 0.350232$
With $h = 0.025$: $y(1) \approx 0.350232$

The table of numerical results is

x	y with $h = 0.2$	y with $h = 0.1$	y with $h = 0.05$	y with $h = 0.025$
0.0	0.000000	0.000000	0.000000	0.000000
0.2	0.002667	0.002667	0.002667	0.002667
0.4	0.021360	0.021359	0.021359	0.021359
0.6	0.072451	0.072448	0.072448	0.072448
0.8	0.174090	0.174081	0.174080	0.174080
1.0	0.350258	0.350234	0.350232	0.350232

In Problems 18–24 we give only the final approximate values of y obtained using the Runge-Kutta method with step sizes $h = 0.2$, $h = 0.1$, $h = 0.05$, and $h = 0.025$.

18. With $h = 0.2$: $y(2) \approx 1.679513$
With $h = 0.1$: $y(2) \approx 1.679461$
With $h = 0.05$: $y(2) \approx 1.679459$
With $h = 0.025$: $y(2) \approx 1.679459$

19. With $h = 0.2$: $y(2) \approx 6.411464$
With $h = 0.1$: $y(2) \approx 6.411474$
With $h = 0.05$: $y(2) \approx 6.411474$
With $h = 0.025$: $y(2) \approx 6.411474$

20. With $h = 0.2$: $y(2) \approx -1.259990$
With $h = 0.1$: $y(2) \approx -1.259992$
With $h = 0.05$: $y(2) \approx -1.259993$
With $h = 0.025$: $y(2) \approx -1.259993$

21. With $h = 0.2$: $y(2) \approx 2.872467$
With $h = 0.1$: $y(2) \approx 2.872468$

With $h = 0.05$: $y(2) \approx 2.872468$
With $h = 0.025$: $y(2) \approx 2.872468$

22. With $h = 0.2$: $y(2) \approx 7.326761$
With $h = 0.1$: $y(2) \approx 7.328452$
With $h = 0.05$: $y(2) \approx 7.328971$
With $h = 0.025$: $y(2) \approx 7.329134$

23. With $h = 0.2$: $y(1) \approx 1.230735$
With $h = 0.1$: $y(1) \approx 1.230731$
With $h = 0.05$: $y(1) \approx 1.230731$
With $h = 0.025$: $y(1) \approx 1.230731$

24. With $h = 0.2$: $y(1) \approx 1.000000$
With $h = 0.1$: $y(1) \approx 1.000000$
With $h = 0.05$: $y(1) \approx 1.000000$
With $h = 0.025$: $y(1) \approx 1.000000$

25. Here $f(t,v) = 32 - 1.6v$ and $t_0 = 0,\ v_0 = 0.$

With $h = 0.1,$ 10 iterations of

$$\begin{aligned} k_1 &= f(t_n, v_n), & k_2 &= f(t_n + \tfrac{1}{2}h, v_n + \tfrac{1}{2}hk_1), \\ k_3 &= f(t_n + \tfrac{1}{2}h, v_n + \tfrac{1}{2}hk_2), & k_4 &= f(t_n + h, v_n + hk_3), \\ k &= \tfrac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4), & v_{n+1} &= v_n + hk \end{aligned}$$

yield $v(1) \approx 15.9620,$ and 20 iterations with $h = 0.05$ yield $v(1) \approx 15.9621.$ Thus we observe an approximate velocity of 15.962 ft/sec after 1 second — 80% of the limiting velocity of 20 ft/sec.

With $h = 0.1,$ 20 iterations yield $v(2) \approx 19.1847,$ and 40 iterations with $h = 0.05$ yield $v(2) \approx 19.1848.$ Thus we observe an approximate velocity of 19.185 ft/sec after 2 seconds — 96% of the limiting velocity of 20 ft/sec.

26. Here $f(t,P) = 0.0225P - 0.003P^2$ and $t_0 = 0,\ P_0 = 25.$

With $h = 6,$ 10 iterations of

$$\begin{aligned} k_1 &= f(t_n, P_n), & k_2 &= f(t_n + \tfrac{1}{2}h, P_n + \tfrac{1}{2}hk_1), \\ k_3 &= f(t_n + \tfrac{1}{2}h, v_n + \tfrac{1}{2}hk_2), & k_4 &= f(t_n + h, P_n + hk_3), \\ k &= \tfrac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4), & P_{n+1} &= P_n + hk \end{aligned}$$

yield $P(60) \approx 49.3915,$ as do 20 iterations with $h = 3.$ Thus we observe an approximate population of 49.3915 deer after 5 years — 65% of the limiting population of 75 deer.

With $h = 6$, 20 iterations yield $P(120) \approx 66.1136$, as do 40 iterations with $h = 3$. Thus we observe an approximate population of 66.1136 deer after 10 years — 88% of the limiting population of 75 deer.

27. Here $f(x, y) = x^2 + y^2 - 1$ and $x_0 = 0$, $y_0 = 0$. The following table gives the approximate values for the successive step sizes h and corresponding numbers n of steps. It appears likely that $y(2) = 1.00445$ rounded off accurate to 5 decimal places.

h	1	0.1	0.01	0.001
n	2	20	200	2000
$y(2)$	1.05722	1.00447	1.00445	1.00445

28. Here $f(x, y) = x + \frac{1}{2}y^2$ and $x_0 = -2$, $y_0 = 0$. The following table gives the approximate values for the successive step sizes h and corresponding numbers n of steps. It appears likely that $y(2) = 1.46331$ rounded off accurate to 5 decimal places.

h	1	0.1	0.01	0.001
n	4	40	00	40000
$y(2)$	1.48990	1.46332	1.46331	1.46331

In the solutions for Problems 29 and 30 we use the general MATLAB solver **ode** that was listed prior to the Problem 29 solution in Section 2.5. To use the Runge-Kutta method, we call as **'method'** the following function.

```
function [t,y] = rk(yp, t0,t1, y0)

%  [t, y] = rk(yp, t0, t1, y0)
%  Takes one Runge-Kutta step for
%
%       y' = yp( t,y ),
%
%  from t0  to  t1  with initial value  the
%  column vector  y0.

h = t1 - t0;
k1 = feval(yp, t0      , y0            );
k2 = feval(yp, t0 + h/2, y0 + (h/2)*k1 );
k3 = feval(yp, t0 + h/2, y0 + (h/2)*k2 );
k4 = feval(yp, t0 + h   ,y0 +    h *k3 );
k  = (1/6)*(k1 + 2*k2 + 2*k3 + k4);
t = t1;
y = y0 + h*k;
```

29. Here our differential equation is described by the MATLAB function

```
function  vp = vpbolt1(t,v)
vp = -0.04*v - 9.8;
```

Then the commands

```
n = 100;
[t1,v1] = ode('rk','vpbolt1',0,10,49,n);
n = 200;
[t2,v] = ode('rk','vpbolt1',0,10,49,n);
t = (0:10)';
ve = 294*exp(-t/25)-245;
[t, v1(1:n/20:1+n/2), v(1:n/10:n+1), ve]
```

generate the table

t	with $n = 100$	with $n = 200$	actual v
0	49.0000	49.0000	49.0000
1	37.4721	37.4721	37.4721
2	26.3962	26.3962	26.3962
3	15.7546	15.7546	15.7546
4	5.5303	5.5303	5.5303
5	-4.2932	-4.2932	-4.2932
6	-13.7314	-13.7314	-13.7314
7	-22.7996	-22.7996	-22.7996
8	-31.5122	-31.5122	-31.5122
9	-39.8832	-39.8832	-39.8832
10	-47.9259	-47.9259	-47.9259

We notice first that the final three columns agree to the 4 displayed decimal places. Scanning the last column for sign changes in v, we suspect that $v = 0$ (at the bolt's apex) occurs just after $t = 4.5$ sec. Then interpolation between $t = 4.55$ and $t = 4.60$ in the table

```
[t2(91:95),v(91:95)]

   4.5000    0.5694
   4.5500    0.0788
   4.6000   -0.4109
   4.6500   -0.8996
   4.7000   -1.3873
```

indicates that $t = 4.56$ at the bolt's apex. Now the commands

```
y = zeros(n+1,1);
h = 10/n;
```

```
for j = 2:n+1
   y(j) = y(j-1) + v(j-1)*h +
                  0.5*(-.04*v(j-1) - 9.8)*h^2;
end
ye = 7350*(1 - exp(-t/25)) - 245*t;
[t, y(1:n/10:n+1), ye]
```

generate the table

t	Approx y	Actual y
0	0	0
1	43.1974	43.1976
2	75.0945	75.0949
3	96.1342	96.1348
4	106.7424	106.7432
5	107.3281	107.3290
6	98.2842	98.2852
7	79.9883	79.9895
8	52.8032	52.8046
9	17.0775	17.0790
10	-26.8540	-26.8523

We see at least 2-decimal place agreement between approximate and actual values of y. Finally, interpolation between $t = 9$ and $t = 10$ here suggests that $y = 0$ just after $t = 9.4$. Then interpolation between $t = 9.40$ and $t = 9.45$ in the table

```
[t2(187:191),y(187:191)]

   9.3000    4.7448
   9.3500    2.6182
   9.4000    0.4713
   9.4500   -1.6957
   9.5000   -3.8829
```

indicates that the bolt is aloft for about 9.41 seconds.

30. Now our differential equation is described by the MATLAB function

```
function  vp = vpbolt2(t,v)
vp = -0.0011*v.*abs(v) - 9.8;
```

Then the commands

```
n = 200;
[t1,v1] = ode('rk','vpbolt2',0,10,49,n);
n = 2*n;
[t2,v] = ode('rk','vpbolt2',0,10,49,n);
```

```
t = (0:10)';
ve = zeros(size(t));
ve(1:5)= 94.388*tan(0.478837 - 0.103827*t(1:5));
ve(6:11)= -94.388*tanh(0.103827*(t(6:11)-4.6119));

[t, v1(1:n/20:1+n/2), v(1:n/10:n+1), ve]
```

generate the table

t	with $n = 200$	with $n = 400$	actual v
0	49.0000	49.0000	49.0000
1	37.1548	37.1548	37.1547
2	26.2430	26.2430	26.2429
3	15.9456	15.9456	15.9455
4	6.0046	6.0046	6.0045
5	-3.8015	-3.8015	-3.8013
6	13.5101	-13.5101	-13.5100
7	-22.9354	-22.9354	-22.9353
8	-31.8985	-31.8985	-31.8984
9	-40.2559	-40.2559	-40.2559
10	-47.9071	-47.9071	-47.9071

We notice first that the final three columns almost agree to the 4 displayed decimal places. Scanning the last colmun for sign changes in v, we suspect that $v = 0$ (at the bolt's apex) occurs just after $t = 4.6$ sec. Then interpolation between $t = 4.600$ and $t = 4.625$ in the table

```
[t2(185:189),v(185:189)]
```

4.6000	0.1165
4.6250	-0.1285
4.6500	-0.3735
4.6750	-0.6185
4.7000	-0.8635

indicates that $t = 4.61$ at the bolt's apex. Now the commands

```
y = zeros(n+1,1);
h = 10/n;
for j = 2:n+1
   y(j) = y(j-1) + v(j-1)*h + 0.5*(-.04*v(j-1) - 9.8)*h^2;
end
ye = zeros(size(t));
ye(1:5)= 108.465+909.091*log(cos(0.478837 -
0.103827*t(1:5)));
ye(6:11)= 108.465-909.091*log(cosh(0.103827
                     *(t(6:11)-4.6119)));
[t, y(1:n/10:n+1), ye]
```

generate the table

t	Approx y	Actual y
0	0	0.0001
1	42.9881	42.9841
2	74.6217	74.6197
3	95.6719	95.6742
4	106.6232	106.6292
5	107.7206	107.7272
6	99.0526	99.0560
7	80.8027	80.8018
8	53.3439	53.3398
9	17.2113	17.2072
10	-26.9369	-26.9363

We see almost 2-decimal place agreement between approximate and actual values of y. Finally, interpolation between $t = 9$ and $t = 10$ here suggests that $y = 0$ just after $t = 9.4$. Then interpolation between $t = 9.400$ and $t = 9.425$ in the table

```
[t2(377:381),y(377:381)]

   9.4000    0.4740
   9.4250   -0.6137
   9.4500   -1.7062
   9.4750   -2.8035
   9.5000   -3.9055
```

indicates that the bolt is aloft for about 9.41 seconds.

CHAPTER 3

LINEAR SYSTEMS AND MATRICES

SECTION 3.1

INTRODUCTION TO LINEAR SYSTEMS

This initial section takes account of the fact that some students remember only hazily the method of elimination for 2×2 and 3×3 systems. Moreover, high school algebra courses generally emphasize only the case in which a unique solution exists. Here we treat on an equal footing the other two cases — in which either no solution exists or infinitely many solutions exist.

1. Subtraction of twice the first equation from the second equation gives $-5y=-10$, so $y=2$, and it follows that $x=3$.

2. Subtraction of three times the second equation from the first equation gives $5y=-15$, so $y=-3$, and it follows that $x=5$.

3. Subtraction of $3/2$ times the first equation from the second equation gives $\frac{1}{2}y=\frac{3}{2}$, so $y=3$, and it follows that $x=-4$.

4. Subtraction of $6/5$ times the first equation from the second equation gives $\frac{11}{5}y=\frac{44}{5}$, so $y=4$, and it follows that $x=5$.

5. Subtraction of twice the first equation from the second equation gives $0=1$, so no solution exists.

6. Subtraction of $3/2$ times the first equation from the second equation gives $0=1$, so no solution exists.

7. The second equation is -2 times the first equation, so we can choose $y=t$ arbitrarily. The first equation then gives $x=-10+4t$.

8. The second equation is $2/3$ times the first equation, so we can choose $y=t$ arbitrarily. The first equation then gives $x=4+2t$.

9. Subtraction of twice the first equation from the second equation gives $-9y-4z=-3$. Subtraction of the first equation from the third equation gives $2y+z=1$. Solution of these latter two equations gives $y = -1,\ z = 3$. Finally substitution in the first equation gives $x = 4$.

10. Subtraction of twice the first equation from the second equation gives $y + 3z = -5$. Subtraction of twice the first equation from the third equation gives $-y - 2z = 3$. Solution of these latter two equations gives $y = 1,\ z = -2$. Finally substitution in the first equation gives $x = 3$.

11. First we interchange the first and second equations. Then subtraction of twice the new first equation from the new second equation gives $y - z = 7$, and subtraction of three times the new first equation from the third equation gives $-2y + 3z = -18$. Solution of these latter two equations gives $y = 3,\ z = -4$. Finally substitution in the (new) first equation gives $x = 1$.

12. First we interchange the first and third equations. Then subtraction of twice the new first equation from the second equation gives $-7y - 3z = -36$, and subtraction of twice the new first equation from the new third equation gives $-16y - 7z = -83$. Solution of these latter two equations gives $y = 3,\ z = 5$. Finally substitution in the (new) first equation gives $x = 1$.

13. First we subtract the second equation from the first equation to get the new first equation $x + 2y + 3z = 0$. Then subtraction of twice the new first equation from the second equation gives $3y - 2z = 0$, and subtraction of twice the new first equation from the third equation gives $2y - z = 0$. Solution of these latter two equations gives $y = 0,\ z = 0$. Finally substitution in the (new) first equation gives $x = 0$ also.

14. First we subtract the second equation from the first equation to get the new first equation $x + 8y - 4z = 45$. Then subtraction of twice the new first equation from the second equation gives $-23y + 28z = -181$, and subtraction of twice the new first equation from the third equation gives $-9y + 11z = -71$. Solution of these latter two equations gives $y = 3,\ z = -4$. Finally substitution in the (new) first equation gives $x = 5$.

15. Subtraction of the first equation from the second equation gives $-4y + z = -2$. Subtraction of three times the first equation from the third equation gives (after division by 2) $-4y + z = -5/2$. These latter two equations obviously are inconsistent, so the original system has no solution.

16. Subtraction of the first equation from the second equation gives $7y - 3z = -2$. Subtraction of three times the first equation from the third equation gives (after division by 3) $7y - 3z = -10/3$. These latter two equations obviously are inconsistent, so the original system has no solution.

17. First we subtract the first equation from the second equation to get the new first equation $x+3y-6z=-4$. Then subtraction of three times the new first equation from the second equation gives $-7y+16z=15$, and subtraction of five times the new first equation from the third equation gives (after division by 2) $-7y+16z=35/2$. These latter two equations obviously are inconsistent, so the original system has no solution.

18. Subtraction of the five times the first equation from the second equation gives $-23y-40z=-14$. Subtraction of eight times the first equation from the third equation gives $-23y-40z=-19$. These latter two equations obviously are inconsistent, so the original system has no solution.

19. Subtraction of twice the first equation from the second equation gives $3y-6z=9$. Subtraction of the first equation from the third equation gives $y-2z=3$. Obviously these latter two equations are scalar multiples of each other, so we can choose $z=t$ arbitrarily. It follows first that $y=3+2t$ and then that $x=8+3t$.

20. First we subtract the second equation from the first equation to get the new first equation $x-y+6z=-5$. Then subtraction of the new first equation from the second equation gives $5y-5z=25$, and subtraction of the new first equation from the third equation gives $3y-3z=15$. Obviously these latter two equations are both scalar multiples of the equation $y-z=5$, so we can choose $z=t$ arbitrarily. It follows first that $y=5+t$ and then that $x=-5t$.

21. Subtraction of three times the first equation from the second equation gives $3y-6z=9$. Subtraction of four times the first equation from the third equation gives $-3y+9z=-6$. Obviously these latter two equations are both scalar multiples of the equation $y-3z=2$, so we can choose $z=t$ arbitrarily. It follows first that $y=2+3t$ and then that $x=3-2t$.

22. Subtraction of four times the second equation from the first equation gives $2y+10z=0$. Subtraction of twice the second equation from the third equation gives $y+5z=0$. Obviously the first of these latter two equations is twice the second one, so we can choose $z=t$ arbitrarily. It follows first that $y=-5t$ and then that $x=-4t$.

23. The initial conditions $y(0)=3$ and $y'(0)=8$ yield the equations $A=3$ and $2B=8$, so $A=3$ and $B=4$. It follows that $y(x) = 3\cos 2x+4\sin 2x$.

24. The initial conditions $y(0)=5$ and $y'(0)=12$ yield the equations $A=5$ and $3B=12$, so $A=5$ and $B=4$. It follows that $y(x) = 5\cosh 3x+4\sinh 3x$.

25. The initial conditions $y(0)=10$ and $y'(0)=20$ yield the equations $A+B=10$ and $5A-5B=20$ with solution $A=7$, $B=3$. Thus $y(x) = 7e^{5x}+3e^{-5x}$.

26. The initial conditions $y(0) = 44$ and $y'(0) = 22$ yield the equations $A + B = 44$ and $11A - 11B = 22$ with solution $A = 23,\ B = 21$. Thus $y(x) = 23e^{11x} + 21e^{-11x}$.

27. The initial conditions $y(0) = 40$ and $y'(0) = -16$ yield the equations $A + B = 40$ and $3A - 5B = -16$ with solution $A = 23,\ B = 17$. Thus $y(x) = 23e^{3x} + 17e^{-5x}$.

28. The initial conditions $y(0) = 15$ and $y'(0) = 13$ yield the equations $A + B = 15$ and $3A + 7B = -13$ with solution $A = 23,\ B = -8$. Thus $y(x) = 23e^{3x} - 8e^{7x}$.

29. The initial conditions $y(0) = 7$ and $y'(0) = 11$ yield the equations $A + B = 7$ and $\frac{1}{2}A + \frac{1}{3}B = 11$ with solution $A = 52,\ B = -45$. Thus $y(x) = 52e^{x/2} - 45e^{x/3}$.

30. The initial conditions $y(0) = 41$ and $y'(0) = 164$ yield the equations $A + B = 41$ and $\dfrac{4}{3}A - \dfrac{7}{5}B = 164$ with solution $A = 81,\ B = -40$. Thus $y(x) = 81e^{4x/3} - 40e^{-7x/5}$.

31. The graph of each of these linear equations in x and y is a straight line through the origin (0, 0) in the xy-plane. If these two lines are distinct then they intersect only at the origin, so the two equations have the unique solution $x = y = 0$. If the two lines coincide, then each of the infinitely many different points (x, y) on this common line provides a solution of the system.

32. The graph of each of these linear equations in x, y, and z is a plane in xyz-space. If these two planes are parallel — that is, do not intersect — then the equations have no solution. Otherwise, they intersect in a straight line, and each of the infinitely many different points (x, y, z) on this line provides a solution of the system.

33. **(a)** The three lines have no common point of intersection, so the system has no solution.

(b) The three lines have a single point of intersection, so the system has a unique solution.

(c) The three lines — two of them parallel — have no common point of intersection, so the system has no solution.

(d) The three distinct parallel lines have no common point of intersection, so the system has no solution.

(e) Two of the lines coincide and intersect the third line in a single point, so the system has a unique solution.

(f) The three lines coincide, and each of the infinitely many different points (x, y, z) on this common line provides a solution of the system.

34. **(a)** If the three planes are parallel and distinct, then they have no common point of intersection, so the system has no solution.

(b) If the three planes coincide, then each of the infinitely many different points (x, y, z) of this common plane provides a solution of the system.

(c) If two of the planes coincide and are parallel to the third plane, then the three planes have no common point of intersection, so the system has no solution.

(d) If two of the planes intersect in a line that is parallel to the third plane, then the three planes have no common point of intersection, so the system has no solution.

(e) If two of the planes intersect in a line that lies in the third plane, then each of the infinitely many different points (x, y, z) of this line provides a solution of the system.

(f) If two of the planes intersect in a line that intersects the third plane in a single point, then this point (x, y, z) provides the unique solution of the system.

SECTION 3.2

MATRICES AND GAUSSIAN ELIMINATION

Because the linear systems in Problems 1–10 are already in echelon form, we need only start at the end of the list of unknowns and work backwards.

1. Starting with $x_3 = 2$ from the third equation, the second equation gives $x_2 = 0,$ and then the first equation gives $x_1 = 1.$

2. Starting with $x_3 = -3$ from the third equation, the second equation gives $x_2 = 1,$ and then the first equation gives $x_1 = 5.$

3. If we set $x_3 = t$ then the second equation gives $x_2 = 2 + 5t,$ and next the first equation gives $x_1 = 13 + 11t.$

4. If we set $x_3 = t$ then the second equation gives $x_2 = 5 + 7t,$ and next the first equation gives $x_1 = 35 + 33t.$

5. If we set $x_4 = t$ then the third equation gives $x_3 = 5 + 3t,$ next the second equation gives $x_2 = 6 + t,$ and finally the first equation gives $x_1 = 13 + 4t.$

6. If we set $x_3 = t$ and $x_4 = -4$ from the third equation, then the second equation gives $x_2 = 11 + 3t,$ and next the first equation gives $x_1 = 17 + t.$

7. If we set $x_3 = s$ and $x_4 = t$, then the second equation gives $x_2 = 7 + 2s - 7t$, and next the first equation gives $x_1 = 3 - 8s + 19t$.

8. If we set $x_2 = s$ and $x_4 = t$, then the second equation gives $x_3 = 10 - 3t$, and next the first equation gives $x_1 = -25 + 10s + 22t$.

9. Starting with $x_4 = 6$ from the fourth equation, the third equation gives $x_3 = -5$, next the second equation gives $x_2 = 3$, and finally the first equation gives $x_1 = 1$.

10. If we set $x_3 = s$ and $x_5 = t$, then the third equation gives $x_4 = 5t$, next the second equation gives $x_2 = 13s - 8t$, and finally the first equation gives $x_1 = 63s - 16t$.

In each of Problems 11–22, we give just the first two or three steps in the reduction. Then we display a resulting echelon form **E** of the augmented coefficient matrix **A** of the given linear system, and finally list the resulting solution (if any). The student should understand that the echelon matrix **E** is not unique, so a different sequence of elementary row operations may produce a different echelon matrix.

11. Begin by interchanging rows 1 and 2 of **A**. Then subtract twice row 1 both from row 2 and from row 3.

$$\mathbf{E} = \begin{bmatrix} 1 & 3 & 2 & 5 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 4 \end{bmatrix}; \quad x_1 = 3,\ x_2 = -2,\ x_3 = 4$$

12. Begin by subtracting row 2 of **A** from row 1. Then subtract twice row 1 both from row 2 and from row 3.

$$\mathbf{E} = \begin{bmatrix} 1 & -6 & -4 & 15 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 2 \end{bmatrix}; \quad x_1 = 5,\ x_2 = -3,\ x_3 = 2$$

13. Begin by subtracting twice row 1 of **A** both from row 2 and from row 3. Then add row 2 to row 3.

$$\mathbf{E} = \begin{bmatrix} 1 & 3 & 3 & 13 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}; \quad x_1 = 4 + 3t,\ x_2 = 3 - 2t,\ x_3 = t$$

14. Begin by interchanging rows 1 and 3 of **A**. Then subtract twice row 1 from row 2, and three times row 1 from row 3.

$$\mathbf{E} = \begin{bmatrix} 1 & -2 & -2 & -9 \\ 0 & 0 & 1 & 7 \\ 0 & 0 & 0 & 0 \end{bmatrix}; \quad x_1 = 5+2t,\ x_2 = t,\ x_3 = 7$$

15. Begin by interchanging rows 1 and 2 of **A**. Then subtract three times row 1 from row 2, and five times row 1 from row 3.

$$\mathbf{E} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

The system has no solution.

16. Begin by subtracting row 1 from row 2 of **A**. Then interchange rows 1 and 2. Next subtract twice row 1 from row 2, and five times row 1 from row 3.

$$\mathbf{E} = \begin{bmatrix} 1 & -4 & -7 & 6 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

The system has no solution.

17.

$$\begin{aligned} x_1 - 4x_2 - 3x_3 - 3x_4 &= 4 \\ 2x_1 - 6x_2 - 5x_3 - 5x_4 &= 5 \\ 3x_1 - \ \ x_2 - 4x_3 - 5x_4 &= -7 \end{aligned}$$

Begin by subtracting twice row 1 from row 2 of **A**, and three times row 1 from row 3.

$$\mathbf{E} = \begin{bmatrix} 1 & -4 & -3 & -3 & 4 \\ 0 & 1 & 0 & -1 & -4 \\ 0 & 0 & 1 & 3 & 5 \end{bmatrix}; \quad x_1 = 3-2t,\ x_2 = -4+t,\ x_3 = 5-3t,\ x_4 = t$$

18. Begin by subtracting row 3 from row 1 of **A**. Then subtract 3 times row 1 from row 2, and twice row 1 from row 3.

$$\mathbf{E} = \begin{bmatrix} 1 & -2 & -4 & -13 & -8 \\ 0 & 0 & 1 & 4 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}; \quad x_1 = 4+2s-3t,\ x_2 = s,\ x_3 = 3-4t,\ x_4 = t$$

19. Begin by interchanging rows 1 and 2 of **A**. Then subtract three times row 1 from row 2, and four times row 1 from row 3.

$$\mathbf{E} = \begin{bmatrix} 1 & -2 & 5 & -5 & -7 \\ 0 & 1 & -2 & 3 & 5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}; \quad x_1 = 3 - s - t,\ x_2 = 5 + 2s - 3t,\ x_3 = s,\ x_4 = t$$

20. Begin by interchanging rows 1 and 2 of **A**. Then subtract twice row 1 from row 2, and five times row 1 from row 3.

$$\mathbf{E} = \begin{bmatrix} 1 & 3 & 2 & -7 & 3 & 9 \\ 0 & 1 & 3 & -7 & 2 & 7 \\ 0 & 0 & 1 & -2 & 0 & 2 \end{bmatrix}; \quad x_1 = 2 + 3t,\ x_2 = 1 + s - 2t,\ x_3 = 2 + 2s,\ x_4 = s,\ x_5 = t$$

21. Begin by subtracting twice row 1 from row 2, three times row 1 from row 3, and four times row 1 from row 4.

$$\mathbf{E} = \begin{bmatrix} 1 & 1 & 1 & 0 & 6 \\ 0 & 1 & 5 & 1 & 20 \\ 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 1 & 4 \end{bmatrix}; \quad x_1 = 2,\ x_2 = 1,\ x_3 = 3,\ x_4 = 4$$

22. Begin by subtracting row 4 from row 1. Then subtracting twice row 1 from row 2, four times row 1 from row 3, and three times row 1 from row 4.

$$\mathbf{E} = \begin{bmatrix} 1 & -2 & -4 & 0 & -9 \\ 0 & 1 & 6 & 1 & 21 \\ 0 & 0 & 1 & 0 & 4 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix}; \quad x_1 = 3,\ x_2 = -2,\ x_3 = 4,\ x_4 = -1$$

23. If we subtract twice the first row from the second row, we obtain the echelon form

$$\mathbf{E} = \begin{bmatrix} 3 & 2 & 1 \\ 0 & 0 & k-2 \end{bmatrix}$$

of the augmented coefficient matrix. It follows that the given system has no solutions unless $k = 2$, in which case it has infinitely many solutions given by $x_1 = \frac{1}{3}(1 - 2t),\ x_2 = t.$

24. If we subtract twice the first row from the second row, we obtain the echelon form

$$\mathbf{E} = \begin{bmatrix} 3 & 2 & 0 \\ 0 & k-4 & 0 \end{bmatrix}$$

of the augmented coefficient matrix. It follows that the given system has only the trivial solution $x_1 = x_2 = 0$ unless $k = 4$, in which case it has infinitely many solutions given by $x_1 = -\frac{2}{3}t,\ x_2 = t.$

25. If we subtract twice the first row from the second row, we obtain the echelon form

$$\mathbf{E} = \begin{bmatrix} 3 & 2 & 11 \\ 0 & k-4 & -1 \end{bmatrix}$$

of the augmented coefficient matrix. It follows that the given system has a unique solution if $k \neq 4$, but no solution if $k = 4$.

26. If we first subtract twice the first row from the second row, then interchange the two rows, and finally subtract 3 times the first row from the second row, then we obtain the echelon form

$$\mathbf{E} = \begin{bmatrix} 1 & 1 & k-2 \\ 0 & 1 & 3k-7 \end{bmatrix}$$

of the augmented coefficient matrix. It follows that the given system has a unique solution whatever the value of k.

27. If we first subtract twice the first row from the second row, then subtract 4 times the first row from the third row, and finally subtract the second row from the third row , we obtain the echelon form

$$\mathbf{E} = \begin{bmatrix} 1 & 2 & 1 & 3 \\ 0 & 5 & 5 & 1 \\ 0 & 0 & 0 & k-11 \end{bmatrix}$$

of the augmented coefficient matrix. It follows that the given system has no solution unless $k = 11$, in which case it has infinitely many solutions with x_3 arbitrary.

28. If we first interchange rows 1 and 2, then subtract twice the first row from the second row, next subtract 7 times the first row from the third row, and finally subtract twice the second row from the third row , we obtain the echelon form

$$\mathbf{E} = \begin{bmatrix} 1 & 2 & 1 & b \\ 0 & -5 & 1 & a-2b \\ 0 & 0 & 0 & c-2a-3b \end{bmatrix}$$

of the augmented coefficient matrix. It follows that the given system has no solution unless $c = 2a + 3b$, in which case it has infinitely many solutions with x_3 arbitrary.

29. In each of parts (a)-(c), we start with a typical 2×2 matrix $\mathbf{A}$ and carry out two row successive operations as indicated, observing that we wind up with the original matrix $\mathbf{A}$.

(a) $$\mathbf{A} = \begin{bmatrix} s & t \\ u & v \end{bmatrix} \xrightarrow{cR2} \begin{bmatrix} s & t \\ cu & cv \end{bmatrix} \xrightarrow{(1/c)R2} \begin{bmatrix} s & t \\ u & v \end{bmatrix} = \mathbf{A}$$

(b) $$\mathbf{A} = \begin{bmatrix} s & t \\ u & v \end{bmatrix} \xrightarrow{SWAP(R1,R2)} \begin{bmatrix} u & v \\ s & t \end{bmatrix} \xrightarrow{SWAP(R1,R2)} \begin{bmatrix} s & t \\ u & v \end{bmatrix} = \mathbf{A}$$

(c) $$\mathbf{A} = \begin{bmatrix} s & t \\ u & v \end{bmatrix} \xrightarrow{cR1+R2} \begin{bmatrix} u & v \\ cu+s & cv+t \end{bmatrix} \xrightarrow{(-c)R1+R2} \begin{bmatrix} s & t \\ u & v \end{bmatrix} = \mathbf{A}$$

Since we therefore can "reverse" any single elementary row operation, it follows that we can reverse any finite sequence of such operations — on at a time — so part (d) follows.

30. **(a)** This part is essentially obvious, because a multiple of an equation that is satisfied is also satisfied, and the sum of two equations that are satisfied is one that is also satisfied.

(b) Let us write $\mathbf{A}_1 = \mathbf{B}_1, \mathbf{B}_2, \cdots, \mathbf{B}_n, \mathbf{B}_{n+1} = \mathbf{A}_2$ where each matrix $\mathbf{B}_{k+1}$ is obtained from $\mathbf{B}_k$ by a single elementary row operation (for $k = 1, 2, \cdots, n$). Then it follows by n applications of part (a) that every solution of the system LS_1 associated with the matrix $\mathbf{A}_1$ is also a solution of the system LS_2 associated with the matrix $\mathbf{A}_2$. But part (d) of Problem 29 implies that $\mathbf{A}_1$ also can be obtained by $\mathbf{A}_2$ by elementary row operations, so by the same token every solution of LS_2 is also a solution of LS_1.

SECTION 3.3

REDUCED ROW-ECHELON MATRICES

Each of the matrices in Problems 1-20 can be transformed to reduced echelon form without the appearance of any fractions. The main thing is to get started right. Generally our first goal is to get a 1 in the upper left corner of $\mathbf{A}$, then clear out the rest of the first column. In each problem we first give at least the initial steps, and then the final result $\mathbf{E}$. The particular sequence of elementary row operations used is not unique; you might find $\mathbf{E}$ in a quite different way.

1. $$\begin{bmatrix} 1 & 2 \\ 3 & 7 \end{bmatrix} \xrightarrow{R2-3R1} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \xrightarrow{R1-2R2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

2. $\begin{bmatrix} 3 & 7 \\ 2 & 5 \end{bmatrix} \overset{R1-R2}{\longrightarrow} \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix} \overset{R2-2R1}{\longrightarrow} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \overset{R1-2R2}{\longrightarrow} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

3. $\begin{bmatrix} 3 & 7 & 15 \\ 2 & 5 & 11 \end{bmatrix} \overset{R1-R2}{\longrightarrow} \begin{bmatrix} 1 & 2 & 4 \\ 2 & 5 & 11 \end{bmatrix} \overset{R2-2R1}{\longrightarrow} \begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 3 \end{bmatrix} \overset{R1-2R2}{\longrightarrow} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \end{bmatrix}$

4. $\begin{bmatrix} 3 & 7 & -1 \\ 5 & 2 & 8 \end{bmatrix} \overset{R2-R1}{\longrightarrow} \begin{bmatrix} 3 & 7 & -1 \\ 2 & -5 & 9 \end{bmatrix} \overset{R1-R2}{\longrightarrow} \begin{bmatrix} 1 & 12 & -10 \\ 2 & -5 & 9 \end{bmatrix}$

$\overset{R2-2R1}{\longrightarrow} \begin{bmatrix} 1 & 12 & -10 \\ 0 & -29 & 29 \end{bmatrix} \overset{(-1/29)R2}{\longrightarrow} \begin{bmatrix} 1 & 12 & -10 \\ 0 & 1 & -1 \end{bmatrix} \overset{R1-12\,R2}{\longrightarrow} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \end{bmatrix}$

5. $\begin{bmatrix} 1 & 2 & -11 \\ 2 & 3 & -19 \end{bmatrix} \overset{R2-2R1}{\longrightarrow} \begin{bmatrix} 1 & 2 & -11 \\ 0 & -1 & 3 \end{bmatrix} \overset{(-1)R2}{\longrightarrow} \begin{bmatrix} 1 & 2 & -11 \\ 0 & 1 & -3 \end{bmatrix} \overset{R1-2R2}{\longrightarrow} \begin{bmatrix} 1 & 0 & -5 \\ 0 & 1 & -3 \end{bmatrix}$

6. $\begin{bmatrix} 1 & -2 & 19 \\ 4 & -7 & 70 \end{bmatrix} \overset{R2-4\,R1}{\longrightarrow} \begin{bmatrix} 1 & -2 & 19 \\ 0 & 1 & -6 \end{bmatrix} \overset{R1+2\,R2}{\longrightarrow} \begin{bmatrix} 1 & 0 & 7 \\ 0 & 1 & -6 \end{bmatrix}$

7. $\begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 1 \\ 2 & 1 & 9 \end{bmatrix} \overset{R2-R1}{\longrightarrow} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & -2 \\ 2 & 1 & 9 \end{bmatrix} \overset{R3-2R1}{\longrightarrow} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & -2 \\ 0 & -3 & 3 \end{bmatrix}$

$\overset{(1/2)R2}{\longrightarrow} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & -1 \\ 0 & -3 & 3 \end{bmatrix} \overset{R3+3R2}{\longrightarrow} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \overset{R1-2R2}{\longrightarrow} \begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$

8. $\begin{bmatrix} 1 & -4 & -5 \\ 3 & -9 & 3 \\ 1 & -2 & 3 \end{bmatrix} \overset{R2-3R1}{\longrightarrow} \begin{bmatrix} 1 & -4 & -5 \\ 0 & 3 & 18 \\ 1 & -2 & 3 \end{bmatrix} \overset{R3-R1}{\longrightarrow} \begin{bmatrix} 1 & -4 & -5 \\ 0 & 3 & 18 \\ 0 & 2 & 8 \end{bmatrix}$

$\overset{R2-R3}{\longrightarrow} \begin{bmatrix} 1 & -4 & -5 \\ 0 & 1 & 10 \\ 0 & 2 & 8 \end{bmatrix} \overset{R3-2R2}{\longrightarrow} \begin{bmatrix} 1 & -4 & -5 \\ 0 & 1 & 10 \\ 0 & 0 & -12 \end{bmatrix} \overset{(-1/12)R3}{\longrightarrow} \cdots \rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

9. $\begin{bmatrix} 5 & 2 & 18 \\ 0 & 1 & 4 \\ 4 & 1 & 12 \end{bmatrix} \xrightarrow{R1-R3} \begin{bmatrix} 1 & 1 & 6 \\ 0 & 1 & 4 \\ 4 & 1 & 12 \end{bmatrix} \xrightarrow{R3-4R1} \begin{bmatrix} 1 & 1 & 6 \\ 0 & 1 & 4 \\ 0 & -3 & -12 \end{bmatrix}$

$\xrightarrow{R3+3R2} \begin{bmatrix} 1 & 1 & 6 \\ 0 & 1 & 4 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{R1-R2} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 4 \\ 0 & 0 & 0 \end{bmatrix}$

10. $\begin{bmatrix} 5 & 2 & -5 \\ 9 & 4 & -7 \\ 4 & 1 & -7 \end{bmatrix} \xrightarrow{R1-R3} \begin{bmatrix} 1 & 1 & 2 \\ 9 & 4 & -7 \\ 4 & 1 & -7 \end{bmatrix} \xrightarrow{R2-9R1} \begin{bmatrix} 1 & 1 & 2 \\ 0 & -5 & -25 \\ 4 & 1 & -7 \end{bmatrix}$

$\xrightarrow{R3-4R1} \begin{bmatrix} 1 & 1 & 2 \\ 0 & -5 & -25 \\ 0 & -3 & -15 \end{bmatrix} \xrightarrow{(-1/5)R2} \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 5 \\ 0 & -3 & -15 \end{bmatrix} \xrightarrow{R3+3R2} \cdots \to \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 5 \\ 0 & 0 & 0 \end{bmatrix}$

11. $\begin{bmatrix} 3 & 9 & 1 \\ 2 & 6 & 7 \\ 1 & 3 & -6 \end{bmatrix} \xrightarrow{SWAP(R1,R3)} \begin{bmatrix} 1 & 3 & -6 \\ 2 & 6 & 7 \\ 3 & 9 & 1 \end{bmatrix} \xrightarrow{R2-2R1} \begin{bmatrix} 1 & 3 & -6 \\ 0 & 0 & 19 \\ 3 & 9 & 1 \end{bmatrix}$

$\xrightarrow{R3-3R1} \begin{bmatrix} 1 & 3 & -6 \\ 0 & 0 & 19 \\ 0 & 0 & 19 \end{bmatrix} \xrightarrow{(1/19)R2} \begin{bmatrix} 1 & 3 & -6 \\ 0 & 0 & 1 \\ 0 & 0 & 19 \end{bmatrix} \xrightarrow{R3-19R2} \cdots \begin{bmatrix} 1 & 3 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

12. $\begin{bmatrix} 1 & -4 & -2 \\ 3 & -12 & 1 \\ 2 & -8 & 5 \end{bmatrix} \xrightarrow{R2-3R1} \begin{bmatrix} 1 & -4 & -2 \\ 0 & 0 & 7 \\ 2 & -8 & 5 \end{bmatrix} \xrightarrow{R3-2R1} \begin{bmatrix} 1 & -4 & -2 \\ 0 & 0 & 7 \\ 0 & 0 & 9 \end{bmatrix}$

$\xrightarrow{(1/7)R2} \cdots \to \begin{bmatrix} 1 & -4 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

13. $\begin{bmatrix} 2 & 7 & 4 & 0 \\ 1 & 3 & 2 & 1 \\ 2 & 6 & 5 & 4 \end{bmatrix} \xrightarrow{SWAP(R1,R2)} \begin{bmatrix} 1 & 3 & 2 & 1 \\ 2 & 7 & 4 & 0 \\ 2 & 6 & 5 & 4 \end{bmatrix} \xrightarrow{R2-2R1} \begin{bmatrix} 1 & 3 & 2 & 1 \\ 0 & 1 & 0 & -2 \\ 2 & 6 & 5 & 4 \end{bmatrix}$

$$\xrightarrow{R2-2R1} \begin{bmatrix} 1 & 3 & 2 & 1 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 2 \end{bmatrix} \xrightarrow{R1-3R2} \cdots \rightarrow \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

14. $$\begin{bmatrix} 1 & 3 & 2 & 5 \\ 2 & 5 & 2 & 3 \\ 2 & 7 & 7 & 22 \end{bmatrix} \xrightarrow{R2-2R1} \begin{bmatrix} 1 & 3 & 2 & 5 \\ 0 & -1 & -2 & -7 \\ 2 & 7 & 7 & 22 \end{bmatrix} \xrightarrow{R3-2R1} \begin{bmatrix} 1 & 3 & 2 & 5 \\ 0 & -1 & -2 & -7 \\ 0 & 1 & 3 & 12 \end{bmatrix}$$

$$\xrightarrow{R3+R2} \begin{bmatrix} 1 & 3 & 2 & 5 \\ 0 & -1 & -2 & -7 \\ 0 & 0 & 1 & 5 \end{bmatrix} \xrightarrow{(-1)R2} \cdots \rightarrow \begin{bmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 5 \end{bmatrix}$$

15. $$\begin{bmatrix} 2 & 2 & 4 & 2 \\ 1 & -1 & -4 & 3 \\ 2 & 7 & 19 & -3 \end{bmatrix} \xrightarrow{SWAP(R1,R2)} \begin{bmatrix} 1 & -1 & -4 & 3 \\ 2 & 2 & 4 & 2 \\ 2 & 7 & 19 & -3 \end{bmatrix} \xrightarrow{R2-2R1} \begin{bmatrix} 1 & -1 & -4 & 3 \\ 0 & 4 & 12 & -4 \\ 2 & 7 & 19 & -3 \end{bmatrix}$$

$$\xrightarrow{R3-2R1} \begin{bmatrix} 1 & -1 & -4 & 3 \\ 0 & 4 & 12 & -4 \\ 0 & 9 & 27 & -9 \end{bmatrix} \xrightarrow{(1/4)R2} \begin{bmatrix} 1 & -1 & -4 & 3 \\ 0 & 1 & 3 & -1 \\ 0 & 9 & 27 & -9 \end{bmatrix}$$

$$\xrightarrow{R3-9R2} \begin{bmatrix} 1 & -1 & -4 & 3 \\ 0 & 1 & 3 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{R1+R2} \begin{bmatrix} 1 & 0 & -1 & 2 \\ 0 & 1 & 3 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

16. $$\begin{bmatrix} 1 & 3 & 15 & 7 \\ 2 & 4 & 22 & 8 \\ 2 & 7 & 34 & 17 \end{bmatrix} \xrightarrow{R2-2R1} \begin{bmatrix} 1 & 3 & 15 & 7 \\ 0 & -2 & -8 & -6 \\ 2 & 7 & 34 & 17 \end{bmatrix} \xrightarrow{R3-2R1} \begin{bmatrix} 1 & 3 & 15 & 7 \\ 0 & -2 & -8 & -6 \\ 0 & 1 & 4 & 3 \end{bmatrix}$$

$$\xrightarrow{(-1/2)R2} \begin{bmatrix} 1 & 3 & 15 & 7 \\ 0 & 1 & 4 & 3 \\ 0 & 1 & 4 & 3 \end{bmatrix} \xrightarrow{R3-R2} \cdots \rightarrow \begin{bmatrix} 1 & 0 & 3 & -2 \\ 0 & 1 & 4 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

17. $$\begin{bmatrix} 1 & 1 & 1 & -1 & -4 \\ 1 & -2 & -2 & 8 & -1 \\ 2 & 3 & -1 & 3 & 11 \end{bmatrix} \xrightarrow{R2-R1} \begin{bmatrix} 1 & 1 & 1 & -1 & -4 \\ 0 & -3 & -3 & 9 & 3 \\ 2 & 3 & -1 & 3 & 11 \end{bmatrix} \xrightarrow{R3-2R1} \begin{bmatrix} 1 & 1 & 1 & -1 & -4 \\ 0 & -3 & -3 & 9 & 3 \\ 0 & 1 & -3 & 5 & 19 \end{bmatrix}$$

$$\xrightarrow{(-1/3)R2} \begin{bmatrix} 1 & 1 & 1 & -1 & -4 \\ 0 & 1 & 1 & -3 & -1 \\ 0 & 1 & -3 & 5 & 19 \end{bmatrix} \xrightarrow{R3-R2} \begin{bmatrix} 1 & 1 & 1 & -1 & -4 \\ 0 & 1 & 1 & -3 & -1 \\ 0 & 0 & -4 & 8 & 20 \end{bmatrix}$$

$$\xrightarrow{(-1/4)R3} \begin{bmatrix} 1 & 1 & 1 & -1 & -4 \\ 0 & 1 & 1 & -3 & -1 \\ 0 & 0 & 1 & -2 & -5 \end{bmatrix} \xrightarrow{R1-R2} \cdots \rightarrow \begin{bmatrix} 1 & 0 & 0 & 2 & -3 \\ 0 & 1 & 0 & -1 & 4 \\ 0 & 0 & 1 & -2 & -5 \end{bmatrix}$$

18. $$\begin{bmatrix} 1 & -2 & -5 & -12 & 1 \\ 2 & 3 & 18 & 11 & 9 \\ 2 & 5 & 26 & 21 & 11 \end{bmatrix} \xrightarrow{R2-2R1} \begin{bmatrix} 1 & -2 & -5 & -12 & 1 \\ 0 & 7 & 28 & 35 & 7 \\ 2 & 5 & 26 & 21 & 11 \end{bmatrix} \xrightarrow{R3-2R1} \begin{bmatrix} 1 & -2 & -5 & -12 & 1 \\ 0 & 7 & 28 & 35 & 7 \\ 0 & 9 & 36 & 45 & 9 \end{bmatrix}$$

$$\xrightarrow{(1/7)R2} \begin{bmatrix} 1 & -2 & -5 & -12 & 1 \\ 0 & 1 & 4 & 5 & 1 \\ 0 & 9 & 36 & 45 & 9 \end{bmatrix} \xrightarrow{(1/7)R2} \begin{bmatrix} 1 & -2 & -5 & -12 & 1 \\ 0 & 1 & 4 & 5 & 1 \\ 0 & 9 & 36 & 45 & 9 \end{bmatrix}$$

$$\xrightarrow{(1/9)R3} \begin{bmatrix} 1 & -2 & -5 & -12 & 1 \\ 0 & 1 & 4 & 5 & 1 \\ 0 & 1 & 4 & 5 & 1 \end{bmatrix} \xrightarrow{R3-R2} \cdots \rightarrow \begin{bmatrix} 1 & 0 & 3 & -2 & 3 \\ 0 & 1 & 4 & 5 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

19. $$\begin{bmatrix} 2 & 7 & -10 & -19 & 13 \\ 1 & 3 & -4 & -8 & 6 \\ 1 & 0 & 2 & 1 & 3 \end{bmatrix} \xrightarrow{SWAP(R1,R3)} \begin{bmatrix} 1 & 0 & 2 & 1 & 3 \\ 1 & 3 & -4 & -8 & 6 \\ 2 & 7 & -10 & -19 & 13 \end{bmatrix}$$

$$\xrightarrow{R2-R1} \begin{bmatrix} 1 & 0 & 2 & 1 & 3 \\ 0 & 3 & -6 & -9 & 3 \\ 2 & 7 & -10 & -19 & 13 \end{bmatrix} \xrightarrow{R3-2R1} \begin{bmatrix} 1 & 0 & 2 & 1 & 3 \\ 0 & 3 & -6 & -9 & 3 \\ 0 & 7 & -14 & -21 & 7 \end{bmatrix}$$

$$\xrightarrow{(1/3)R2} \begin{bmatrix} 1 & 0 & 2 & 1 & 3 \\ 0 & 1 & -2 & -3 & 1 \\ 0 & 7 & -14 & -21 & 7 \end{bmatrix} \xrightarrow{R3-7R2} \begin{bmatrix} 1 & 0 & 2 & 1 & 3 \\ 0 & 1 & -2 & -3 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

20. $$\begin{bmatrix} 3 & 6 & 1 & 7 & 13 \\ 5 & 10 & 8 & 18 & 47 \\ 2 & 4 & 5 & 9 & 26 \end{bmatrix} \xrightarrow{R1-R3} \begin{bmatrix} 1 & 2 & -4 & -2 & -13 \\ 5 & 10 & 8 & 18 & 47 \\ 2 & 4 & 5 & 9 & 26 \end{bmatrix}$$

$$\xrightarrow{R2-5R1} \begin{bmatrix} 1 & 2 & -4 & -2 & -13 \\ 0 & 0 & 28 & 28 & 112 \\ 2 & 4 & 5 & 9 & 26 \end{bmatrix} \xrightarrow{R3-2R1} \begin{bmatrix} 1 & 2 & -4 & -2 & -13 \\ 0 & 0 & 28 & 28 & 112 \\ 0 & 0 & 13 & 13 & 52 \end{bmatrix}$$

$$\xrightarrow{(1/28)R2} \begin{bmatrix} 1 & 2 & -4 & -2 & -13 \\ 0 & 0 & 1 & 1 & 4 \\ 0 & 0 & 13 & 13 & 52 \end{bmatrix} \xrightarrow{R3-13R2} \cdots \rightarrow \begin{bmatrix} 1 & 2 & 0 & 2 & 3 \\ 0 & 0 & 1 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

In each of Problems 21–30, we give just the first two or three steps in the reduction. Then we display the resulting reduced echelon form **E** of the augmented coefficient matrix **A** of the given linear system, and finally list the resulting solution (if any).

21. Begin by interchanging rows 1 and 2 of **A**. Then subtract twice row 1 both from row 2 and from row 3.

$$\mathbf{E} = \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 4 \end{bmatrix}; \quad x_1 = 3,\ x_2 = -2,\ x_3 = 4$$

22. Begin by subtracting row 2 of **A** from row 1. Then subtract twice row 1 both from row 2 and from row 3.

$$\mathbf{E} = \begin{bmatrix} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 2 \end{bmatrix}; \quad x_1 = 5,\ x_2 = -3,\ x_3 = 2$$

23. Begin by subtracting twice row 1 of **A** both from row 2 and from row 3. Then add row 2 to row 3.

$$\mathbf{E} = \begin{bmatrix} 1 & 0 & -3 & 14 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}; \quad x_1 = 4+3t,\ x_2 = 3-2t,\ x_3 = t$$

24. Begin by interchanging rows 1 and 3 of **A**. Then subtract twice row 1 from row 2, and three times row 1 from row 3.

$$\mathbf{E} = \begin{bmatrix} 1 & -2 & 0 & 5 \\ 0 & 0 & 1 & 7 \\ 0 & 0 & 0 & 0 \end{bmatrix}; \quad x_1 = 5+2t,\ x_2 = t,\ x_3 = 7$$

25. Begin by interchanging rows 1 and 2 of **A**. Then subtract three times row 1 from row 2, and five times row 1 from row 3.

$$\mathbf{E} = \begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \qquad \text{The system has no solution.}$$

26. Begin by subtracting row 1 from row 2 of **A**. Then interchange rows 1 and 2. Next subtract twice row 1 from row 2, and five times row 1 from row 3.

$$\mathbf{E} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \qquad \text{The system has no solution.}$$

27.
$$\begin{aligned} x_1 - 4x_2 - 3x_3 - 3x_4 &= 4 \\ 2x_1 - 6x_2 - 5x_3 - 5x_4 &= 5 \\ 3x_1 - x_2 - 4x_3 - 5x_4 &= -7 \end{aligned}$$

Begin by subtracting twice row 1 from row 2 of **A**, and three times row 1 from row 3.

$$\mathbf{E} = \begin{bmatrix} 1 & 0 & 0 & 2 & 3 \\ 0 & 1 & 0 & -1 & -4 \\ 0 & 0 & 1 & 3 & 5 \end{bmatrix}; \quad x_1 = 3 - 2t,\ x_2 = -4 + t,\ x_3 = 5 - 3t,\ x_4 = t$$

28. Begin by subtracting row 3 from row 1 of **A**. Then subtract 3 times row 1 from row 2, and twice row 1 from row 3.

$$\mathbf{E} = \begin{bmatrix} 1 & -2 & 0 & 3 & 4 \\ 0 & 0 & 1 & 4 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}; \quad x_1 = 4 + 2s - 3t,\ x_2 = s,\ x_3 = 3 - 4t,\ x_4 = t$$

29. Begin by interchanging rows 1 and 2 of **A**. Then subtract three times row 1 from row 2, and four times row 1 from row 3.

$$\mathbf{E} = \begin{bmatrix} 1 & 0 & 1 & 1 & 3 \\ 0 & 1 & -2 & 3 & 5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}; \quad x_1 = 3 - s - t,\ x_2 = 5 + 2s - 3t,\ x_3 = s,\ x_4 = t$$

30. Begin by interchanging rows 1 and 2 of $\mathbf{A}$. Then subtract twice row 1 from row 2, and five times row 1 from row 3.

$$\mathbf{E} = \begin{bmatrix} 1 & 0 & 0 & 0 & -3 & 2 \\ 0 & 1 & 0 & -1 & 2 & 1 \\ 0 & 0 & 1 & -2 & 0 & 2 \end{bmatrix}; \quad x_1 = 2+3t,\ x_2 = 1+s-2t,\ x_3 = 2+2s,\ x_4 = s,\ x_5 = t$$

31. $$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix} \xrightarrow{(1/6)R3} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{R2-5R3} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{(1/4)R2} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow{R1-2R2} \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{R1-3R3} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

32. If $ad - bc \neq 0,$ then not both a and b can be zero. If, for instance, $a \neq 0,$ then

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \xrightarrow{(1/a)R1} \begin{bmatrix} 1 & b/a \\ c & d \end{bmatrix} \xrightarrow{R2-cR1} \begin{bmatrix} 1 & b/a \\ 0 & d-bc/a \end{bmatrix} \xrightarrow{aR2} \begin{bmatrix} 1 & b/a \\ 0 & ad-bc \end{bmatrix}$$

$$\xrightarrow{(1/(ad-bc))R2} \begin{bmatrix} 1 & b/a \\ 0 & 1 \end{bmatrix} \xrightarrow{R1-(b/a)R2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

33. If the upper left element of a 2×2 reduced echelon matrix is $1,$ then the possibilities are $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 & * \\ 0 & 0 \end{bmatrix}$, depending on whether there is a nonzero element in the second row. If the upper left element is zero — so both elements of the second row are also 0, then the possibilities are $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

34. If the upper left element of a 3×3 reduced echelon matrix is $1,$ then the possibilities are

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & * \\ 0 & 1 & * \\ 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & * & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad \text{and} \quad \begin{bmatrix} 1 & * & * \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

depending on whether the second and third row contain any nonzero elements. If the upper left element is zero — so the first column and third row contain no nonzero elements — then use of the four 2×2 reduced echelon matrices of Problem 33 (for the upper right 2×2 submatrix of our reduced 3×3 matrix) gives the additional possibilities

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 & * \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \text{and} \quad \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

35. **(a)** If (x_0, y_0) is a solution, then it follows that

$$\begin{aligned} a(kx_0) + b(ky_0) &= k(ax_0 + by_0) = k \cdot 0 = 0, \\ c(kx_0) + d(ky_0) &= k(cx_0 + dy_0) = k \cdot 0 = 0 \end{aligned}$$

so (kx_0, ky_0) is also a solution.

(b) If (x_1, y_1) and (x_2, y_2) are solutions, then it follows that

$$\begin{aligned} a(x_1 + x_2) + b(y_1 + y_2) &= (ax_1 + by_1) + (ax_2 + by_2) = 0 + 0 = 0, \\ c(x_1 + x_2) + d(y_1 + y_2) &= (cx_1 + dy_1) + (cx_2 + dy_2) = 0 + 0 = 0 \end{aligned}$$

so $(x_1 + x_2, y_1 + y_2)$ is also a solution.

36. By Problem 32, the coefficient matrix of the given homogeneous 2×2 system is row-equivalent to the 2×2 identity matrix. Therefore, Theorem 4 implies that the given system has only the trivial solution.

37. If $ad - bc = 0$ then, much as in Problem 32, we see that the second row of the reduced echelon form of the coefficient matrix is allzero. Hence there is a free variable, and thus the given homogeneous system has a nontrivial solution involving a parameter t.

38. By Problem 37, there is a nontrivial solution if and only if

$$(c+2)(c-3) - (2)(3) = c^2 - c - 12 = (c-4)(c+3) = 0,$$

that is, either $c = 4$ or $c = -3$.

39. It is given that the augmented coefficient matrix of the homogeneous 3×3 system has the form

$$\begin{bmatrix} a_1 & b_1 & c_1 & 0 \\ a_2 & b_2 & c_2 & 0 \\ pa_1 + qa_2 & pb_1 + qb_2 & pc_1 + qc_2 & 0 \end{bmatrix}.$$

Upon subtracting both p times row 1 and q times row 2 from row 3, we get the matrix

$$\begin{bmatrix} a_1 & b_1 & c_1 & 0 \\ a_2 & b_2 & c_2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

corresponding to two homogeneous linear equations in three unknowns. Hence there is at least one free variable, and thus the system has a nontrivial family of solutions.

40. In reducing further from the echelon matrix **E** to the matrix **E***, the leading entries of **E** become the leading ones in the reduced echelon matrix **E***. Thus the nonzero rows of **E*** come precisely from the nonzero rows of **E**. We therefore are talking about the same rows — and in particular about the same *number* of rows — in either case.

SECTION 3.4

MATRIX OPERATIONS

The objective of this section is simple to state. It is not merely knowledge of, but complete mastery of matrix addition and multiplication (particularly the latter). Matrix multiplication must be practiced until it is carried out not only accurately but quickly and with confidence — until you can hardly look at two matrices **A** and **B** without thinking of "pouring" the *i*th row of **A** down the *j*th column of **B**.

1. $$3\begin{bmatrix} 3 & -5 \\ 2 & 7 \end{bmatrix} + 4\begin{bmatrix} -1 & 0 \\ 3 & -4 \end{bmatrix} = \begin{bmatrix} 9 & -15 \\ 6 & 21 \end{bmatrix} + \begin{bmatrix} -4 & 0 \\ 12 & -16 \end{bmatrix} = \begin{bmatrix} 5 & -15 \\ 18 & 5 \end{bmatrix}$$

2. $$5\begin{bmatrix} 2 & 0 & -3 \\ -1 & 5 & 6 \end{bmatrix} - 3\begin{bmatrix} -2 & 3 & 1 \\ 7 & 1 & 5 \end{bmatrix} = \begin{bmatrix} 10 & 0 & -15 \\ -5 & 25 & 30 \end{bmatrix} + \begin{bmatrix} 6 & -9 & -3 \\ -21 & -3 & -15 \end{bmatrix} = \begin{bmatrix} 16 & -9 & -18 \\ -26 & 22 & 15 \end{bmatrix}$$

3. $$-2\begin{bmatrix} 5 & 0 \\ 0 & 7 \\ 3 & -1 \end{bmatrix} + 4\begin{bmatrix} -4 & 5 \\ 3 & 2 \\ 7 & 4 \end{bmatrix} = \begin{bmatrix} -10 & 0 \\ 0 & -14 \\ -6 & 2 \end{bmatrix} + \begin{bmatrix} -16 & 20 \\ 12 & 8 \\ 28 & 16 \end{bmatrix} = \begin{bmatrix} -26 & 20 \\ 12 & -6 \\ 22 & 18 \end{bmatrix}$$

4. $$7\begin{bmatrix} 2 & -1 & 0 \\ 4 & 0 & -3 \\ 5 & -2 & 7 \end{bmatrix} + 5\begin{bmatrix} 6 & -3 & -4 \\ 5 & 2 & -1 \\ 0 & 7 & 9 \end{bmatrix}$$

$$= \begin{bmatrix} 14 & -7 & 0 \\ 28 & 0 & -21 \\ 35 & -14 & 49 \end{bmatrix} + \begin{bmatrix} 30 & -15 & -20 \\ 25 & 10 & -5 \\ 0 & 35 & 45 \end{bmatrix} = \begin{bmatrix} 44 & -22 & -20 \\ 53 & 10 & -26 \\ 35 & 21 & 94 \end{bmatrix}$$

5. $\begin{bmatrix} 2 & -1 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} -4 & 2 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} -9 & 1 \\ -10 & 12 \end{bmatrix}$; $\quad \begin{bmatrix} -4 & 2 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} -2 & 8 \\ 11 & 5 \end{bmatrix}$

6. $\begin{bmatrix} 1 & 0 & -3 \\ 3 & 2 & 4 \\ 2 & -3 & 5 \end{bmatrix}\begin{bmatrix} 7 & -4 & 3 \\ 1 & 5 & -2 \\ 0 & 3 & 9 \end{bmatrix} = \begin{bmatrix} 7 & -13 & -24 \\ 23 & 10 & 41 \\ 11 & -8 & 57 \end{bmatrix}$

$\begin{bmatrix} 7 & -4 & 3 \\ 1 & 5 & -2 \\ 0 & 3 & 9 \end{bmatrix}\begin{bmatrix} 1 & 0 & -3 \\ 3 & 2 & 4 \\ 2 & -3 & 5 \end{bmatrix} = \begin{bmatrix} 1 & -17 & -22 \\ 12 & 16 & 7 \\ 27 & -21 & 57 \end{bmatrix}$

7. $\begin{bmatrix} 1 & 2 & 3 \end{bmatrix}\begin{bmatrix} 3 \\ 4 \\ 6 \end{bmatrix} = \begin{bmatrix} 26 \end{bmatrix}$; $\quad \begin{bmatrix} 3 \\ 4 \\ 6 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 \end{bmatrix} = \begin{bmatrix} 3 & 6 & 9 \\ 4 & 8 & 12 \\ 5 & 10 & 15 \end{bmatrix}$

8. $\begin{bmatrix} 1 & 0 & 3 \\ 2 & -5 & 4 \end{bmatrix}\begin{bmatrix} 3 & 0 \\ -1 & 4 \\ 6 & 5 \end{bmatrix} = \begin{bmatrix} 21 & 15 \\ 35 & 0 \end{bmatrix}$; $\quad \begin{bmatrix} 3 & 0 \\ -1 & 4 \\ 6 & 5 \end{bmatrix}\begin{bmatrix} 1 & 0 & 3 \\ 2 & -5 & 4 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 9 \\ 7 & -20 & 13 \\ 16 & -25 & 38 \end{bmatrix}$

9. $\begin{bmatrix} 0 & -2 \\ 3 & 1 \\ -4 & 5 \end{bmatrix}\begin{bmatrix} 3 \\ -2 \end{bmatrix} = \begin{bmatrix} 4 \\ 7 \\ -22 \end{bmatrix}$ but the product $\begin{bmatrix} 3 \\ -2 \end{bmatrix}\begin{bmatrix} 0 & -2 \\ 3 & 1 \\ -4 & 5 \end{bmatrix}$ is not defined.

10. $\mathbf{AB} = \begin{bmatrix} 2 & 1 \\ 4 & 3 \end{bmatrix}\begin{bmatrix} -1 & 0 & 4 \\ 3 & -2 & 5 \end{bmatrix} = \begin{bmatrix} 1 & -2 & 13 \\ 5 & -6 & 31 \end{bmatrix}$ but the product $\mathbf{BA}$ is not defined.

11. $\mathbf{AB} = \begin{bmatrix} 3 & -5 \end{bmatrix}\begin{bmatrix} 2 & 7 & 5 & 6 \\ -1 & 4 & 2 & 3 \end{bmatrix} = \begin{bmatrix} 11 & 1 & 5 & 3 \end{bmatrix}$ but the product $\mathbf{BA}$ is not defined.

12. Neither product matrix $\mathbf{AB}$ or $\mathbf{BA}$ is defined.

13. $\mathbf{A}(\mathbf{BC}) = \begin{bmatrix} 3 & 1 \\ -1 & 4 \end{bmatrix}\left(\begin{bmatrix} 2 & 5 \\ -3 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 2 & 3 \end{bmatrix}\right) = \begin{bmatrix} 3 & 1 \\ -1 & 4 \end{bmatrix}\begin{bmatrix} 10 & 17 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} 32 & 51 \\ -2 & -17 \end{bmatrix}$

$(\mathbf{AB})\mathbf{C} = \left(\begin{bmatrix} 3 & 1 \\ -1 & 4 \end{bmatrix}\begin{bmatrix} 2 & 5 \\ -3 & 1 \end{bmatrix}\right)\begin{bmatrix} 0 & 1 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 3 & 16 \\ -14 & -1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 32 & 51 \\ -2 & -17 \end{bmatrix}$

14. $\mathbf{A}(\mathbf{BC}) = \begin{bmatrix} 2 & -1 \end{bmatrix}\left(\begin{bmatrix} 2 & 5 \\ -3 & 1 \end{bmatrix}\begin{bmatrix} 6 \\ -5 \end{bmatrix}\right) = \begin{bmatrix} 2 & -1 \end{bmatrix}\begin{bmatrix} -13 \\ -23 \end{bmatrix} = \begin{bmatrix} -3 \end{bmatrix}$

$$(\mathbf{AB})\mathbf{C} = \left(\begin{bmatrix} 2 & -1 \end{bmatrix}\begin{bmatrix} 2 & 5 \\ -3 & 1 \end{bmatrix}\right)\begin{bmatrix} 6 \\ -5 \end{bmatrix} = \begin{bmatrix} 7 & 9 \end{bmatrix}\begin{bmatrix} 6 \\ -5 \end{bmatrix} = \begin{bmatrix} -3 \end{bmatrix}$$

15. $\mathbf{A}(\mathbf{BC}) = \begin{bmatrix} 3 \\ 2 \end{bmatrix}\left(\begin{bmatrix} 1 & -1 & 2 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & 3 \\ 1 & 4 \end{bmatrix}\right) = \begin{bmatrix} 3 \\ 2 \end{bmatrix}\begin{bmatrix} 4 & 5 \end{bmatrix} = \begin{bmatrix} 12 & 15 \\ 8 & 10 \end{bmatrix}$

$$(\mathbf{AB})\mathbf{C} = \left(\begin{bmatrix} 3 \\ 2 \end{bmatrix}\begin{bmatrix} 1 & -1 & 2 \end{bmatrix}\right)\begin{bmatrix} 2 & 0 \\ 0 & 3 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} 3 & -3 & 6 \\ 2 & -2 & 4 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & 3 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} 12 & 15 \\ 8 & 10 \end{bmatrix}$$

16. $\mathbf{A}(\mathbf{BC}) = \begin{bmatrix} 2 & 0 \\ 0 & 3 \\ 1 & 4 \end{bmatrix}\left(\begin{bmatrix} 1 & -1 \\ 3 & -2 \end{bmatrix}\begin{bmatrix} 1 & 0 & -1 & 2 \\ 3 & 2 & 0 & 1 \end{bmatrix}\right)$

$$= \begin{bmatrix} 2 & 0 \\ 0 & 3 \\ 1 & 4 \end{bmatrix}\begin{bmatrix} -2 & -2 & -1 & 1 \\ -3 & -4 & -3 & 4 \end{bmatrix} = \begin{bmatrix} -4 & -4 & -2 & 2 \\ -9 & -12 & -9 & 12 \\ -14 & -18 & -13 & 17 \end{bmatrix}$$

$$(\mathbf{AB})\mathbf{C} = \left(\begin{bmatrix} 2 & 0 \\ 0 & 3 \\ 1 & 4 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 3 & -2 \end{bmatrix}\right)\begin{bmatrix} 1 & 0 & -1 & 2 \\ 3 & 2 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & -2 \\ 9 & -6 \\ 13 & -9 \end{bmatrix}\begin{bmatrix} 1 & 0 & -1 & 2 \\ 3 & 2 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -4 & -4 & -2 & 2 \\ -9 & -12 & -9 & 12 \\ -14 & -18 & -13 & 17 \end{bmatrix}$$

Each of the homogeneous linear systems in Problems 17–22 is already in echelon form, so it remains only to write (by back substitution) the solution, first in parametric form and then in vector form.

17. $x_3 = s, \quad x_4 = t, \quad x_1 = 5s - 4t, \quad x_2 = -2s + 7t$

$$\mathbf{x} = s(5,-2,1,0) + t(-4,7,0,1)$$

18. $x_2 = s, \quad x_4 = t, \quad x_1 = 3s - 6t, \quad x_3 = -9t$

$$\mathbf{x} = s(3,1,0,0) + t(-6,0,-9,1)$$

19. $x_4 = s, \quad x_5 = t, \quad x_1 = -3s+t, \quad x_2 = 2s-6t, \quad x_3 = -s+8t$

$\mathbf{x} = s(-3,2,-1,1,0)+t(1,-6,8,0,1)$

20. $x_2 = s, \quad x_5 = t, \quad x_1 = 3s-7t, \quad x_3 = 2t, \quad x_4 = 10t$

$\mathbf{x} = s(3,1,0,0,0)+t(-7,0,2,10,0)$

21. $x_3 = r, \quad x_4 = s, \quad x_5 = t, \quad x_1 = r-2s-7t, \quad x_2 = -2r+3s-4t$

$\mathbf{x} = r(1,-2,1,0,0)+s(-2,3,0,1,0)+t(-7,-4,0,0,1)$

22. $x_2 = r, \quad x_4 = s, \quad x_5 = t, \quad x_1 = r-7s-3t, \quad x_3 = s+2t$

$\mathbf{x} = r(1,1,0,0,0)+s(-7,0,1,1,0)+t(-3,0,2,0,1)$

23. The matrix equation $\begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix}=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ entails the four scalar equations

$$\begin{aligned} 2a+c &= 1 \qquad & 2b+d &= 0 \\ 3a+2c &= 0 \qquad & 3b+2d &= 1 \end{aligned}$$

that we readily solve for $a=2,\ b=-1,\ c=-3,\ d=2$. Hence the apparent inverse matrix of $\mathbf{A}$, such that $\mathbf{AB}=\mathbf{I}$, is $\mathbf{B} = \begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix}$. Indeed, we find that $\mathbf{BA}=\mathbf{I}$ as well.

24. The matrix equation $\begin{bmatrix} 3 & 4 \\ 5 & 7 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix}=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ entails the four scalar equations

$$\begin{aligned} 3a+4c &= 1 \qquad & 3b+4d &= 0 \\ 5a+7c &= 0 \qquad & 5b+7d &= 1 \end{aligned}$$

that we readily solve for $a=7,\ b=-4,\ c=-5,\ d=3$. Hence the apparent inverse matrix of $\mathbf{A}$, such that $\mathbf{AB}=\mathbf{I}$, is $\mathbf{B} = \begin{bmatrix} 7 & -4 \\ -5 & 3 \end{bmatrix}$. Indeed, we find that $\mathbf{BA}=\mathbf{I}$ as well.

25. The matrix equation $\begin{bmatrix} 5 & 7 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix}=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ entails the four scalar equations

$$\begin{aligned} 5a+7c &= 1 \qquad & 5b+7d &= 0 \\ 2a+3c &= 0 \qquad & 2b+3d &= 1 \end{aligned}$$

that we readily solve for $a=3,\ b=-7,\ c=-2,\ d=5$. Hence the apparent inverse matrix of $\mathbf{A}$, such that $\mathbf{AB}=\mathbf{I}$, is $\mathbf{B} = \begin{bmatrix} 3 & -7 \\ -2 & 5 \end{bmatrix}$. Indeed, we find that $\mathbf{BA}=\mathbf{I}$ as well.

26. The matrix equation $\begin{bmatrix} 1 & -2 \\ -2 & 4 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ entails the four scalar equations

$$\begin{aligned} a-2c &= 1 \qquad & b-2d &= 0 \\ -2a+4c &= 0 \qquad & -2b+4d &= 1. \end{aligned}$$

But the two equations in a and c obviously are inconsistent, because $(-1)(1) \neq 0$, and the two equations in b and d are similarly inconsistent. Therefore the given matrix $\mathbf{A}$ has no inverse matrix.

27.

$$\begin{bmatrix} a_1 & 0 & 0 & \cdots & 0 \\ 0 & a_2 & 0 & \cdots & 0 \\ 0 & 0 & a_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_n \end{bmatrix}\begin{bmatrix} b_1 & 0 & 0 & \cdots & 0 \\ 0 & b_2 & 0 & \cdots & 0 \\ 0 & 0 & b_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & b_n \end{bmatrix} = \begin{bmatrix} a_1b_1 & 0 & 0 & \cdots & 0 \\ 0 & a_2b_2 & 0 & \cdots & 0 \\ 0 & 0 & a_3b_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_nb_n \end{bmatrix}$$

Thus the product of two diagonal matrices of the same size is obtained simply by multiplying corresponding diagonal elements. Then the commutativity of scalar multiplication immediately implies that $\mathbf{AB}=\mathbf{BA}$ for diagonal matrices.

28. The matrix power $\mathbf{A}^n$ is simply the product $\mathbf{AAA}\cdots\mathbf{A}$ of n copies of $\mathbf{A}$. It follows (by associativity) that parentheses don't matter:

$$\mathbf{A}^r\mathbf{A}^s = \underbrace{(\mathbf{AAA}\cdots\mathbf{A})}_{r \text{ copies}}\underbrace{(\mathbf{AAA}\cdots\mathbf{A})}_{s \text{ copies}} = \underbrace{(\mathbf{AAA}\cdots\mathbf{A})}_{r+s \text{ copies}} = \mathbf{A}^{r+s},$$

the product of $r+s$ copies of $\mathbf{A}$ in either case.

29.

$$\begin{aligned} (a+d)\mathbf{A}-(ad-bc)\mathbf{I} &= (a+d)\begin{bmatrix} a & b \\ c & d \end{bmatrix} - (ad-bc)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} (a^2+ad)-(ad-bc) & ab+bd \\ ac+cd & (ad+d^2)-(ad-bc) \end{bmatrix} = \begin{bmatrix} a^2+bc & ab+bd \\ ac+cd & bc+d^2 \end{bmatrix} \end{aligned}$$

$$= \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \mathbf{A}^2$$

30. If $\mathbf{A} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ then $a+d = 4$ and $ad-bc = 3$. Hence

$$\mathbf{A}^2 = 4\mathbf{A} - 3\mathbf{I} = 4\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} - 3\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 4 \\ 4 & 5 \end{bmatrix};$$

$$\mathbf{A}^3 = 4\mathbf{A}^2 - 3\mathbf{A} = 4\begin{bmatrix} 5 & 4 \\ 4 & 5 \end{bmatrix} - 3\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 14 & 13 \\ 13 & 14 \end{bmatrix};$$

$$\mathbf{A}^4 = 4\mathbf{A}^3 - 3\mathbf{A}^2 = 4\begin{bmatrix} 14 & 13 \\ 13 & 14 \end{bmatrix} - 3\begin{bmatrix} 5 & 4 \\ 4 & 5 \end{bmatrix} = \begin{bmatrix} 41 & 40 \\ 40 & 41 \end{bmatrix};$$

$$\mathbf{A}^5 = 4\mathbf{A}^5 - 3\mathbf{A}^3 = 4\begin{bmatrix} 41 & 40 \\ 40 & 41 \end{bmatrix} - 3\begin{bmatrix} 14 & 13 \\ 13 & 14 \end{bmatrix} = \begin{bmatrix} 122 & 121 \\ 121 & 122 \end{bmatrix}.$$

31. **(a)** If $\mathbf{A} = \begin{bmatrix} 2 & -1 \\ -4 & 3 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 1 & 5 \\ 3 & 7 \end{bmatrix}$ then

$$(\mathbf{A}+\mathbf{B})(\mathbf{A}-\mathbf{B}) = \begin{bmatrix} 3 & 4 \\ -1 & 10 \end{bmatrix} \begin{bmatrix} 1 & -6 \\ -7 & -4 \end{bmatrix} = \begin{bmatrix} -25 & -34 \\ -71 & -34 \end{bmatrix}$$

but

$$\mathbf{A}^2 - \mathbf{B}^2 = \begin{bmatrix} 8 & -5 \\ -20 & 13 \end{bmatrix} - \begin{bmatrix} 16 & 40 \\ 24 & 64 \end{bmatrix} = \begin{bmatrix} -8 & -45 \\ -44 & -51 \end{bmatrix}.$$

(b) If $\mathbf{AB} = \mathbf{BA}$ then

$$\begin{aligned} (\mathbf{A}+\mathbf{B})(\mathbf{A}-\mathbf{B}) &= \mathbf{A}(\mathbf{A}-\mathbf{B}) + \mathbf{B}(\mathbf{A}-\mathbf{B}) \\ &= \mathbf{A}^2 - \mathbf{AB} + \mathbf{BA} - \mathbf{B}^2 = \mathbf{A}^2 - \mathbf{B}^2. \end{aligned}$$

32. **(a)** If $\mathbf{A} = \begin{bmatrix} 2 & -1 \\ -4 & 3 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 1 & 5 \\ 3 & 7 \end{bmatrix}$ then

$$(\mathbf{A}+\mathbf{B})^2 = \begin{bmatrix} 3 & 4 \\ -1 & 10 \end{bmatrix} \begin{bmatrix} 3 & 4 \\ -1 & 10 \end{bmatrix} = \begin{bmatrix} 5 & 52 \\ -13 & 96 \end{bmatrix}$$

but

$$\begin{aligned} \mathbf{A}^2 + 2\mathbf{AB} + \mathbf{B}^2 &= \begin{bmatrix} 8 & -5 \\ -20 & 13 \end{bmatrix} + 2\begin{bmatrix} 2 & -1 \\ -4 & 3 \end{bmatrix}\begin{bmatrix} 1 & 5 \\ 3 & 7 \end{bmatrix} + \begin{bmatrix} 16 & 40 \\ 24 & 64 \end{bmatrix} \\ &= \begin{bmatrix} 8 & -5 \\ -20 & 13 \end{bmatrix} + 2\begin{bmatrix} -1 & 3 \\ 5 & 1 \end{bmatrix} + \begin{bmatrix} 16 & 40 \\ 24 & 64 \end{bmatrix} = \begin{bmatrix} 22 & 41 \\ 14 & 79 \end{bmatrix}. \end{aligned}$$

(b) If $\mathbf{AB} = \mathbf{BA}$ then

$$\begin{aligned}(\mathbf{A}+\mathbf{B})(\mathbf{A}-\mathbf{B}) &= \mathbf{A}(\mathbf{A}-\mathbf{B})+\mathbf{B}(\mathbf{A}-\mathbf{B}) \\ &= \mathbf{A}^2-\mathbf{AB}+\mathbf{BA}-\mathbf{B}^2 = \mathbf{A}^2-\mathbf{B}^2.\end{aligned}$$

33. Four different 2×2 matrices $\mathbf{A}$ with $\mathbf{A}^2 = \mathbf{I}$ are

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \text{ and } \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}.$$

34. If $\mathbf{A} = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} \neq \mathbf{0}$ then $\mathbf{A}^2 = (0)\mathbf{A}-(0)\mathbf{I} = \mathbf{0}$.

35. If $\mathbf{A} = \begin{bmatrix} 2 & -1 \\ 2 & -1 \end{bmatrix} \neq \mathbf{0}$ then $\mathbf{A}^2 = (1)\mathbf{A}-(0)\mathbf{I} = \mathbf{A}$.

36. If $\mathbf{A} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \neq \mathbf{0}$ then $\mathbf{A}^2 = (0)\mathbf{A}-(-1)\mathbf{I} = \mathbf{I}$.

37. If $\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \neq \mathbf{0}$ then $\mathbf{A}^2 = (0)\mathbf{A}-(1)\mathbf{I} = -\mathbf{I}$.

38. If $\mathbf{A} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \neq \mathbf{0}$ is the matrix of Problem 36 and $\mathbf{B} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \neq \mathbf{0}$ is the matrix of Problem 37, then $\mathbf{A}^2+\mathbf{B}^2 = (\mathbf{I})+(-\mathbf{I}) = \mathbf{0}$.

39. If $\mathbf{Ax}_1 = \mathbf{Ax}_2 = \mathbf{0}$, then

$$\mathbf{A}(c_1\mathbf{x}_1+c_2\mathbf{x}_2) = c_1(\mathbf{Ax}_1)+c_2(\mathbf{Ax}_2) = c_1(\mathbf{0})+c_2(\mathbf{0}) = \mathbf{0}.$$

40. **(a)** If $\mathbf{Ax}_0 = \mathbf{0}$ and $\mathbf{Ax}_1 = \mathbf{b}$, then $\mathbf{A}(\mathbf{x}_0+\mathbf{x}_1) = \mathbf{Ax}_0+\mathbf{Ax}_1 = \mathbf{0}+\mathbf{b} = \mathbf{b}$.

(b) If $\mathbf{Ax}_1 = \mathbf{b}$ and $\mathbf{Ax}_2 = \mathbf{b}$, then $\mathbf{A}(\mathbf{x}_1-\mathbf{x}_2) = \mathbf{Ax}_1-\mathbf{Ax}_2 = \mathbf{b}-\mathbf{b} = \mathbf{0}$.

41. If $\mathbf{AB} = \mathbf{BA}$ then

$$\begin{aligned}(\mathbf{A}+\mathbf{B})^3 &= (\mathbf{A}+\mathbf{B})(\mathbf{A}+\mathbf{B})^2 = (\mathbf{A}+\mathbf{B})(\mathbf{A}^2+2\mathbf{AB}+\mathbf{B}^2) \\ &= \mathbf{A}(\mathbf{A}^2+2\mathbf{AB}+\mathbf{B}^2)+\mathbf{B}(\mathbf{A}^2+2\mathbf{AB}+\mathbf{B}^2) \\ &= (\mathbf{A}^3+2\mathbf{A}^2\mathbf{B}+\mathbf{AB}^2)+(\mathbf{A}^2\mathbf{B}+2\mathbf{AB}^2+\mathbf{B}^3)\end{aligned}$$

$$= \mathbf{A}^3 + 3\mathbf{A}^2\mathbf{B} + 3\mathbf{A}\mathbf{B}^2 + \mathbf{B}^3.$$

To compute $(\mathbf{A}+\mathbf{B})^4$, write $(\mathbf{A}+\mathbf{B})^4 = (\mathbf{A}+\mathbf{B})(\mathbf{A}+\mathbf{B})^3$ and proceed similarly, substituting the expansion of $(\mathbf{A}+\mathbf{B})^3$ just obtained.

42. **(a)** Matrix multiplication gives $\mathbf{N}^2 = \begin{bmatrix} 0 & 0 & 4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ and $\mathbf{N}^3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

(b) $$\mathbf{A}^2 = \mathbf{I} + 2\mathbf{N} + \mathbf{N}^2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + 2\begin{bmatrix} 0 & 2 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 4 & 4 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{A}^3 = \mathbf{I} + 3\mathbf{N} + 3\mathbf{N}^2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + 3\begin{bmatrix} 0 & 2 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix} + 3\begin{bmatrix} 0 & 0 & 4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 6 & 12 \\ 0 & 1 & 6 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{A}^4 = \mathbf{I} + 4\mathbf{N} + 6\mathbf{N}^2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + 4\begin{bmatrix} 0 & 2 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix} + 6\begin{bmatrix} 0 & 0 & 4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 8 & 24 \\ 0 & 1 & 8 \\ 0 & 0 & 1 \end{bmatrix}$$

43. First, matrix multiplication gives $\mathbf{A}^2 = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 6 & -3 & -3 \\ -3 & 6 & -3 \\ -3 & -3 & 6 \end{bmatrix} = 3\mathbf{A}$. Then

$$\mathbf{A}^3 = \mathbf{A}^2 \cdot \mathbf{A} = 3\mathbf{A} \cdot \mathbf{A} = 3\mathbf{A}^2 = 3 \cdot 3\mathbf{A} = 9\mathbf{A},$$
$$\mathbf{A}^4 = \mathbf{A}^3 \cdot \mathbf{A} = 9\mathbf{A} \cdot \mathbf{A} = 9\mathbf{A}^2 = 9 \cdot 3\mathbf{A} = 27\mathbf{A},$$

and so forth.

SECTION 3.5

INVERSES OF MATRICES

The computational objective of this section is clearcut — to find the inverse of a given invertible matrix. From a more general viewpoint, Theorem 7 on the properties of nonsingular matrices summarizes most of the basic theory of this chapter.

In Problems 1–8 we first give the inverse matrix $\mathbf{A}^{-1}$ and then calculate the solution vector $\mathbf{x}$.

1. $\mathbf{A}^{-1} = \begin{bmatrix} 3 & -2 \\ -4 & 3 \end{bmatrix}; \quad \mathbf{x} = \begin{bmatrix} 3 & -2 \\ -4 & 3 \end{bmatrix}\begin{bmatrix} 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$

2. $\mathbf{A}^{-1} = \begin{bmatrix} 5 & -7 \\ -2 & 3 \end{bmatrix}; \quad \mathbf{x} = \begin{bmatrix} 5 & -7 \\ -2 & 3 \end{bmatrix}\begin{bmatrix} -1 \\ 3 \end{bmatrix} = \begin{bmatrix} -26 \\ 11 \end{bmatrix}$

3. $\mathbf{A}^{-1} = \begin{bmatrix} 6 & -7 \\ -5 & 6 \end{bmatrix}; \quad \mathbf{x} = \begin{bmatrix} 6 & -7 \\ -5 & 6 \end{bmatrix}\begin{bmatrix} 2 \\ -3 \end{bmatrix} = \begin{bmatrix} 33 \\ -28 \end{bmatrix}$

4. $\mathbf{A}^{-1} = \begin{bmatrix} 17 & -12 \\ -7 & 5 \end{bmatrix}; \quad \mathbf{x} = \begin{bmatrix} 17 & -12 \\ -7 & 5 \end{bmatrix}\begin{bmatrix} 5 \\ 5 \end{bmatrix} = \begin{bmatrix} 25 \\ -10 \end{bmatrix}$

5. $\mathbf{A}^{-1} = \frac{1}{2}\begin{bmatrix} 4 & -2 \\ -5 & 3 \end{bmatrix}; \quad \mathbf{x} = \frac{1}{2}\begin{bmatrix} 4 & -2 \\ -5 & 3 \end{bmatrix}\begin{bmatrix} 5 \\ 6 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 8 \\ -7 \end{bmatrix}$

6. $\mathbf{A}^{-1} = \frac{1}{3}\begin{bmatrix} 6 & -7 \\ -3 & 4 \end{bmatrix}; \quad \mathbf{x} = \frac{1}{3}\begin{bmatrix} 6 & -7 \\ -3 & 4 \end{bmatrix}\begin{bmatrix} 10 \\ 5 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 25 \\ -10 \end{bmatrix}$

7. $\mathbf{A}^{-1} = \frac{1}{4}\begin{bmatrix} 7 & -9 \\ -5 & 7 \end{bmatrix}; \quad \mathbf{x} = \frac{1}{4}\begin{bmatrix} 7 & -9 \\ -5 & 7 \end{bmatrix}\begin{bmatrix} 3 \\ 2 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 3 \\ -1 \end{bmatrix}$

8. $\mathbf{A}^{-1} = \frac{1}{5}\begin{bmatrix} 10 & -15 \\ -5 & 8 \end{bmatrix}; \quad \mathbf{x} = \frac{1}{5}\begin{bmatrix} 10 & -15 \\ -5 & 8 \end{bmatrix}\begin{bmatrix} 7 \\ 3 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 25 \\ -11 \end{bmatrix}$

In Problems 9–22 we give at least the first few steps in the reduction of the augmented matrix whose right half is the identity matrix of appropriate size. We wind up with its echelon form, whose left half is an identity matrix and whose right half is the desired inverse matrix.

9. $$\begin{bmatrix} 5 & 6 & 1 & 0 \\ 4 & 5 & 0 & 1 \end{bmatrix} \xrightarrow{R1-R2} \begin{bmatrix} 1 & 1 & 1 & -1 \\ 4 & 5 & 0 & 1 \end{bmatrix} \xrightarrow{R2-4R1} \begin{bmatrix} 1 & 1 & 1 & -1 \\ 0 & 1 & -4 & 5 \end{bmatrix}$$

$$\xrightarrow{R1-R2} \begin{bmatrix} 1 & 0 & 5 & -6 \\ 0 & 1 & -4 & 5 \end{bmatrix}; \quad \text{thus } \mathbf{A}^{-1} = \begin{bmatrix} 5 & -6 \\ -4 & 5 \end{bmatrix}$$

10. $$\begin{bmatrix} 5 & 7 & 1 & 0 \\ 4 & 6 & 0 & 1 \end{bmatrix} \xrightarrow{R1-R2} \begin{bmatrix} 1 & 1 & 1 & -1 \\ 4 & 6 & 0 & 1 \end{bmatrix} \xrightarrow{R2-4R1} \begin{bmatrix} 1 & 1 & 1 & -1 \\ 0 & 2 & -4 & 5 \end{bmatrix}$$

$$\xrightarrow{(1/2)R2} \begin{bmatrix} 1 & 1 & 1 & -1 \\ 0 & 1 & -2 & \frac{5}{2} \end{bmatrix} \xrightarrow{R1-R2} \begin{bmatrix} 1 & 0 & 3 & -\frac{7}{2} \\ 0 & 1 & -2 & \frac{5}{2} \end{bmatrix}; \quad \text{thus } \mathbf{A}^{-1} = \frac{1}{2}\begin{bmatrix} 6 & -7 \\ -4 & 5 \end{bmatrix}$$

11. $$\begin{bmatrix} 1 & 5 & 1 & 1 & 0 & 0 \\ 2 & 5 & 0 & 0 & 1 & 0 \\ 2 & 7 & 1 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{R2-2R1} \begin{bmatrix} 1 & 5 & 1 & 1 & 0 & 0 \\ 0 & -5 & -2 & -2 & 1 & 0 \\ 2 & 7 & 1 & 0 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow{R3-2R1} \begin{bmatrix} 1 & 5 & 1 & 1 & 0 & 0 \\ 0 & -5 & -2 & -2 & 1 & 0 \\ 0 & -3 & -1 & -2 & 0 & 1 \end{bmatrix} \xrightarrow{R1+R3} \begin{bmatrix} 1 & 2 & 0 & -1 & 0 & 1 \\ 0 & -5 & -2 & -2 & 1 & 0 \\ 0 & -3 & -1 & -2 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow{R2-2R3} \begin{bmatrix} 1 & 2 & 0 & -1 & 0 & 1 \\ 0 & 1 & 0 & 2 & 1 & -2 \\ 0 & -3 & -1 & -2 & 0 & 1 \end{bmatrix} \xrightarrow{R3+3R2} \begin{bmatrix} 1 & 2 & 0 & -1 & 0 & 1 \\ 0 & 1 & 0 & 2 & 1 & -2 \\ 0 & 0 & -1 & 4 & 3 & -5 \end{bmatrix}$$

$$\xrightarrow{(-1)R3} \cdots \xrightarrow{R1-2R2} \begin{bmatrix} 1 & 0 & 0 & -5 & -2 & 5 \\ 0 & 1 & 0 & 2 & 1 & -2 \\ 0 & 0 & 1 & -4 & -3 & 5 \end{bmatrix}; \quad \text{thus } \mathbf{A}^{-1} = \begin{bmatrix} -5 & -2 & 5 \\ 2 & 1 & -2 \\ -4 & -3 & 5 \end{bmatrix}$$

12. $$\begin{bmatrix} 1 & 3 & 2 & 1 & 0 & 0 \\ 2 & 8 & 3 & 0 & 1 & 0 \\ 3 & 10 & 6 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{R2-2R1} \begin{bmatrix} 1 & 3 & 2 & 1 & 0 & 0 \\ 0 & 2 & -1 & -2 & 1 & 0 \\ 3 & 10 & 6 & 0 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow{R3-2R1} \begin{bmatrix} 1 & 3 & 2 & 1 & 0 & 0 \\ 0 & 2 & -1 & -2 & 1 & 0 \\ 0 & 1 & 0 & -3 & 0 & 1 \end{bmatrix} \xrightarrow{R2-R3} \begin{bmatrix} 1 & 3 & 2 & 1 & 0 & 0 \\ 0 & 1 & -1 & 1 & 1 & -1 \\ 0 & 1 & 0 & -3 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow{R3-R2} \begin{bmatrix} 1 & 3 & 2 & 1 & 0 & 0 \\ 0 & 1 & -1 & 1 & 1 & -1 \\ 0 & 0 & 1 & -4 & -1 & 2 \end{bmatrix} \xrightarrow{R1-3R2} \cdots \rightarrow \begin{bmatrix} 1 & 0 & 0 & 18 & 2 & -7 \\ 0 & 1 & 0 & -3 & 0 & 1 \\ 0 & 0 & 1 & -4 & -2 & 2 \end{bmatrix};$$

$$\text{thus } \mathbf{A}^{-1} = \begin{bmatrix} 18 & 2 & -7 \\ -3 & 0 & 1 \\ -4 & -1 & 2 \end{bmatrix}$$

13. $$\begin{bmatrix} 2 & 7 & 3 & 1 & 0 & 0 \\ 1 & 3 & 2 & 0 & 1 & 0 \\ 3 & 7 & 9 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{SWAP(R1,R2)} \begin{bmatrix} 1 & 3 & 2 & 0 & 1 & 0 \\ 2 & 7 & 3 & 1 & 0 & 0 \\ 3 & 7 & 9 & 0 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow{R2-2R1} \left[\begin{array}{ccc|ccc} 1 & 3 & 2 & 0 & 1 & 0 \\ 0 & 1 & -1 & 1 & -2 & 0 \\ 3 & 7 & 9 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R3-3R1} \left[\begin{array}{ccc|ccc} 1 & 3 & 2 & 0 & 1 & 0 \\ 0 & 1 & -1 & 1 & -2 & 0 \\ 0 & -2 & 3 & 0 & -3 & 1 \end{array}\right]$$

$$\xrightarrow{R3+2R2} \cdots \rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -13 & 42 & -5 \\ 0 & 1 & 0 & 3 & -9 & 1 \\ 0 & 0 & 1 & 2 & -7 & 1 \end{array}\right]; \quad \text{thus } \mathbf{A}^{-1} = \begin{bmatrix} -13 & 42 & -5 \\ 3 & -9 & 1 \\ 2 & -7 & 1 \end{bmatrix}$$

14. $$\left[\begin{array}{ccc|ccc} 3 & 5 & 6 & 1 & 0 & 0 \\ 2 & 4 & 3 & 0 & 1 & 0 \\ 2 & 3 & 5 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R1-R2} \left[\begin{array}{ccc|ccc} 1 & 1 & 3 & 1 & -1 & 0 \\ 2 & 4 & 3 & 0 & 1 & 0 \\ 2 & 3 & 5 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R2-R3} \left[\begin{array}{ccc|ccc} 1 & 1 & 3 & 1 & -1 & 0 \\ 0 & 1 & -2 & 0 & 1 & -1 \\ 2 & 3 & 5 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R3-2R1} \left[\begin{array}{ccc|ccc} 1 & 1 & 3 & 1 & -1 & 0 \\ 0 & 1 & -2 & 0 & 1 & -1 \\ 0 & 1 & -1 & -2 & 2 & 1 \end{array}\right]$$

$$\xrightarrow{R3-R2} \cdots \rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 11 & -7 & -9 \\ 0 & 1 & 0 & -4 & 3 & 3 \\ 0 & 0 & 1 & -2 & 1 & 2 \end{array}\right]; \quad \text{thus } \mathbf{A}^{-1} = \begin{bmatrix} 11 & -7 & -9 \\ -4 & 3 & 3 \\ -2 & 1 & 2 \end{bmatrix}$$

15. $$\left[\begin{array}{ccc|ccc} 1 & 1 & 5 & 1 & 0 & 0 \\ 1 & 4 & 13 & 0 & 1 & 0 \\ 3 & 2 & 12 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R2-R1} \left[\begin{array}{ccc|ccc} 1 & 1 & 5 & 1 & 0 & 0 \\ 0 & 3 & 8 & -1 & 1 & 0 \\ 3 & 2 & 12 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R3-3R1} \left[\begin{array}{ccc|ccc} 1 & 1 & 5 & 1 & 0 & 0 \\ 0 & 3 & 8 & -1 & 1 & 0 \\ 0 & -1 & -3 & -3 & 0 & 1 \end{array}\right] \xrightarrow{R2+2R3} \left[\begin{array}{ccc|ccc} 1 & 1 & 5 & 1 & 0 & 0 \\ 0 & 1 & 2 & -7 & 1 & 2 \\ 0 & -1 & -3 & -3 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R3+R2} \cdots \rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -22 & 2 & 7 \\ 0 & 1 & 0 & -27 & 3 & 8 \\ 0 & 0 & 1 & 10 & -1 & -3 \end{array}\right]; \quad \text{thus } \mathbf{A}^{-1} = \begin{bmatrix} -22 & 2 & 7 \\ -27 & 3 & 8 \\ 10 & -1 & -3 \end{bmatrix}$$

16. $$\left[\begin{array}{ccc|ccc} 1 & -3 & -3 & 1 & 0 & 0 \\ -1 & 1 & 2 & 0 & 1 & 0 \\ 2 & -3 & -3 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R2+R1} \left[\begin{array}{ccc|ccc} 1 & -3 & -3 & 1 & 0 & 0 \\ 0 & -2 & -1 & 1 & 1 & 0 \\ 2 & -3 & -3 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R3-2R1}\left[\begin{array}{cccccc}1&-3&-3&1&0&0\\0&-2&-1&1&1&0\\0&3&3&-2&0&1\end{array}\right]\xrightarrow{R2+R3}\left[\begin{array}{cccccc}1&-3&-3&1&0&0\\0&1&2&-1&1&1\\0&3&3&-2&0&1\end{array}\right]$$

$$\xrightarrow{R3-3R2}\left[\begin{array}{cccccc}1&-3&-3&1&0&0\\0&1&2&-1&1&1\\0&0&-3&1&-3&-2\end{array}\right]\xrightarrow{(-1/3(R3)}\left[\begin{array}{cccccc}1&-3&-3&1&0&0\\0&1&2&-1&1&1\\0&0&1&-\frac{1}{3}&1&\frac{2}{3}\end{array}\right]$$

$$\xrightarrow{R1+3R2}\cdots\rightarrow\left[\begin{array}{cccccc}1&0&0&-1&0&1\\0&1&0&-\frac{1}{3}&-1&-\frac{1}{3}\\0&0&1&-\frac{1}{3}&1&\frac{2}{3}\end{array}\right];\quad \text{thus } \mathbf{A}^{-1}=\frac{1}{3}\left[\begin{array}{ccc}-3&0&3\\-1&-3&-1\\-1&3&2\end{array}\right]$$

17. $$\left[\begin{array}{cccccc}1&-3&0&1&0&0\\-1&2&-1&0&1&0\\0&-2&2&0&0&1\end{array}\right]\xrightarrow{R2+R1}\left[\begin{array}{cccccc}1&-3&0&1&0&0\\0&-1&-1&1&1&0\\0&-2&2&0&0&1\end{array}\right]$$

$$\xrightarrow{(-1)R2}\left[\begin{array}{cccccc}1&-3&0&1&0&0\\0&1&1&-1&-1&0\\0&-2&2&0&0&1\end{array}\right]\xrightarrow{R3+2R2}\left[\begin{array}{cccccc}1&-3&0&1&0&0\\0&1&1&-1&-1&0\\0&0&4&-2&-2&1\end{array}\right]$$

$$\xrightarrow{(1/4)R3}\cdots\rightarrow\left[\begin{array}{cccccc}1&0&0&-\frac{1}{2}&-\frac{3}{2}&-\frac{3}{4}\\0&1&0&-\frac{1}{2}&-\frac{1}{2}&-\frac{1}{4}\\0&0&1&-\frac{1}{2}&-\frac{1}{2}&\frac{1}{4}\end{array}\right];\quad \text{thus } \mathbf{A}^{-1}=\frac{1}{4}\left[\begin{array}{ccc}-2&-6&-3\\-2&-2&-1\\-2&-2&1\end{array}\right]$$

18. $$\left[\begin{array}{cccccc}1&-2&2&1&0&0\\3&0&1&0&1&0\\1&-1&2&0&0&1\end{array}\right]\xrightarrow{R2-3R1}\left[\begin{array}{cccccc}1&-2&2&1&0&0\\0&6&-5&-3&1&0\\1&-1&2&0&0&1\end{array}\right]$$

$$\xrightarrow{R3-R1}\left[\begin{array}{cccccc}1&-2&2&1&0&0\\0&6&-5&-3&1&0\\0&1&0&-1&0&1\end{array}\right]\xrightarrow{R2-5R3}\left[\begin{array}{cccccc}1&-2&2&1&0&0\\0&1&-5&2&1&-5\\0&1&0&-1&0&1\end{array}\right]$$

$$\xrightarrow{R3-R2}\left[\begin{array}{cccccc}1&-2&2&1&0&0\\0&1&-5&2&1&-5\\0&0&5&-3&-1&6\end{array}\right]\xrightarrow{(1/5)R3}\cdots\rightarrow\left[\begin{array}{cccccc}1&0&0&\frac{1}{5}&\frac{2}{5}&-\frac{2}{5}\\0&1&0&-1&0&1\\0&0&1&-\frac{3}{5}&-\frac{1}{5}&\frac{6}{5}\end{array}\right];$$

$$\text{thus } \mathbf{A}^{-1}=\frac{1}{5}\left[\begin{array}{ccc}1&2&-2\\-5&0&5\\-3&-1&6\end{array}\right].$$

19. $\begin{bmatrix} 1 & 4 & 3 & 1 & 0 & 0 \\ 1 & 4 & 5 & 0 & 1 & 0 \\ 2 & 5 & 1 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{R2-R1} \begin{bmatrix} 1 & 4 & 3 & 1 & 0 & 0 \\ 0 & 0 & 2 & -1 & 1 & 0 \\ 2 & 5 & 1 & 0 & 0 & 1 \end{bmatrix}$

$\xrightarrow{SWAP(R2,R3)} \begin{bmatrix} 1 & 4 & 3 & 1 & 0 & 0 \\ 2 & 5 & 1 & 0 & 0 & 1 \\ 0 & 0 & 2 & -1 & 1 & 0 \end{bmatrix} \xrightarrow{R2-2R1} \begin{bmatrix} 1 & 4 & 3 & 1 & 0 & 0 \\ 0 & -3 & -5 & -2 & 0 & 1 \\ 0 & 0 & 2 & -1 & 1 & 0 \end{bmatrix}$

$\xrightarrow{(-1/3)R2} \begin{bmatrix} 1 & 4 & 3 & 1 & 0 & 0 \\ 0 & 1 & \frac{5}{3} & \frac{2}{3} & 0 & -\frac{1}{3} \\ 0 & 0 & 2 & -1 & 1 & 0 \end{bmatrix} \xrightarrow{(1/2)R3} \cdots \rightarrow \begin{bmatrix} 1 & 0 & 0 & -\frac{7}{2} & \frac{11}{6} & \frac{4}{3} \\ 0 & 1 & 0 & \frac{3}{2} & -\frac{5}{6} & -\frac{1}{3} \\ 0 & 0 & 1 & -\frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix};$

thus $\mathbf{A}^{-1} = \frac{1}{6}\begin{bmatrix} -21 & 11 & 8 \\ 9 & -5 & -2 \\ -3 & 3 & 0 \end{bmatrix}$

20. $\begin{bmatrix} 2 & 0 & -1 & 1 & 0 & 0 \\ 1 & 0 & 3 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{R1-R2} \begin{bmatrix} 1 & 0 & -4 & 1 & -1 & 0 \\ 1 & 0 & 3 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \end{bmatrix}$

$\xrightarrow{R3-R1} \begin{bmatrix} 1 & 0 & -4 & 1 & -1 & 0 \\ 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 5 & -1 & 1 & 0 \end{bmatrix} \xrightarrow{SWAP(R2,R3)} \begin{bmatrix} 1 & 0 & -4 & 1 & -1 & 0 \\ 0 & 1 & 5 & -1 & 1 & 1 \\ 1 & 0 & 3 & 0 & 1 & 0 \end{bmatrix}$

$\xrightarrow{R3-R1} \begin{bmatrix} 1 & 0 & -4 & 1 & -1 & 0 \\ 0 & 1 & 5 & -1 & 1 & 1 \\ 0 & 0 & 7 & -1 & 2 & 0 \end{bmatrix} \xrightarrow{(1/7)R3} \cdots \rightarrow \begin{bmatrix} 1 & 0 & 0 & \frac{3}{7} & \frac{1}{7} & 0 \\ 0 & 1 & 0 & -\frac{2}{7} & -\frac{3}{7} & 1 \\ 0 & 0 & 1 & -\frac{1}{7} & \frac{2}{7} & 0 \end{bmatrix};$

thus $\mathbf{A}^{-1} = \frac{1}{7}\begin{bmatrix} 3 & 1 & 0 \\ -2 & -3 & 7 \\ -1 & 2 & 0 \end{bmatrix}$

21. $\begin{bmatrix} 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 & 0 & 1 & 0 \\ 3 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{SWAP(R1,R2)} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 & 0 & 1 & 0 \\ 3 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}$

$$\xrightarrow{SWAP(R2,R3)} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 3 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{R4-3R1} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & -3 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow{R2-2R3} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & -2 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & -3 & 0 & 1 \end{bmatrix}; \quad \text{thus } \mathbf{A}^{-1} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -2 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & -3 & 0 & 1 \end{bmatrix}$$

22. $$\begin{bmatrix} 4 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 3 & 1 & 3 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 & 0 & 1 & 0 \\ 3 & 2 & 4 & 1 & 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{R1-R2} \begin{bmatrix} 1 & -1 & -2 & 0 & 1 & -1 & 0 & 0 \\ 3 & 1 & 3 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 & 0 & 1 & 0 \\ 3 & 2 & 4 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow{R2-3R1} \begin{bmatrix} 1 & -1 & -2 & 0 & 1 & -1 & 0 & 0 \\ 0 & 4 & 9 & 1 & -3 & 4 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 & 0 & 1 & 0 \\ 3 & 2 & 4 & 1 & 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{R4-3R1} \begin{bmatrix} 1 & -1 & -2 & 0 & 1 & -1 & 0 & 0 \\ 0 & 4 & 9 & 1 & -3 & 4 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 & 0 & 1 & 0 \\ 0 & 5 & 10 & 1 & -3 & 3 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow{R2-3R3} \begin{bmatrix} 1 & -1 & -2 & 0 & 1 & -1 & 0 & 0 \\ 0 & 1 & 3 & 1 & -3 & 4 & -3 & 0 \\ 0 & 1 & 2 & 0 & 0 & 0 & 1 & 0 \\ 0 & 5 & 10 & 1 & -3 & 3 & 0 & 1 \end{bmatrix} \xrightarrow{R3-R2} \begin{bmatrix} 1 & -1 & -2 & 0 & 1 & -1 & 0 & 0 \\ 0 & 1 & 3 & 1 & -3 & 4 & -3 & 0 \\ 0 & 0 & -1 & -1 & 3 & -4 & 4 & 0 \\ 0 & 5 & 10 & 1 & -3 & 3 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow{R4-5R2} \cdots \rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & -1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & -2 & -1 & 2 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 1 & -3 & 3 & -5 & 1 \end{bmatrix}; \text{ thus } \mathbf{A}^{-1} = \begin{bmatrix} 1 & -1 & 1 & 0 \\ 0 & -2 & -1 & 2 \\ 0 & 1 & 1 & -1 \\ -3 & 3 & -5 & 1 \end{bmatrix}$$

In Problems 23–28 we first give the inverse matrix $\mathbf{A}^{-1}$ and then calculate the solution matrix $\mathbf{X}$.

23. $$\mathbf{A}^{-1} = \begin{bmatrix} 4 & -3 \\ -5 & 4 \end{bmatrix}; \quad \mathbf{X} = \begin{bmatrix} 4 & -3 \\ -5 & 4 \end{bmatrix}\begin{bmatrix} 1 & 3 & -5 \\ -1 & -2 & 5 \end{bmatrix} = \begin{bmatrix} 7 & 18 & -35 \\ -9 & -23 & 45 \end{bmatrix}$$

24. $$\mathbf{A}^{-1} = \begin{bmatrix} 7 & -6 \\ -8 & 7 \end{bmatrix}; \quad \mathbf{X} = \begin{bmatrix} 7 & -6 \\ -8 & 7 \end{bmatrix}\begin{bmatrix} 2 & 0 & 4 \\ 0 & 5 & -3 \end{bmatrix} = \begin{bmatrix} 14 & -30 & 46 \\ -16 & 35 & -53 \end{bmatrix}$$

25. $\mathbf{A}^{-1} = \begin{bmatrix} 11 & -9 & 4 \\ -2 & 2 & -1 \\ -2 & 1 & 0 \end{bmatrix}$; $\mathbf{X} = \begin{bmatrix} 11 & -9 & 4 \\ -2 & 2 & -1 \\ -2 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 3 \\ 0 & 2 & 2 \\ -1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 7 & -14 & 15 \\ -1 & 3 & -2 \\ -2 & 2 & -4 \end{bmatrix}$

26. $\mathbf{A}^{-1} = \begin{bmatrix} -16 & 3 & 11 \\ 6 & -1 & -4 \\ -13 & 2 & 9 \end{bmatrix}$; $\mathbf{X} = \begin{bmatrix} -16 & 3 & 11 \\ 6 & -1 & -4 \\ -13 & 2 & 9 \end{bmatrix} \begin{bmatrix} 2 & 0 & 1 \\ 0 & 3 & 0 \\ 1 & 0 & 2 \end{bmatrix} = \begin{bmatrix} -21 & 9 & 6 \\ 8 & -3 & -2 \\ -17 & 6 & 5 \end{bmatrix}$

27. $\mathbf{A}^{-1} = \begin{bmatrix} 7 & -20 & 17 \\ 0 & -1 & 1 \\ -2 & 6 & -5 \end{bmatrix}$; $\mathbf{X} = \begin{bmatrix} 7 & -20 & 17 \\ 0 & -1 & 1 \\ -2 & 6 & -5 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 17 & -20 & 24 & -13 \\ 1 & -1 & 1 & -1 \\ -5 & 6 & -7 & 4 \end{bmatrix}$

28. $\mathbf{A}^{-1} = \begin{bmatrix} -5 & 5 & 10 \\ -8 & 8 & 15 \\ 24 & -23 & -45 \end{bmatrix}$; $\mathbf{X} = \begin{bmatrix} -5 & 5 & 10 \\ -8 & 8 & 15 \\ 24 & -23 & -45 \end{bmatrix} \begin{bmatrix} 2 & 1 & 0 & 2 \\ -1 & 3 & 5 & 0 \\ 1 & 1 & 0 & 5 \end{bmatrix} = \begin{bmatrix} -5 & 5 & 10 & 1 \\ -8 & 8 & 15 & 7 \\ 24 & -23 & -45 & -13 \end{bmatrix}$

29. **(a)** The fact that $\mathbf{A}^{-1}$ is the inverse of $\mathbf{A}$ means that $\mathbf{AA}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$. That is, that when $\mathbf{A}^{-1}$ is multiplied either on the right or on the left by $\mathbf{A}$, the result is the identity matrix $\mathbf{I}$. By the same token, this means that $\mathbf{A}$ is the inverse of $\mathbf{A}^{-1}$.

(b) $\mathbf{A}^n(\mathbf{A}^{-1})^n = \mathbf{A}^{n-1} \cdot \mathbf{AA}^{-1} \cdot (\mathbf{A}^{-1})^{n-1} = \mathbf{A}^{n-1} \cdot \mathbf{I} \cdot (\mathbf{A}^{-1})^{n-1} = \cdots = \mathbf{I}$. Similarly, $(\mathbf{A}^{-1})^n \mathbf{A}^n = \mathbf{I}$, so it follows that $(\mathbf{A}^{-1})^n$ is the inverse of $\mathbf{A}^n$.

30. $\mathbf{ABC} \cdot \mathbf{C}^{-1}\mathbf{B}^{-1}\mathbf{A}^{-1} = \mathbf{AB} \cdot \mathbf{I} \cdot \mathbf{B}^{-1}\mathbf{A}^{-1} = \mathbf{A} \cdot \mathbf{I} \cdot \mathbf{A}^{-1} = \mathbf{I}$, and we see is a similar way that $\mathbf{C}^{-1}\mathbf{B}^{-1}\mathbf{A}^{-1} \cdot \mathbf{ABC} = \mathbf{I}$.

31. Let $p = -r > 0$, $q = -s > 0$, and $\mathbf{B} = \mathbf{A}^{-1}$. Then

$$\begin{aligned} \mathbf{A}^r\mathbf{A}^s &= \mathbf{A}^{-p}\mathbf{A}^{-q} = (\mathbf{A}^{-1})^p(\mathbf{A}^{-1})^q \\ &= \mathbf{B}^p\mathbf{B}^q = \mathbf{B}^{p+q} \qquad (\text{because } p, q > 0) \\ &= (\mathbf{A}^{-1})^{p+q} = \mathbf{A}^{-p-q} = \mathbf{A}^{r+s} \end{aligned}$$

as desired, and $(\mathbf{A}^r)^s = (\mathbf{A}^{-p})^{-q} = (\mathbf{B}^p)^{-q} = \mathbf{B}^{-pq} = \mathbf{A}^{pq} = \mathbf{A}^{rs}$ similarly.

32. Multiplication of $\mathbf{AB} = \mathbf{AC}$ on the left by $\mathbf{A}^{-1}$ yields $\mathbf{B} = \mathbf{C}$.

33. In particular, $\mathbf{Ae}_j = \mathbf{e}_j$ where $\mathbf{e}_j$ denotes the jth column vector of the identity matrix $\mathbf{I}$. Hence it follows from Fact 2 that $\mathbf{AI} = \mathbf{I}$, and therefore $\mathbf{A} = \mathbf{I}^{-1} = \mathbf{I}$.

34. The invertibility of a diagonal matrix with nonzero diagonal elements follows immediately from the rule for multiplying diagonal matrices (Problem 27 in Section 3.4). The inverse of such a diagonal matrix is gotten simply by inverting each diagonal element.

35. If the *j*th column of $\mathbf{A}$ is all zeros and $\mathbf{B}$ is any $n \times n$ matrix, then the *j*th column of $\mathbf{BA}$ is all zeros, so $\mathbf{BA} \neq \mathbf{I}$. Hence $\mathbf{A}$ has no inverse matrix. Similarly, if the *i*th row of $\mathbf{A}$ is all zeros, then so is the *i*th row of $\mathbf{AB}$.

36. If $ad - bc = 0$, then it follows easily that one row of $\mathbf{A}$ is a multiple of the other. Hence the reduced echelon form of $\mathbf{A}$ is of the form $\begin{bmatrix} * & * \\ 0 & 0 \end{bmatrix}$ rather than the 2×2 identity matrix. Therefore $\mathbf{A}$ is not invertible.

37. Direct multiplication shows that $\mathbf{AA}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$.

38. $$\mathbf{EA} = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 3a & 3b \\ c & d \end{bmatrix}$$

39. $$\mathbf{EA} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} + 2a_{11} & a_{32} + a_{12} & a_{33} + a_{13} \end{bmatrix}$$

40. $$\mathbf{EA} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} a_{21} & a_{22} & a_{23} \\ a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

41. This follows immediately from the fact that the *ij*th element of $\mathbf{AB}$ is the product of the *i*th row of $\mathbf{A}$ and the *j*th column of $\mathbf{B}$.

42. Let $\mathbf{e}_i$ denote the *i*th ***row*** of $\mathbf{I}$. Then $\mathbf{e}_i\mathbf{B} = \mathbf{B}_i$, the *i*th row of $\mathbf{B}$. Hence the result in Problem 41 yields

$$\mathbf{IB} = \begin{bmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \vdots \\ \mathbf{e}_m \end{bmatrix}\mathbf{B} = \begin{bmatrix} \mathbf{e}_1\mathbf{B} \\ \mathbf{e}_2\mathbf{B} \\ \vdots \\ \mathbf{e}_m\mathbf{B} \end{bmatrix} = \begin{bmatrix} \mathbf{B}_1 \\ \mathbf{B}_2 \\ \vdots \\ \mathbf{B}_m \end{bmatrix} = \mathbf{B}.$$

43. Let $\mathbf{E}_1, \mathbf{E}_2, \cdots, \mathbf{E}_k$ be the elementary matrices corresponding to the elementary row operations that reduce $\mathbf{A}$ to $\mathbf{B}$. Then Theorem 5 gives $\mathbf{B} = \mathbf{E}_k\mathbf{E}_{k-1}\cdots\mathbf{E}_2\mathbf{E}_1\mathbf{A} = \mathbf{GA}$ where $\mathbf{G} = \mathbf{E}_k\mathbf{E}_{k-1}\cdots\mathbf{E}_2\mathbf{E}_1$.

44. This follows immediately from the result in Problem 43, because an invertible matrix is row-equivalent to the identity matrix.

45. One can simply photocopy the portion of the proof of Theorem 7 that follows Equation (20). Starting only with the assumption that $\mathbf{A}$ and $\mathbf{B}$ are square matrices with $\mathbf{AB} = \mathbf{I}$, it is proved there that $\mathbf{A}$ and $\mathbf{B}$ are then invertible.

46. If $\mathbf{C} = \mathbf{AB}$ is invertible, so $\mathbf{C}^{-1}$ exists, then $\mathbf{A}(\mathbf{BC}^{-1}) = \mathbf{I}$ and $(\mathbf{C}^{-1}\mathbf{A})\mathbf{B} = \mathbf{I}$. Hence the fact that $\mathbf{A}$ and $\mathbf{B}$ are invertible follows immediately from Problem 45.

SECTION 3.6

DETERMINANTS

1. $$\begin{vmatrix} 0 & 0 & 3 \\ 4 & 0 & 0 \\ 0 & 5 & 0 \end{vmatrix} = +(3)\begin{vmatrix} 4 & 0 \\ 0 & 5 \end{vmatrix} = 3\cdot 4\cdot 5 = 60$$

2. $$\begin{vmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{vmatrix} = +(2)\begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} - (1)\begin{vmatrix} 1 & 1 \\ 0 & 2 \end{vmatrix} = 2(4-1)-(2-0) = 4$$

3. $$\begin{vmatrix} 1 & 0 & 0 & 0 \\ 2 & 0 & 5 & 0 \\ 3 & 6 & 9 & 8 \\ 4 & 0 & 10 & 7 \end{vmatrix} = +(1)\begin{vmatrix} 0 & 5 & 0 \\ 6 & 9 & 8 \\ 0 & 10 & 7 \end{vmatrix} = -(5)\begin{vmatrix} 6 & 8 \\ 0 & 7 \end{vmatrix} = -5(42-0) = -210$$

4. $$\begin{vmatrix} 5 & 11 & 8 & 7 \\ 3 & -2 & 6 & 23 \\ 0 & 0 & 0 & -3 \\ 0 & 4 & 0 & 17 \end{vmatrix} = -(-3)\begin{vmatrix} 5 & 11 & 8 \\ 3 & -2 & 6 \\ 0 & 4 & 0 \end{vmatrix} = 3(-4)\begin{vmatrix} 5 & 8 \\ 3 & 6 \end{vmatrix} = -12(30-24) = -72$$

5. $$\begin{vmatrix} 0 & 0 & 1 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 4 \\ 0 & 5 & 0 & 0 & 0 \end{vmatrix} = +1\begin{vmatrix} 2 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 \\ 0 & 5 & 0 & 0 \end{vmatrix} = +2\begin{vmatrix} 0 & 3 & 0 \\ 0 & 0 & 4 \\ 5 & 0 & 0 \end{vmatrix} = 2(+5)\begin{vmatrix} 3 & 0 \\ 0 & 4 \end{vmatrix} = 2\cdot 5\cdot 3\cdot 4 = 120$$

6. $\begin{vmatrix} 3 & 0 & 11 & -5 & 0 \\ -2 & 4 & 13 & 6 & 5 \\ 0 & 0 & 5 & 0 & 0 \\ 7 & 6 & -9 & 17 & 7 \\ 0 & 0 & 8 & 2 & 0 \end{vmatrix} = +5\begin{vmatrix} 3 & 0 & -5 & 0 \\ -2 & 4 & 6 & 5 \\ 7 & 6 & 17 & 7 \\ 0 & 0 & 2 & 0 \end{vmatrix} = 5(-2)\begin{vmatrix} 3 & 0 & 0 \\ -2 & 4 & 5 \\ 7 & 6 & 7 \end{vmatrix} = -10(+3)\begin{vmatrix} 4 & 5 \\ 6 & 7 \end{vmatrix} = 60$

7. $\begin{vmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \end{vmatrix} \overset{R2-2R1}{=} \begin{vmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 3 & 3 & 3 \end{vmatrix} = 0$

8. $\begin{vmatrix} 2 & 3 & 4 \\ -2 & -3 & 1 \\ 3 & 2 & 7 \end{vmatrix} \overset{R2+R1}{=} \begin{vmatrix} 2 & 3 & 4 \\ 0 & 0 & 5 \\ 3 & 2 & 7 \end{vmatrix} = -5\begin{vmatrix} 2 & 3 \\ 3 & 2 \end{vmatrix} = -5(4-9) = 25$

9. $\begin{vmatrix} 3 & -2 & 5 \\ 0 & 5 & 17 \\ 6 & -4 & 12 \end{vmatrix} \overset{R3-2R1}{=} \begin{vmatrix} 3 & -2 & 5 \\ 0 & 5 & 17 \\ 0 & 0 & 2 \end{vmatrix} = +2\begin{vmatrix} 3 & 5 \\ 0 & 2 \end{vmatrix} = 5(6-0) = 30$

10. $\begin{vmatrix} -3 & 6 & 5 \\ 1 & -2 & -4 \\ 2 & -5 & 12 \end{vmatrix} \overset{R1+3R2}{=} \begin{vmatrix} 0 & 0 & -7 \\ 1 & -2 & -4 \\ 2 & -5 & 12 \end{vmatrix} = +(-7)\begin{vmatrix} 1 & -2 \\ 2 & -5 \end{vmatrix} = -7(-5+4) = 7$

11. $\begin{vmatrix} 1 & 2 & 3 & 4 \\ 0 & 5 & 6 & 7 \\ 0 & 0 & 8 & 9 \\ 2 & 4 & 6 & 9 \end{vmatrix} \overset{R4-2R1}{=} \begin{vmatrix} 1 & 2 & 3 & 4 \\ 0 & 5 & 6 & 7 \\ 0 & 0 & 8 & 9 \\ 0 & 0 & 0 & 1 \end{vmatrix} = +1\begin{vmatrix} 5 & 6 & 7 \\ 0 & 8 & 9 \\ 0 & 0 & 1 \end{vmatrix} = +5\begin{vmatrix} 8 & 9 \\ 0 & 1 \end{vmatrix} = 5\cdot 8 = 40$

12. $\begin{vmatrix} 2 & 0 & 0 & -3 \\ 0 & 1 & 11 & 12 \\ 0 & 0 & 5 & 13 \\ -4 & 0 & 0 & 7 \end{vmatrix} \overset{R4+2R1}{=} \begin{vmatrix} 2 & 0 & 0 & -3 \\ 0 & 1 & 11 & 12 \\ 0 & 0 & 5 & 13 \\ 0 & 0 & 0 & 1 \end{vmatrix} = +2\begin{vmatrix} 1 & 11 & 12 \\ 0 & 5 & 13 \\ 0 & 0 & 1 \end{vmatrix} = +2\begin{vmatrix} 5 & 13 \\ 0 & 1 \end{vmatrix} = 2\cdot 5 = 10$

13. $\begin{vmatrix} -4 & 4 & -1 \\ -1 & -2 & 2 \\ 1 & 4 & 3 \end{vmatrix} \overset{R2+R3}{=} \begin{vmatrix} -4 & 4 & -1 \\ 0 & 2 & 5 \\ 1 & 4 & 3 \end{vmatrix} \overset{R1+4R3}{=} \begin{vmatrix} 0 & 20 & 11 \\ 0 & 2 & 5 \\ 1 & 4 & 3 \end{vmatrix} = +1\begin{vmatrix} 20 & 11 \\ 2 & 5 \end{vmatrix} = 100-22 = 78$

14. $\begin{vmatrix} 4 & 2 & -2 \\ 3 & 1 & -5 \\ -5 & -4 & 3 \end{vmatrix} \overset{R1-R2}{=} \begin{vmatrix} 1 & 1 & 3 \\ 3 & 1 & -5 \\ -5 & -4 & 3 \end{vmatrix} \overset{\substack{R2-3R1 \\ R3+5R1}}{=} \begin{vmatrix} 1 & 1 & 3 \\ 0 & -2 & -14 \\ 0 & 1 & 18 \end{vmatrix} = +1\begin{vmatrix} -2 & -14 \\ 1 & 18 \end{vmatrix} = -22$

15. $\begin{vmatrix} -2 & 5 & 4 \\ 5 & 3 & 1 \\ 1 & 4 & 5 \end{vmatrix} \overset{R1+2R3}{=} \begin{vmatrix} 0 & 13 & 14 \\ 5 & 3 & 1 \\ 1 & 4 & 5 \end{vmatrix} \overset{R2-5R3}{=} \begin{vmatrix} 0 & 13 & 14 \\ 0 & -17 & -24 \\ 1 & 4 & 5 \end{vmatrix} = +1\begin{vmatrix} 13 & 14 \\ -17 & -24 \end{vmatrix} = -74$

16. $\begin{vmatrix} 2 & 4 & -2 \\ -5 & -4 & -1 \\ -4 & 2 & 1 \end{vmatrix} \overset{R1-2R3}{=} \begin{vmatrix} 10 & 0 & -4 \\ -5 & -4 & -1 \\ -4 & 2 & 1 \end{vmatrix} \overset{R2+2R3}{=} \begin{vmatrix} 10 & 0 & -4 \\ -13 & 0 & 1 \\ -4 & 2 & 1 \end{vmatrix} = -2\begin{vmatrix} 10 & -4 \\ -13 & 1 \end{vmatrix} = 84$

17. $\begin{vmatrix} 2 & 3 & 3 & 1 \\ 0 & 4 & 3 & -3 \\ 2 & -1 & -1 & -3 \\ 0 & -4 & -3 & 2 \end{vmatrix} \overset{R3-R1}{=} \begin{vmatrix} 2 & 3 & 3 & 1 \\ 0 & 4 & 3 & -3 \\ 0 & -4 & -4 & -4 \\ 0 & -4 & -3 & 2 \end{vmatrix} = 2\begin{vmatrix} 4 & 3 & -3 \\ -4 & -4 & -4 \\ -4 & -3 & 2 \end{vmatrix} \overset{\substack{R2+R1 \\ R3+R1}}{=} 2\begin{vmatrix} 4 & 3 & -3 \\ 0 & -1 & -7 \\ 0 & 0 & -1 \end{vmatrix} = 8$

18. $\begin{vmatrix} 1 & 4 & 4 & 1 \\ 0 & 1 & -2 & 2 \\ 3 & 3 & 1 & 4 \\ 0 & 1 & -3 & -2 \end{vmatrix} \overset{R3-3R1}{=} \begin{vmatrix} 1 & 4 & 4 & 1 \\ 0 & 1 & -2 & 2 \\ 0 & -9 & -11 & 1 \\ 0 & 1 & -3 & -2 \end{vmatrix} = 1\begin{vmatrix} 1 & -2 & 2 \\ -9 & -11 & 1 \\ 1 & -3 & -2 \end{vmatrix} \overset{\substack{R2+9R1 \\ R3-R1}}{=} \begin{vmatrix} 1 & -2 & 2 \\ 0 & -29 & 19 \\ 0 & -1 & -4 \end{vmatrix} = 135$

19. $\begin{vmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & -2 & 0 \\ -2 & 3 & -2 & 3 \\ 0 & -3 & 3 & 3 \end{vmatrix} \overset{R3+2R1}{=} \begin{vmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & -2 & 0 \\ 0 & 3 & -2 & 9 \\ 0 & -3 & 3 & 3 \end{vmatrix} = 1\begin{vmatrix} 1 & -2 & 0 \\ 3 & -2 & 9 \\ -3 & 3 & 3 \end{vmatrix} \overset{C2+2C1}{=} \begin{vmatrix} 1 & 0 & 0 \\ 3 & 4 & 9 \\ -3 & -3 & 3 \end{vmatrix} = 39$

20. $\begin{vmatrix} 1 & 2 & 1 & -1 \\ 2 & 1 & 3 & 3 \\ 0 & 1 & -2 & 3 \\ -1 & 4 & -2 & 4 \end{vmatrix} \overset{\substack{R2-2R1 \\ R4+R1}}{=} \begin{vmatrix} 1 & 2 & 1 & -1 \\ 0 & -3 & 1 & 5 \\ 0 & 1 & -2 & 3 \\ 0 & 6 & -1 & 3 \end{vmatrix} = 1\begin{vmatrix} -3 & 1 & 5 \\ 1 & -2 & 3 \\ 6 & -1 & 3 \end{vmatrix} \overset{\substack{R2+2R1 \\ R3+R1}}{=} \begin{vmatrix} -3 & 1 & 5 \\ -5 & 0 & 13 \\ 3 & 0 & 8 \end{vmatrix} = 79$

21. $\Delta = \begin{vmatrix} 3 & 4 \\ 5 & 7 \end{vmatrix} = 1; \quad x = \frac{1}{\Delta}\begin{vmatrix} 2 & 4 \\ 1 & 7 \end{vmatrix} = 10, \quad y = \frac{1}{\Delta}\begin{vmatrix} 3 & 2 \\ 5 & 1 \end{vmatrix} = -7$

22. $\Delta = \begin{vmatrix} 5 & 8 \\ 8 & 13 \end{vmatrix} = 1; \quad x = \frac{1}{\Delta}\begin{vmatrix} 3 & 8 \\ 5 & 13 \end{vmatrix} = -1, \quad y = \frac{1}{\Delta}\begin{vmatrix} 5 & 3 \\ 8 & 5 \end{vmatrix} = 1$

23. $\Delta = \begin{vmatrix} 17 & 7 \\ 12 & 5 \end{vmatrix} = 1; \quad x = \dfrac{1}{\Delta}\begin{vmatrix} 6 & 7 \\ 4 & 5 \end{vmatrix} = 2, \quad y = \dfrac{1}{\Delta}\begin{vmatrix} 17 & 6 \\ 12 & 4 \end{vmatrix} = -4$

24. $\Delta = \begin{vmatrix} 11 & 15 \\ 8 & 11 \end{vmatrix} = 1; \quad x = \dfrac{1}{\Delta}\begin{vmatrix} 10 & 15 \\ 7 & 11 \end{vmatrix} = 5, \quad y = \dfrac{1}{\Delta}\begin{vmatrix} 11 & 10 \\ 8 & 7 \end{vmatrix} = -3$

25. $\Delta = \begin{vmatrix} 5 & 6 \\ 3 & 4 \end{vmatrix} = 2; \quad x = \dfrac{1}{\Delta}\begin{vmatrix} 12 & 6 \\ 6 & 4 \end{vmatrix} = 6, \quad y = \dfrac{1}{\Delta}\begin{vmatrix} 5 & 12 \\ 3 & 6 \end{vmatrix} = -3$

26. $\Delta = \begin{vmatrix} 6 & 7 \\ 8 & 9 \end{vmatrix} = -2; \quad x = \dfrac{1}{\Delta}\begin{vmatrix} 3 & 7 \\ 4 & 9 \end{vmatrix} = \dfrac{1}{2}, \quad y = \dfrac{1}{\Delta}\begin{vmatrix} 6 & 3 \\ 8 & 4 \end{vmatrix} = 0$

27. $\Delta = \begin{vmatrix} 5 & 2 & -2 \\ 1 & 5 & -3 \\ 5 & -3 & 5 \end{vmatrix} = 96; \qquad x_1 = \dfrac{1}{\Delta}\begin{vmatrix} 1 & 2 & -2 \\ -2 & 5 & -3 \\ 2 & -3 & 5 \end{vmatrix} = \dfrac{1}{3},$

$x_2 = \dfrac{1}{\Delta}\begin{vmatrix} 5 & 1 & -2 \\ 1 & -2 & -3 \\ 5 & 2 & 5 \end{vmatrix} = -\dfrac{2}{3}, \qquad x_3 = \dfrac{1}{\Delta}\begin{vmatrix} 5 & 2 & 1 \\ 1 & 5 & -2 \\ 5 & -3 & 2 \end{vmatrix} = -\dfrac{1}{3}$

28. $\Delta = \begin{vmatrix} 5 & 4 & -2 \\ 2 & 0 & 3 \\ 2 & -1 & 1 \end{vmatrix} = 35; \qquad x_1 = \dfrac{1}{\Delta}\begin{vmatrix} 4 & 4 & -2 \\ 2 & 0 & 3 \\ 1 & -1 & 1 \end{vmatrix} = \dfrac{4}{7},$

$x_2 = \dfrac{1}{\Delta}\begin{vmatrix} 5 & 4 & -2 \\ 2 & 2 & 3 \\ 2 & 1 & 1 \end{vmatrix} = \dfrac{3}{7}, \qquad x_3 = \dfrac{1}{\Delta}\begin{vmatrix} 5 & 4 & 4 \\ 2 & 0 & 2 \\ 2 & -1 & 1 \end{vmatrix} = \dfrac{2}{7}$

29. $\Delta = \begin{vmatrix} 3 & -1 & -5 \\ 4 & -4 & -3 \\ 1 & 0 & -5 \end{vmatrix} = 23; \qquad x_1 = \dfrac{1}{\Delta}\begin{vmatrix} 3 & -1 & -5 \\ -4 & -4 & -3 \\ 2 & 0 & -5 \end{vmatrix} = 2,$

$x_2 = \dfrac{1}{\Delta}\begin{vmatrix} 3 & 3 & -5 \\ 4 & -4 & -3 \\ 1 & 2 & -5 \end{vmatrix} = 3, \qquad x_3 = \dfrac{1}{\Delta}\begin{vmatrix} 3 & -1 & 3 \\ 4 & -4 & -4 \\ 1 & 0 & 2 \end{vmatrix} = 0$

30. $\Delta = \begin{vmatrix} 1 & 4 & 2 \\ 4 & 2 & 1 \\ 2 & -2 & -5 \end{vmatrix} = 56;\qquad x_1 = \dfrac{1}{\Delta}\begin{vmatrix} 3 & 4 & 2 \\ 1 & 2 & 1 \\ -3 & -2 & -5 \end{vmatrix} = -\dfrac{1}{7},$

$x_2 = \dfrac{1}{\Delta}\begin{vmatrix} 1 & 3 & 2 \\ 4 & 1 & 1 \\ 2 & -3 & -5 \end{vmatrix} = \dfrac{9}{14},\qquad x_3 = \dfrac{1}{\Delta}\begin{vmatrix} 1 & 4 & 3 \\ 4 & 2 & 1 \\ 2 & -2 & -3 \end{vmatrix} = \dfrac{2}{7}$

31. $\Delta = \begin{vmatrix} 2 & 0 & -5 \\ 4 & -5 & 3 \\ -2 & 1 & 1 \end{vmatrix} = 14;\qquad x_1 = \dfrac{1}{\Delta}\begin{vmatrix} -3 & 0 & -5 \\ 3 & -5 & 3 \\ 1 & 1 & 1 \end{vmatrix} = -\dfrac{8}{7},$

$x_2 = \dfrac{1}{\Delta}\begin{vmatrix} 2 & -3 & -5 \\ 4 & 3 & 3 \\ -2 & 1 & 1 \end{vmatrix} = -\dfrac{10}{7},\qquad x_3 = \dfrac{1}{\Delta}\begin{vmatrix} 2 & 0 & -3 \\ 4 & -5 & 3 \\ -2 & 1 & 1 \end{vmatrix} = \dfrac{1}{7}$

32. $\Delta = \begin{vmatrix} 3 & 4 & -3 \\ 3 & -2 & 4 \\ 3 & 2 & -1 \end{vmatrix} = 6;\qquad x_1 = \dfrac{1}{\Delta}\begin{vmatrix} 5 & 4 & -3 \\ 7 & -2 & 4 \\ 3 & 2 & -1 \end{vmatrix} = -\dfrac{7}{3},$

$x_2 = \dfrac{1}{\Delta}\begin{vmatrix} 3 & 5 & -3 \\ 3 & 7 & 4 \\ 3 & 3 & -1 \end{vmatrix} = 9,\qquad x_3 = \dfrac{1}{\Delta}\begin{vmatrix} 3 & 4 & 5 \\ 3 & -2 & 7 \\ 3 & 2 & 3 \end{vmatrix} = 8$

33. $\det \mathbf{A} = -4,\qquad \mathbf{A}^{-1} = \dfrac{1}{4}\begin{bmatrix} 4 & 4 & 4 \\ 16 & 15 & 13 \\ 28 & 25 & 23 \end{bmatrix}$

34. $\det \mathbf{A} = 35,\qquad \mathbf{A}^{-1} = \dfrac{1}{35}\begin{bmatrix} -2 & -3 & 12 \\ 9 & -4 & -19 \\ 13 & 2 & -8 \end{bmatrix}$

35. $\det \mathbf{A} = 35,\qquad \mathbf{A}^{-1} = \dfrac{1}{35}\begin{bmatrix} -15 & 25 & -26 \\ 10 & -5 & 8 \\ 15 & -25 & 19 \end{bmatrix}$

36. $\det \mathbf{A} = 23, \qquad \mathbf{A}^{-1} = \dfrac{1}{23}\begin{bmatrix} 5 & 20 & -17 \\ 10 & 17 & -11 \\ 1 & 4 & -8 \end{bmatrix}$

37. $\det \mathbf{A} = 29, \qquad \mathbf{A}^{-1} = \dfrac{1}{29}\begin{bmatrix} 11 & -14 & -15 \\ -17 & 19 & 10 \\ 18 & -15 & -14 \end{bmatrix}$

38. $\det \mathbf{A} = 6, \qquad \mathbf{A}^{-1} = \dfrac{1}{6}\begin{bmatrix} -6 & 10 & 2 \\ 15 & -21 & -6 \\ 12 & -18 & -6 \end{bmatrix}$

39. $\det \mathbf{A} = 37, \qquad \mathbf{A}^{-1} = \dfrac{1}{37}\begin{bmatrix} -21 & -1 & -13 \\ 4 & 9 & 6 \\ -6 & 5 & -9 \end{bmatrix}$

40. $\det \mathbf{A} = 107, \qquad \mathbf{A}^{-1} = \dfrac{1}{107}\begin{bmatrix} 9 & 12 & -13 \\ 11 & -21 & -4 \\ -15 & -20 & -14 \end{bmatrix}$

41. If $\mathbf{A} = \begin{bmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 \end{bmatrix}$ in terms of the two row vectors of $\mathbf{A}$ and the two column vectors of $\mathbf{B}$, then $\mathbf{AB} = \begin{bmatrix} \mathbf{a}_1\mathbf{b}_1 & \mathbf{a}_1\mathbf{b}_2 \\ \mathbf{a}_2\mathbf{b}_1 & \mathbf{a}_2\mathbf{b}_2 \end{bmatrix}$, so

$$(\mathbf{AB})^T = \begin{bmatrix} \mathbf{a}_1\mathbf{b}_1 & \mathbf{a}_2\mathbf{b}_1 \\ \mathbf{a}_1\mathbf{b}_2 & \mathbf{a}_2\mathbf{b}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{b}_1^T \\ \mathbf{b}_2^T \end{bmatrix}\begin{bmatrix} \mathbf{a}_1^T & \mathbf{a}_1^T \end{bmatrix} = \mathbf{B}^T\mathbf{A}^T,$$

because the rows of $\mathbf{A}$ are the columns of $\mathbf{A}^T$ and the columns of $\mathbf{B}$ are the rows of $\mathbf{B}^T$.

42. $\det \mathbf{AB} = \det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} \mathbf{x} \\ \mathbf{y} \end{bmatrix}\right) = \begin{vmatrix} a\mathbf{x}+b\mathbf{y} \\ c\mathbf{x}+d\mathbf{y} \end{vmatrix} = \begin{vmatrix} a\mathbf{x} \\ c\mathbf{x}+d\mathbf{y} \end{vmatrix} + \begin{vmatrix} b\mathbf{y} \\ c\mathbf{x}+d\mathbf{y} \end{vmatrix}$

$$= ac\begin{vmatrix} \mathbf{x} \\ \mathbf{x} \end{vmatrix} + ad\begin{vmatrix} \mathbf{x} \\ \mathbf{y} \end{vmatrix} + bc\begin{vmatrix} \mathbf{y} \\ \mathbf{x} \end{vmatrix} + bd\begin{vmatrix} \mathbf{y} \\ \mathbf{y} \end{vmatrix} = ad\begin{vmatrix} \mathbf{x} \\ \mathbf{y} \end{vmatrix} + bc\begin{vmatrix} \mathbf{y} \\ \mathbf{x} \end{vmatrix}$$

$$= ad\begin{vmatrix} \mathbf{x} \\ \mathbf{y} \end{vmatrix} - bc\begin{vmatrix} \mathbf{x} \\ \mathbf{y} \end{vmatrix} = (ad-bc)\begin{vmatrix} \mathbf{x} \\ \mathbf{y} \end{vmatrix} = (\det \mathbf{A})(\det \mathbf{B})$$

43. We expand the left-hand determinant along its first column:

$$\begin{aligned}
&\begin{vmatrix} ka_{11} & a_{12} & a_{13} \\ ka_{21} & a_{22} & a_{23} \\ ka_{31} & a_{32} & a_{33} \end{vmatrix} \\
&\quad = ka_{11}\left(a_{12}a_{23} - a_{22}a_{13}\right) - ka_{21}\left(a_{12}a_{33} - a_{32}a_{13}\right) + ka_{31}\left(a_{12}a_{23} - a_{22}a_{13}\right) \\
&\quad = k\left[a_{11}\left(a_{12}a_{23} - a_{22}a_{13}\right) - a_{21}\left(a_{12}a_{33} - a_{32}a_{13}\right) + a_{31}\left(a_{12}a_{23} - a_{22}a_{13}\right)\right] \\
&\quad = k\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}
\end{aligned}$$

44. We expand the left-hand determinant along its third row:

$$\begin{aligned}
&\begin{vmatrix} a_{21} & a_{22} & a_{23} \\ a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \\
&\quad = a_{31}\left(a_{22}a_{13} - a_{23}a_{12}\right) - a_{32}\left(a_{21}a_{13} - a_{23}a_{11}\right) + a_{33}\left(a_{21}a_{13} - a_{22}a_{11}\right) \\
&\quad = -\left[a_{31}\left(a_{23}a_{12} - a_{22}a_{13}\right) - a_{32}\left(a_{23}a_{11} - a_{21}a_{13}\right) + a_{33}\left(a_{22}a_{11} - a_{21}a_{13}\right)\right] \\
&\quad = k\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}
\end{aligned}$$

45. We expand the left-hand determinant along its third column:

$$\begin{aligned}
&\begin{vmatrix} a_1 & b_1 & c_1 + d_1 \\ a_2 & b_2 & c_2 + d_2 \\ a_3 & b_3 & c_3 + d_3 \end{vmatrix} \\
&\quad = \left(c_1 + d_1\right)\left(a_2b_3 - a_3b_2\right) - \left(c_2 + d_2\right)\left(a_1b_3 - a_3b_1\right) + \left(c_3 + d_3\right)\left(a_1b_2 - a_2b_1\right) \\
&\quad = c_1\left(a_2b_3 - a_3b_2\right) - c_2\left(a_1b_3 - a_3b_1\right) + c_3\left(a_1b_2 - a_2b_1\right) \\
&\qquad + d_1\left(a_2b_3 - a_3b_2\right) - d_2\left(a_1b_3 - a_3b_1\right) + d_3\left(a_1b_2 - a_2b_1\right) \\
&\quad = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}
\end{aligned}$$

46. We expand the left-hand determinant along its first column:

$$\begin{vmatrix} a_1+kb_1 & b_1 & c_1 \\ a_2+kb_2 & b_2 & c_2 \\ a_3+kb_3 & b_3 & c_3 \end{vmatrix}$$

$$= (a_1+kb_1)(b_2c_3-b_3c_2)-(a_2+kb_2)(b_1c_3-c_3b_3)+(a_3+kb_3)(b_1c_2-b_2c_1)$$

$$= \left[a_1(b_2c_3-b_3c_2)-a_2(b_1c_3-c_3b_3)+a_3(b_1c_2-b_2c_1)\right]$$

$$+k\left[b_1(b_2c_3-b_3c_2)-b_2(b_1c_3-c_3b_3)+b_3(b_1c_2-b_2c_1)\right]$$

$$= \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + k\begin{vmatrix} b_1 & b_1 & d_1 \\ b_2 & b_2 & d_2 \\ b_3 & b_3 & d_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

47. We illustrate these properties with 2×2 matrices $\mathbf{A}=\left[a_{ij}\right]$ and $\mathbf{B}=\left[b_{ij}\right]$.

(a) $$\left(\mathbf{A}^T\right)^T = \begin{bmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{bmatrix}^T = \begin{bmatrix} a_{11} & a_{12} \\ a_{22} & a_{22} \end{bmatrix} = \mathbf{A}$$

(b) $$(c\mathbf{A})^T = \begin{bmatrix} ca_{11} & ca_{12} \\ ca_{21} & ca_{22} \end{bmatrix}^T = \begin{bmatrix} ca_{11} & ca_{21} \\ ca_{12} & ca_{22} \end{bmatrix} = c\begin{bmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{bmatrix} = c\mathbf{A}^T$$

(c) $$(\mathbf{A}+\mathbf{B})^T = \begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} \\ a_{21}+b_{21} & a_{22}+b_{22} \end{bmatrix}^T = \begin{bmatrix} a_{11}+b_{11} & a_{21}+b_{21} \\ a_{12}+b_{12} & a_{22}+b_{22} \end{bmatrix}$$

$$= \begin{bmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{bmatrix} + \begin{bmatrix} b_{11} & b_{21} \\ b_{12} & b_{22} \end{bmatrix} = \mathbf{A}^T+\mathbf{B}^T$$

48. The *ij*th element of $(\mathbf{AB})^T$ is the *ji*th element of $\mathbf{AB}$, and hence is the product of the *j*th row of $\mathbf{A}$ and the *i*th column of $\mathbf{B}$. The *ij*th element of $\mathbf{B}^T\mathbf{A}^T$ is the product of the *i*th row of $\mathbf{B}^T$ and the jth column of $\mathbf{A}^T$. Because transposition of a matrix changes the *i*th row to the *i*th column and vice versa, it follows that the *ij*th element of $\mathbf{B}^T\mathbf{A}^T$ is the product of the *j*th row of $\mathbf{A}$ and the *i*th column of $\mathbf{B}$. Thus the matrices $(\mathbf{AB})^T$ and $\mathbf{B}^T\mathbf{A}^T$ have the same *ij*th elements, and hence are equal matrices.

49. If we write $\mathbf{A} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ and $\mathbf{A}^T = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$, then expansion of $|\mathbf{A}|$ along its first row and of $\left|\mathbf{A}^T\right|$ along its first column both give the result $a_1(b_2c_3-b_3c_2)+b_1(a_2c_3-a_3c_2)+c_1(a_2b_3-a_3b_2)$.

50. If $\mathbf{A}^2 = \mathbf{A}$ then $|\mathbf{A}|^2 = |\mathbf{A}|$, so $|\mathbf{A}|^2 - |\mathbf{A}| = |\mathbf{A}|(|\mathbf{A}|-1) = 0$, and hence it follows immediately that either $|\mathbf{A}| = 0$ or $|\mathbf{A}| = 1$.

51. If $\mathbf{A}^n = \mathbf{0}$ then $|\mathbf{A}|^n = \mathbf{0}$, so it follows immediately that $|\mathbf{A}| = \mathbf{0}$.

52. If $\mathbf{A}^T = \mathbf{A}^{-1}$ then $|\mathbf{A}| = |\mathbf{A}^T| = |\mathbf{A}^{-1}| = |\mathbf{A}|^{-1}$. Hence $|\mathbf{A}|^2 = 1$, so it follows that $|\mathbf{A}| = \pm 1$.

53. If $\mathbf{A} = \mathbf{P}^{-1}\mathbf{B}\mathbf{P}$ then $|\mathbf{A}| = |\mathbf{P}^{-1}\mathbf{B}\mathbf{P}| = |\mathbf{P}^{-1}||\mathbf{B}||\mathbf{P}| = |\mathbf{P}|^{-1}|\mathbf{B}||\mathbf{P}| = |\mathbf{B}|$.

54. If $\mathbf{A}$ and $\mathbf{B}$ are invertible, then $|\mathbf{A}| \neq 0$ and $|\mathbf{B}| \neq 0$. Hence $|\mathbf{AB}| = |\mathbf{A}||\mathbf{B}| \neq 0$, so it follows that $\mathbf{AB}$ is invertible. Conversely, $\mathbf{AB}$ invertible implies that $|\mathbf{AB}| = |\mathbf{A}||\mathbf{B}| \neq 0$, so it follows that both $|\mathbf{A}| \neq 0$ and $|\mathbf{B}| \neq 0$, and therefore that both $\mathbf{A}$ and $\mathbf{B}$ are invertible.

55. If either $\mathbf{AB} = \mathbf{I}$ or $\mathbf{BA} = \mathbf{I}$ is given, then it follows from Problem 54 that $\mathbf{A}$ and $\mathbf{B}$ are both invertible because their product (one way or the other) is invertible. Hence $\mathbf{A}^{-1}$ exists. So if (for instance) it is $\mathbf{AB} = \mathbf{I}$ that is given, then multiplication by $\mathbf{A}^{-1}$ on the right yields $\mathbf{B} = \mathbf{A}^{-1}$.

56. The matrix $\mathbf{A}^{-1}$ in part (a) and the solution vector $\mathbf{x}$ in part (b) have only integer entries because the only division involved in their calculation — using the adjoint formula for the inverse matrix or Cramer's rule for the solution vector — is by the determinant $|\mathbf{A}| = 1$.

57. If $\mathbf{A} = \begin{bmatrix} a & d & f \\ 0 & b & e \\ 0 & 0 & b \end{bmatrix}$ then $\mathbf{A}^{-1} = \dfrac{1}{abc}\begin{bmatrix} bc & -cd & de-bf \\ 0 & ac & -ae \\ 0 & 0 & ab \end{bmatrix}$.

58. The coefficient determinant of the linear system

$$\begin{aligned} c\cos B + b\cos C &= a \\ c\cos A \qquad\quad + a\cos C &= b \\ b\cos A + a\cos B \qquad\quad &= c \end{aligned}$$

in the unknowns $\{\cos A, \cos B, \cos C\}$ is

$$\begin{vmatrix} 0 & c & b \\ c & 0 & a \\ b & a & 0 \end{vmatrix} = 2abc.$$

Hence Cramer's rule gives

$$\cos A = \frac{1}{2abc}\begin{vmatrix} a & c & b \\ b & 0 & a \\ c & a & 0 \end{vmatrix} = \frac{ab^2 - a^3 + ac^2}{2abc} = \frac{b^2 - a^2 + c^2}{2bc},$$

whence $a^2 = b^2 + c^2 - 2bc\cos A.$

59. These are almost immediate computations.

60. **(a)** In the 4×4 case, expansion along the first row gives

$$\begin{vmatrix} 2 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 2 \end{vmatrix} = 2\begin{vmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{vmatrix} - \begin{vmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 2 \end{vmatrix} = 2\begin{vmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{vmatrix} - \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix},$$

so $B_4 = 2B_3 - B_2 = 2(4) - (3) = 5.$ The general recursion formula $B_n = 2B_{n-1} - B_{n-2}$ results in the same way upon expansion along the first row.

(b) If we assume inductively that

$$B_{n-1} = (n-1) + 1 = n \quad\text{and}\quad B_{n-2} = (n-2) + 1 = n-1,$$

then the recursion formula of part (a) yields

$$B_n = 2B_{n-1} - B_{n-2} = 2(n) - (n-1) = n+1.$$

61. Subtraction of the first row from both the second and the third row gives

$$\begin{aligned}\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} &= \begin{vmatrix} 1 & a & a^2 \\ 0 & b-a & b^2-a^2 \\ 0 & c-a & c^2-a^2 \end{vmatrix} = (b-a)(c^2-a^2) - (c-a)(b^2-a^2) \\ &= (b-a)(c-a)(c+a) - (c-a)(b-a)(b+a) \\ &= (b-a)(c-a)\left[(c+a)-(b+a)\right] = (b-a)(c-a)(c-b).\end{aligned}$$

62. Expansion of the 4×4 determinant defining $P(y)$ along its 4th row yields

$$P(y) = y^3 \begin{vmatrix} 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \\ 1 & x_3 & x_3^2 \end{vmatrix} + \cdots = y^3 V(x_1, x_2, x_3) + \text{ lower-degree terms in } y.$$

Because it is clear from the determinant definition of $P(y)$ that $P(x_1) = P(x_2) = P(x_3) = 0,$ the three roots of the cubic polynomial $P(y)$ are $x_1, x_2, x_3.$ The factor theorem therefore says that $P(y) = k(y - x_1)(y - x_2)(y - x_3)$ for some constant $k,$ and the calculation above implies that

$$k = V(x_1, x_2, x_3) = (x_3 - x_1)(x_3 - x_2)(x_2 - x_1).$$

Finally we see that

$$\begin{aligned} V(x_1, x_2, x_3, x_4) &= P(x_4) = V(x_1, x_2, x_3) \cdot (x_4 - x_1)(x_4 - x_2)(x_4 - x_1) \\ &= (x_4 - x_1)(x_4 - x_2)(x_4 - x_1)(x_3 - x_1)(x_3 - x_2)(x_2 - x_1), \end{aligned}$$

which is the desired formula for $V(x_1, x_2, x_3, x_4)$.

63. The same argument as in Problem 62 yields

$$P(y) = V(x_1, x_2, \cdots, x_{n-1}) \cdot (y - x_1)(y - x_2) \cdot \cdots \cdot (y - x_{n-1}).$$

Therefore

$$\begin{aligned} V(x_1, x_2, \cdots, x_n) &= (x_n - x_1)(x_n - x_2) \cdot \cdots \cdot (x_n - x_{n-1}) V(x_1, x_2, \cdots, x_{n-1}) \\ &= (x_n - x_1)(x_n - x_2) \cdot \cdots \cdot (x_n - x_{n-1}) \prod_{i>j}^{n-1} (x_i - x_j) \\ &= \prod_{i>j}^{n} (x_i - x_j). \end{aligned}$$

64. **(a)** $V(1, 2, 3, 4) = (4 - 1)(4 - 2)(4 - 3)(3 - 1)(3 - 2)(2 - 1) = 12$

(b) $V(-1, 2, -2, 3) =$

$$\big[3 - (-1)\big][3 - 2]\big[3 - (-2)\big]\big[(-2) - (-1)\big]\big[(-2) - 2\big]\big[2 - (-1)\big] = 240$$

SECTION 3.7

LINEAR EQUATIONS AND CURVE FITTING

In Problems 1–10 we first set up the linear system in the coefficients $a, b, \ldots$ that we get by substituting each given point (x_i, y_i) into the desired interpolating polynomial equation $y = a + bx + \cdots$. Then we give the polynomial that results from solution of this linear system.

1. $y(x) = a + bx$

$$\begin{bmatrix} 1 & 1 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1 \\ 7 \end{bmatrix} \quad \Rightarrow \quad a = -2,\ b = 3 \quad \text{so} \quad y(x) = -2 + 3x$$

2. $y(x) = a + bx$

$$\begin{bmatrix} 1 & -1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 11 \\ -10 \end{bmatrix} \quad \Rightarrow \quad a = 4,\ b = -7 \quad \text{so} \quad y(x) = 4 - 7x$$

3. $y(x) = a + bx + cx^2$

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ -5 \end{bmatrix} \quad \Rightarrow \quad a = 3,\ b = 0,\ c = -2 \quad \text{so} \quad y(x) = 3 - 2x^2$$

4. $y(x) = a + bx + cx^2$

$$\begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \\ 16 \end{bmatrix} \quad \Rightarrow \quad a = 0,\ b = 2,\ c = 3 \quad \text{so} \quad y(x) = 2x + 3x^2$$

5. $y(x) = a + bx + cx^2$

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ 5 \end{bmatrix} \quad \Rightarrow \quad a = 5,\ b = -3,\ c = 1 \quad \text{so} \quad y(x) = 5 - 3x + x^2$$

6. $y(x) = a + bx + cx^2$

$$\begin{bmatrix} 1 & -1 & 1 \\ 1 & 3 & 9 \\ 1 & 5 & 25 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} -1 \\ -13 \\ 5 \end{bmatrix}$$

$$\Rightarrow \quad a=-10,\ b=-7,\ c=2 \quad \text{so} \quad y(x) = -10-7x+2x^2$$

7. $y(x) = a+bx+cx^2+dx^3$

$$\begin{bmatrix} 1 & -1 & 1 & -1 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ -4 \end{bmatrix}$$

$$\Rightarrow \quad a=0,\ b=\frac{4}{3},\ c=1,\ d=-\frac{4}{3} \quad \text{so} \quad y(x) = \frac{1}{3}\left(4x+3x^2-4x^3\right)$$

8. $y(x) = a+bx+cx^2+dx^3$

$$\begin{bmatrix} 1 & -1 & 1 & -1 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \\ 7 \\ 3 \end{bmatrix}$$

$$\Rightarrow \quad a=5,\ b=3,\ c=0,\ d=-1 \quad \text{so} \quad y(x) = 5+3x-x^3$$

9. $y(x) = a+bx+cx^2+dx^3$

$$\begin{bmatrix} 1 & -2 & 4 & -8 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \\ 10 \\ 26 \end{bmatrix}$$

$$\Rightarrow \quad a=4,\ b=3,\ c=2,\ d=1 \quad \text{so} \quad y(x) = 4+3x+2x^2+x^3$$

10. $y(x) = a+bx+cx^2+dx^3$

$$\begin{bmatrix} 1 & -1 & 1 & -1 \\ 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \\ 1 & 3 & 9 & 27 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 17 \\ -5 \\ 3 \\ -2 \end{bmatrix}$$

$$\Rightarrow \quad a=17,\ b=-5,\ c=3,\ d=-2 \quad \text{so} \quad y(x) = 17-5x+3x^2-2x^3$$

In Problems 11–14 we first set up the linear system in the coefficients A, B, C that we get by substituting each given point (x_i, y_i) into the circle equation $Ax+By+C=-x^2-y^2$ (see Eq. (9) in the text). Then we give the circle that results from solution of this linear system.

11. $Ax + By + C = -x^2 - y^2$

$$\begin{bmatrix} -1 & -1 & 1 \\ 6 & 6 & 1 \\ 7 & 5 & 1 \end{bmatrix} \begin{bmatrix} A \\ B \\ C \end{bmatrix} = \begin{bmatrix} -2 \\ -72 \\ -74 \end{bmatrix} \quad \Rightarrow \quad A = -6, \;\; B = -4, \;\; C = -12$$

$x^2 + y^2 - 6x - 4y - 12 \; = \; 0$

$(x-3)^2 + (y-2)^2 \; = \; 25$ center $(3, 2)$ and radius 5

12. $Ax + By + C = -x^2 - y^2$

$$\begin{bmatrix} 3 & -4 & 1 \\ 5 & 10 & 1 \\ -9 & 12 & 1 \end{bmatrix} \begin{bmatrix} A \\ B \\ C \end{bmatrix} = \begin{bmatrix} -25 \\ -125 \\ -225 \end{bmatrix} \quad \Rightarrow \quad A = 6, \;\; B = -8, \;\; C = -75$$

$x^2 + y^2 + 6x - 8y - 75 \; = \; 0$

$(x+3)^2 + (y-4)^2 \; = \; 100$ center $(-3, 4)$ and radius 10

13. $Ax + By + C = -x^2 - y^2$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & -5 & 1 \\ -5 & -4 & 1 \end{bmatrix} \begin{bmatrix} A \\ B \\ C \end{bmatrix} = \begin{bmatrix} -1 \\ -25 \\ -41 \end{bmatrix} \quad \Rightarrow \quad A = 4, \;\; B = 4, \;\; C = -5$$

$x^2 + y^2 + 4x + 4y - 5 \; = \; 0$

$(x+2)^2 + (y+2)^2 \; = \; 13$ center $(-3, -2)$ and radius $\sqrt{13}$

14. $Ax + By + C = -x^2 - y^2$

$$\begin{bmatrix} 0 & 0 & 1 \\ 10 & 0 & 1 \\ -7 & 7 & 1 \end{bmatrix} \begin{bmatrix} A \\ B \\ C \end{bmatrix} = \begin{bmatrix} 0 \\ -100 \\ -98 \end{bmatrix} \quad \Rightarrow \quad A = -10, \;\; B = -24, \;\; C = 0$$

$x^2 + y^2 - 10x - 24y \; = \; 0$

$(x-5)^2 + (y-12)^2 \; = \; 169$ center $(5, 12)$ and radius 13

In Problems 15–18 we first set up the linear system in the coefficients A, B, C that we get by substituting each given point (x_i, y_i) into the central conic equation $Ax^2 + Bxy + Cy^2 = 1$ (see Eq. (10) in the text). Then we give the equation that results from solution of this linear system.

15. $Ax^2 + Bxy + Cy^2 = 1$

$$\begin{bmatrix} 0 & 0 & 25 \\ 25 & 0 & 0 \\ 25 & 25 & 25 \end{bmatrix} \begin{bmatrix} A \\ B \\ C \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad \Rightarrow \quad A = \frac{1}{25},\ B = -\frac{1}{25},\ C = \frac{1}{25}$$

$x^2 - xy + y^2 = 25$

16. $Ax^2 + Bxy + Cy^2 = 1$

$$\begin{bmatrix} 0 & 0 & 25 \\ 25 & 0 & 0 \\ 100 & 100 & 100 \end{bmatrix} \begin{bmatrix} A \\ B \\ C \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad \Rightarrow \quad A = \frac{1}{25},\ B = -\frac{7}{100},\ C = \frac{1}{25}$$

$4x^2 - 7xy + 4y^2 = 100$

17. $Ax^2 + Bxy + Cy^2 = 1$

$$\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 100 & 100 & 100 \end{bmatrix} \begin{bmatrix} A \\ B \\ C \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad \Rightarrow \quad A = 1,\ B = -\frac{199}{100},\ C = 1$$

$100x^2 - 199xy + 100y^2 = 100$

18. $Ax^2 + Bxy + Cy^2 = 1$

$$\begin{bmatrix} 0 & 0 & 16 \\ 9 & 0 & 0 \\ 25 & 25 & 25 \end{bmatrix} \begin{bmatrix} A \\ B \\ C \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad \Rightarrow \quad A = \frac{1}{9},\ B = -\frac{481}{3600},\ C = \frac{1}{16}$$

$400x^2 - 481xy + 225y^2 = 3600$

19. We substitute each of the two given points into the equation $y = A + \frac{B}{x}$.

$$\begin{bmatrix} 1 & 1 \\ 1 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 5 \\ 4 \end{bmatrix} \quad \Rightarrow \quad A = 3,\ B = 2 \quad \text{so} \quad y = 3 + \frac{2}{x}$$

20. We substitute each of the three given points into the equation $y = Ax + \frac{B}{x} + \frac{C}{x^2}$.

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & \frac{1}{2} & \frac{1}{4} \\ 4 & \frac{1}{4} & \frac{1}{16} \end{bmatrix} \begin{bmatrix} A \\ B \\ C \end{bmatrix} = \begin{bmatrix} 2 \\ 20 \\ 41 \end{bmatrix} \quad \Rightarrow \quad A = 10,\ B = 8,\ C = -16 \quad \text{so} \quad y = 10x + \frac{8}{x} - \frac{16}{x^2}$$

In Problems 21 and 22 we fit the sphere equation $(x-h)^2 + (y-k)^2 + (z-l)^2 = r^2$ in the expanded form $Ax + By + Cz + D = -x^2 - y^2 - z^2$ that is analogous to Eq. (9) in the text (for a circle).

21. $Ax + By + Cz + D = -x^2 - y^2 - z^2$

$$\begin{bmatrix} 4 & 6 & 15 & 1 \\ 13 & 5 & 7 & 1 \\ 5 & 14 & 6 & 1 \\ 5 & 5 & -9 & 1 \end{bmatrix} \begin{bmatrix} A \\ B \\ C \\ D \end{bmatrix} = \begin{bmatrix} -277 \\ -243 \\ -257 \\ -131 \end{bmatrix} \quad \Rightarrow \quad A = -2,\ B = -4,\ C = -6,\ D = -155$$

$$x^2 + y^2 + z^2 - 2x - 4y - 6z - 155 = 0$$

$(x-1)^2 + (y-2)^2 + (z-3)^2 = 169$ center $(1, 2, 3)$ and radius 13

22. $Ax + By + Cz + D = -x^2 - y^2 - z^2$

$$\begin{bmatrix} 11 & 17 & 17 & 1 \\ 29 & 1 & 15 & 1 \\ 13 & -1 & 33 & 1 \\ -19 & -13 & 1 & 1 \end{bmatrix} \begin{bmatrix} A \\ B \\ C \\ D \end{bmatrix} = \begin{bmatrix} -699 \\ -1067 \\ -1259 \\ -531 \end{bmatrix} \quad \Rightarrow \quad A = -10,\ B = 14,\ C = -18,\ D = -521$$

$$x^2 + y^2 + z^2 - 10x + 14y - 18z - 521 = 0$$

$(x-5)^2 + (y+7)^2 + (z-9)^2 = 676$ center $(5, -7, 9)$ and radius 26

In Problems 23–26 we first take $t = 0$ in 1970 to fit a quadratic polynomial $P(t) = a + bt + ct^2$. Then we write the quadratic polynomial $Q(T) = P(T - 1970)$ that expresses the predicted population in terms of the actual calendar year T.

23. $P(t) = a + bt + ct^2$

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 10 & 100 \\ 1 & 20 & 400 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 49.061 \\ 49.137 \\ 50.809 \end{bmatrix}$$

$$P(t) = 49.061 - 0.0722t + 0.00798t^2$$
$$Q(T) = 31160.9 - 31.5134T + 0.00798T^2$$

24. $P(t) = a + bt + ct^2$

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 10 & 100 \\ 1 & 20 & 400 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 56.590 \\ 58.867 \\ 59.669 \end{bmatrix}$$

$$P(t) = 56.590 + 0.30145t - 0.007375t^2$$
$$Q(T) = -29158.9 + 29.3589T - 0.007375T^2$$

25. $P(t) = a + bt + ct^2$

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 10 & 100 \\ 1 & 20 & 400 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 62.813 \\ 75.367 \\ 85.446 \end{bmatrix}$$

$$P(t) = 62.813 + 1.37915t - 0.012375t^2$$
$$Q(T) = -50680.3 + 50.1367T - 0.012375T^2$$

26. $P(t) = a + bt + ct^2$

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 10 & 100 \\ 1 & 20 & 400 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 34.838 \\ 43.171 \\ 52.786 \end{bmatrix}$$

$$P(t) = 34.838 + 0.7692t + 0.00641t^2$$
$$Q(T) = 23396.1 - 24.4862T + 0.00641T^2$$

In Problems 27–30 we first take $t = 0$ in 1960 to fit a cubic polynomial $P(t) = a + bt + ct^2 + dt^3$. Then we write the cubic polynomial $Q(T) = P(T - 1960)$ that expresses the predicted population in terms of the actual calendar year T.

27. $P(t) = a + bt + ct^2 + dt^3$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 10 & 100 & 1000 \\ 1 & 20 & 400 & 8000 \\ 1 & 30 & 900 & 27000 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 44.678 \\ 49.061 \\ 49.137 \\ 50.809 \end{bmatrix}$$

$P(t) = 44.678 + 0.850417t - 0.05105t^2 + 0.000983833t^3$

$Q(T) = -7.60554 \times 10^6 + 11539.4T - 5.83599T^2 + 0.000983833T^3$

28. $P(t) = a + bt + ct^2 + dt^3$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 10 & 100 & 1000 \\ 1 & 20 & 400 & 8000 \\ 1 & 30 & 900 & 27000 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 51.619 \\ 56.590 \\ 58.867 \\ 59.669 \end{bmatrix}$$

$P(t) = 51.619 + 0.672433t - 0.019565t^2 + 0.000203167t^3$

$Q(T) = -1.60618 \times 10^6 + 2418.82T - 1.21419T^2 + 0.000203167T^3$

29. $P(t) = a + bt + ct^2 + dt^3$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 10 & 100 & 1000 \\ 1 & 20 & 400 & 8000 \\ 1 & 30 & 900 & 27000 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 54.973 \\ 62.813 \\ 75.367 \\ 85.446 \end{bmatrix}$$

$P(t) = 54.973 + 0.308667t + 0.059515t^2 - 0.00119817t^3$

$Q(T) = 9.24972 \times 10^6 - 14041.6T + 7.10474T^2 - 0.00119817T^3$

30. $P(t) = a + bt + ct^2 + dt^3$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 10 & 100 & 1000 \\ 1 & 20 & 400 & 8000 \\ 1 & 30 & 900 & 27000 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 28.053 \\ 34.838 \\ 43.171 \\ 52.786 \end{bmatrix}$$

$P(t) = 28.053 + 0.592233t + 0.00907t^2 - 0.0000443333t^3$

$Q(T) = 367520 - 545.895T + 0.26975T^2 - 0.0000443333T^3$

In Problems 31–34 we take $t = 0$ in 1950 to fit a quartic polynomial $P(t) = a + bt + ct^2 + dt^3 + et^4$. Then we write the quartic polynomial $Q(T) = P(T - 1950)$ that expresses the predicted population in terms of the actual calendar year T.

31. $P(t) = a + bt + ct^2 + dt^3 + et^4$.

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 10 & 100 & 1000 & 10000 \\ 1 & 20 & 400 & 8000 & 160000 \\ 1 & 30 & 900 & 27000 & 810000 \\ 1 & 40 & 1600 & 64000 & 2560000 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \\ e \end{bmatrix} = \begin{bmatrix} 39.478 \\ 44.678 \\ 49.061 \\ 49.137 \\ 50.809 \end{bmatrix}$$

$$P(t) = 39.478 + 0.209692t + 0.0564163t^2 - 0.00292992t^3 + 0.0000391375t^4$$
$$Q(T) = 5.87828\times10^8 - 1.19444\times10^6 T + 910.118T^2 - 0.308202T^3 + 0.0000391375T^4$$

32. $P(t) = a + bt + ct^2 + dt^3 + et^4$.

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 10 & 100 & 1000 & 10000 \\ 1 & 20 & 400 & 8000 & 160000 \\ 1 & 30 & 900 & 27000 & 810000 \\ 1 & 40 & 1600 & 64000 & 2560000 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \\ e \end{bmatrix} = \begin{bmatrix} 44.461 \\ 51.619 \\ 56.590 \\ 58.867 \\ 59.669 \end{bmatrix}$$

$$P(t) = 44.461 + 0.7651t - 0.000489167t^2 - 0.000516t^3 + 7.19167\times10^{-6} t^4$$
$$Q(T) = 1.07807\times10^8 - 219185T + 167.096T^2 - 0.056611T^3 + 7.19167\times10^{-6} T^4$$

33. $P(t) = a + bt + ct^2 + dt^3 + et^4$.

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 10 & 100 & 1000 & 10000 \\ 1 & 20 & 400 & 8000 & 160000 \\ 1 & 30 & 900 & 27000 & 810000 \\ 1 & 40 & 1600 & 64000 & 2560000 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \\ e \end{bmatrix} = \begin{bmatrix} 47.197 \\ 54.973 \\ 62.813 \\ 75.367 \\ 85.446 \end{bmatrix}$$

$$P(t) = 47.197 + 1.22537t - 0.0771921t^2 + 0.00373475t^3 - 0.0000493292t^4$$
$$Q(T) = -7.41239\times10^8 + 1.50598\times10^6 T - 1147.37T^2 + 0.388502T^3 - 0.0000493292T^4$$

34. $P(t) = a + bt + ct^2 + dt^3 + et^4.$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 10 & 100 & 1000 & 10000 \\ 1 & 20 & 400 & 8000 & 160000 \\ 1 & 30 & 900 & 27000 & 810000 \\ 1 & 40 & 1600 & 64000 & 2560000 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \\ e \end{bmatrix} = \begin{bmatrix} 20.190 \\ 28.053 \\ 34.838 \\ 43.171 \\ 52.786 \end{bmatrix}$$

$$P(t) = 20.190 + 1.00003t - 0.031775t^2 + 0.00116067t^3 - 0.00001205t^4$$
$$Q(T) = -1.8296\times 10^8 + 370762T - 281.742T^2 + 0.0951507T^3 - 0.00001205T^4$$

35. Expansion of the determinant along the first row gives an equation of the form $ay + bx^2 + cx + d = 0$ that can be solved for $y = Ax^2 + Bx + C$. If the coordinates of any one of the three given points $(x_1, y_1), (x_2, y_2), (x_3, y_3)$ are substituted in the first row, then the determinant has two identical rows and therefore vanishes.

36. Expansion of the determinant along the first row gives

$$\begin{vmatrix} y & x^2 & x & 1 \\ 3 & 1 & 1 & 1 \\ 3 & 4 & 2 & 1 \\ 7 & 9 & 3 & 1 \end{vmatrix} = y\begin{vmatrix} 1 & 1 & 1 \\ 4 & 2 & 1 \\ 9 & 3 & 1 \end{vmatrix} - x^2\begin{vmatrix} 3 & 1 & 1 \\ 3 & 2 & 1 \\ 7 & 3 & 1 \end{vmatrix} + x\begin{vmatrix} 3 & 1 & 1 \\ 3 & 4 & 1 \\ 7 & 9 & 1 \end{vmatrix} - \begin{vmatrix} 3 & 1 & 1 \\ 3 & 4 & 2 \\ 7 & 9 & 3 \end{vmatrix} =$$
$$-2y + 4x^2 - 12x + 14 = 0.$$

Hence $y = 2x^2 - 6x + 7$ is the parabola that interpolates the three given points.

37. Expansion of the determinant along the first row gives an equation of the form $a(x^2 + y^2) + bx + cy + d = 0$, and we get the desired form of the equation of a circle upon division by a. If the coordinates of any one of the three given points $(x_1, y_1), (x_2, y_2)$, and (x_3, y_3) are substituted in the first row, then the determinant has two identical rows and therefore vanishes.

38. Expansion of the determinant along the first row gives

$$\begin{vmatrix} x^2 + y^2 & x & y & 1 \\ 25 & 3 & -4 & 1 \\ 125 & 5 & 10 & 1 \\ 225 & -9 & 12 & 1 \end{vmatrix} =$$

$$= (x^2+y^2)\begin{vmatrix} 3 & -4 & 1 \\ 5 & 10 & 1 \\ -9 & 12 & 1 \end{vmatrix} - x\begin{vmatrix} 25 & -4 & 1 \\ 125 & 10 & 1 \\ 225 & 12 & 1 \end{vmatrix} + y\begin{vmatrix} 25 & 3 & 1 \\ 125 & 5 & 1 \\ 225 & -9 & 1 \end{vmatrix} - \begin{vmatrix} 25 & 3 & -4 \\ 125 & 5 & 10 \\ 225 & -9 & 12 \end{vmatrix}$$

$$= 200(x^2+y^2)+1200x-1600y-15000 = 0.$$

Division by 200 and completion of squares gives $(x+3)^2+(y-4)^2=100,$ so the circle has center $(-3, 4)$ and radius 10.

39. Expansion of the determinant along the first row gives an equation of the form $ax^2+bxy+cy^2+d=0,$ which can be written in the central conic form $Ax^2+Bxy+Cy^2=1$ upon division by $-d$. If the coordinates of any one of the three given points $(x_1, y_1), (x_2, y_2),$ and (x_3, y_3) are substituted in the first row, then the determinant has two identical rows and therefore vanishes.

40. Expansion of the determinant along the first row gives

$$\begin{vmatrix} x^2 & xy & y^2 & 1 \\ 0 & 0 & 16 & 1 \\ 9 & 0 & 0 & 1 \\ 25 & 25 & 25 & 1 \end{vmatrix} =$$

$$= x^2\begin{vmatrix} 0 & 16 & 1 \\ 0 & 0 & 1 \\ 25 & 25 & 1 \end{vmatrix} - xy\begin{vmatrix} 0 & 16 & 1 \\ 9 & 0 & 1 \\ 25 & 25 & 1 \end{vmatrix} + y\begin{vmatrix} 0 & 0 & 1 \\ 9 & 0 & 1 \\ 25 & 25 & 1 \end{vmatrix} - \begin{vmatrix} 0 & 0 & 16 \\ 9 & 0 & 0 \\ 25 & 25 & 25 \end{vmatrix}$$

$$= 400x^2-481xy+225y^2-3600 = 0.$$

CHAPTER 4

VECTOR SPACES

The treatment of vector spaces in this chapter is very concrete. Prior to the final section of the chapter, almost all of the vector spaces appearing in examples and problems are subspaces of Cartesian coordinate spaces of n-tuples of real numbers. The main motivation throughout is the fact that the solution space of a homogeneous linear system $\mathbf{Ax} = \mathbf{0}$ is precisely such a "concrete" vector space.

SECTION 4.1

THE VECTOR SPACE $\mathbf{R}^3$

Here the fundamental concepts of vectors, linear independence, and vector spaces are introduced in the context of the familiar 2-dimensional coordinate plane $\mathbf{R}^2$ and 3-space $\mathbf{R}^3$. The concept of a subspace of a vector space is illustrated, the proper nontrivial subspaces of $\mathbf{R}^3$ being simply lines and planes through the origin.

1.
$$|\mathbf{a}-\mathbf{b}| = |(2,5,-4)-(1,-2,-3)| = |(1,7,-1)| = \sqrt{51}$$
$$2\mathbf{a}+\mathbf{b} = 2(2,5,-4)+(1,-2,-3) = (4,10,-8)+(1,-2,-3) = (5,8,-11)$$
$$3\mathbf{a}-4\mathbf{b} = 3(2,5,-4)-4(1,-2,-3) = (6,15,-12)-(4,-8,-12) = (2,23,0)$$

2.
$$|\mathbf{a}-\mathbf{b}| = |(-1,0,2)-(3,4,-5)| = |(-4,-4,7)| = \sqrt{81} = 9$$
$$2\mathbf{a}+\mathbf{b} = 2(-1,0,2)+(3,4,-5) = (-2,0,4)+(3,4,-5) = (1,4,-1)$$
$$3\mathbf{a}-4\mathbf{b} = 3(-1,0,2)-4(3,4,-5) = (-3,0,6)-(12,16,-20) = (-15,-16,26)$$

3.
$$|\mathbf{a}-\mathbf{b}| = |(2\mathbf{i}-3\mathbf{j}+5\mathbf{k})-(5\mathbf{i}+3\mathbf{j}-7\mathbf{k})| = |-3\mathbf{i}-6\mathbf{j}+12\mathbf{k}| = \sqrt{189} = 3\sqrt{21}$$
$$2\mathbf{a}+\mathbf{b} = 2(2\mathbf{i}-3\mathbf{j}+5\mathbf{k})+(5\mathbf{i}+3\mathbf{j}-7\mathbf{k})$$
$$= (4\mathbf{i}-6\mathbf{j}+10\mathbf{k})+(5\mathbf{i}+3\mathbf{j}-7\mathbf{k}) = 9\mathbf{i}-3\mathbf{j}+3\mathbf{k}$$
$$3\mathbf{a}-4\mathbf{b} = 3(2\mathbf{i}-3\mathbf{j}+5\mathbf{k})-4(5\mathbf{i}+3\mathbf{j}-7\mathbf{k})$$
$$= (6\mathbf{i}-9\mathbf{j}+15\mathbf{k})-(20\mathbf{i}+12\mathbf{j}-28\mathbf{k}) = -14\mathbf{i}-21\mathbf{j}+43\mathbf{k}$$

4.
$$|\mathbf{a}-\mathbf{b}| = |(2\mathbf{i}-\mathbf{j})-(\mathbf{j}-3\mathbf{k})| = |2\mathbf{i}-2\mathbf{j}+3\mathbf{k}| = \sqrt{17}$$
$$2\mathbf{a}+\mathbf{b} = 2(2\mathbf{i}-\mathbf{j})+(\mathbf{j}-3\mathbf{k}) = (4\mathbf{i}-2\mathbf{j})+(\mathbf{j}-3\mathbf{k}) = 4\mathbf{i}-\mathbf{j}-3\mathbf{k}$$
$$3\mathbf{a}-4\mathbf{b} = 3(2\mathbf{i}-\mathbf{j})-4(\mathbf{j}-3\mathbf{k}) = (6\mathbf{i}-3\mathbf{j})-(4\mathbf{j}-12\mathbf{k}) = 6\mathbf{i}-7\mathbf{j}+12\mathbf{k}$$

5. $\mathbf{v} = \frac{3}{2}\mathbf{u}$, so the vectors $\mathbf{u}$ and $\mathbf{v}$ are linearly dependent.

6. $a\mathbf{u}+b\mathbf{v} = a(0,2)+b(3,0) = (3b,2a) = \mathbf{0}$ implies $a=b=0$, so the vectors $\mathbf{u}$ and $\mathbf{v}$ are linearly independent.

7. $a\mathbf{u}+b\mathbf{v} = a(2,2)+b(2,-2) = (2a+2b,2a-2b) = \mathbf{0}$ implies $a=b=0$, so the vectors $\mathbf{u}$ and $\mathbf{v}$ are linearly independent.

8. $\mathbf{v} = -\mathbf{u}$, so the vectors $\mathbf{u}$ and $\mathbf{v}$ are linearly dependent.

In each of Problems 9–14, we set up and solve (as in Example 2 of this section) the system

$$a\mathbf{u}+b\mathbf{v} = \begin{bmatrix} u_1 & v_1 \\ u_2 & v_2 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \mathbf{w}$$

to find the coefficient values a and b such that $\mathbf{w} = a\mathbf{u}+b\mathbf{v}$,

9. $\begin{bmatrix} 1 & -1 \\ -2 & 3 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \Rightarrow \quad a=3,\ b=2 \quad \text{so} \quad \mathbf{w} = 3\mathbf{u}+2\mathbf{v}$

10. $\begin{bmatrix} 3 & 2 \\ 4 & 3 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix} \quad \Rightarrow \quad a=2,\ b=-3 \quad \text{so} \quad \mathbf{w} = 2\mathbf{u}-3\mathbf{v}$

11. $\begin{bmatrix} 5 & 2 \\ 7 & 3 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \Rightarrow \quad a=1,\ b=-2 \quad \text{so} \quad \mathbf{w} = \mathbf{u}-2\mathbf{v}$

12. $\begin{bmatrix} 4 & -2 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \end{bmatrix} \quad \Rightarrow \quad a=3,\ b=5 \quad \text{so} \quad \mathbf{w} = 3\mathbf{u}+5\mathbf{v}$

13. $\begin{bmatrix} 7 & 3 \\ 5 & 4 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 5 \\ -2 \end{bmatrix} \quad \Rightarrow \quad a=2,\ b=-2 \quad \text{so} \quad \mathbf{w} = 2\mathbf{u}-3\mathbf{v}$

14. $\begin{bmatrix} 5 & -6 \\ -2 & 4 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 5 \\ 6 \end{bmatrix} \quad \Rightarrow \quad a=7,\ b=5 \quad \text{so} \quad \mathbf{w} = 7\mathbf{u}+5\mathbf{v}$

In Problems 15–18, we calculate the determinant $\left|\mathbf{u} \quad \mathbf{v} \quad \mathbf{w}\right|$ so as to determine (using Theorem 4) whether the three vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are linearly dependent (det = 0) or linearly independent (det $\neq$ 0).

15. $\begin{vmatrix} 3 & 5 & 8 \\ -1 & 4 & 3 \\ 2 & -6 & -4 \end{vmatrix} = 0$ so the three vectors are linearly dependent.

16. $\begin{vmatrix} 5 & 2 & 4 \\ -2 & -3 & 5 \\ 4 & 5 & -7 \end{vmatrix} = 0$ so the three vectors are linearly dependent.

17. $\begin{vmatrix} 1 & 3 & 1 \\ -1 & 0 & -2 \\ 2 & 1 & 2 \end{vmatrix} = -5 \neq 0$ so the three vectors are linearly independent.

18. $\begin{vmatrix} 1 & 4 & 3 \\ 1 & 3 & -2 \\ 0 & 1 & -4 \end{vmatrix} = 9 \neq 0$ so the three vectors are linearly independent.

In Problems 19–24, we attempt to solve the homogeneous system $\mathbf{Ax} = \mathbf{0}$ by reducing the coefficient matrix $\mathbf{A} = \begin{bmatrix}\mathbf{u} & \mathbf{v} & \mathbf{w}\end{bmatrix}$ to echelon form $\mathbf{E}$. If we find that the system has only the trivial solution $a = b = c = 0$, this means that the vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are linearly independent. Otherwise, a nontrivial solution $\mathbf{x} = \begin{bmatrix}a & b & c\end{bmatrix}^T \neq \mathbf{0}$ provides us with a nontrivial linear combination $a\mathbf{u} + b\mathbf{v} + c\mathbf{w} \neq \mathbf{0}$ that shows the three vectors are linearly dependent.

19. $\mathbf{A} = \begin{bmatrix} 2 & -3 & 0 \\ 0 & 1 & -2 \\ 1 & -1 & -1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix} = \mathbf{E}$

The nontrivial solution $a = 3$, $b = 2$, $c = 1$ gives $3\mathbf{u} + 2\mathbf{v} + \mathbf{w} = \mathbf{0}$, so the three vectors are linearly dependent.

20. $\mathbf{A} = \begin{bmatrix} 5 & 2 & 4 \\ 5 & 3 & 1 \\ 4 & 1 & 5 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \end{bmatrix} = \mathbf{E}$

The nontrivial solution $a = -2$, $b = 3$, $c = 1$ gives $-2\mathbf{u} + 3\mathbf{v} + \mathbf{w} = \mathbf{0}$, so the three vectors are linearly dependent.

21. $\mathbf{A} = \begin{bmatrix} 1 & -2 & 3 \\ 1 & -1 & 7 \\ -2 & 6 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 11 \\ 0 & 1 & 4 \\ 0 & 0 & 0 \end{bmatrix} = \mathbf{E}$

The nontrivial solution $a = 11,\ b = 4,\ c = -1$ gives $11\mathbf{u} + 4\mathbf{v} - \mathbf{w} = \mathbf{0}$, so the three vectors are linearly dependent.

22. $\mathbf{A} = \begin{bmatrix} 1 & 5 & 0 \\ 1 & 1 & 1 \\ 0 & 3 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \mathbf{E}$

The system $\mathbf{Ax} = \mathbf{0}$ has only the trivial solution $a = b = c = 0$, so the vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are linearly independent.

23. $\mathbf{A} = \begin{bmatrix} 2 & 5 & 2 \\ 0 & 4 & -1 \\ 3 & -2 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \mathbf{E}$

The system $\mathbf{Ax} = \mathbf{0}$ has only the trivial solution $a = b = c = 0$, so the vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are linearly independent.

24. $\mathbf{A} = \begin{bmatrix} 1 & 4 & -3 \\ 4 & 2 & 3 \\ 5 & 5 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \mathbf{E}$

The system $\mathbf{Ax} = \mathbf{0}$ has only the trivial solution $a = b = c = 0$, so the vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are linearly independent.

In Problems 25–28, we solve the nonhomogeneous system $\mathbf{Ax} = \mathbf{t}$ by reducing the augmented coefficient matrix $\mathbf{A} = \begin{bmatrix} \mathbf{u} & \mathbf{v} & \mathbf{w} & \mathbf{t} \end{bmatrix}$ to echelon form $\mathbf{E}$. The solution vector $\mathbf{x} = \begin{bmatrix} a & b & c \end{bmatrix}^T$ appears as the final column of $\mathbf{E}$, and provides us with the desired linear combination $\mathbf{t} = a\mathbf{u} + b\mathbf{v} + c\mathbf{w}$.

25. $\mathbf{A} = \begin{bmatrix} 1 & 3 & 1 & 2 \\ -2 & 0 & -1 & -7 \\ 2 & 1 & 2 & 9 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 3 \end{bmatrix} = \mathbf{E}$

Thus $a = 2,\ b = -1,\ c = 3$ so $\mathbf{t} = 2\mathbf{u} - \mathbf{v} + 3\mathbf{w}$.

26. $\mathbf{A} = \begin{bmatrix} 5 & 1 & 5 & 5 \\ 2 & 5 & -3 & 30 \\ -2 & -3 & 4 & -21 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & -1 \end{bmatrix} = \mathbf{E}$

Thus $a = 1,\ b = 5,\ c = -1$ so $\mathbf{t} = \mathbf{u} + 5\mathbf{v} - \mathbf{w}$.

27. $\mathbf{A} = \begin{bmatrix} 1 & -1 & 4 & 0 \\ 4 & -2 & 4 & 0 \\ 3 & 2 & 1 & 19 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 6 \\ 0 & 0 & 1 & 1 \end{bmatrix} = \mathbf{E}$

Thus $a = 2,\ b = 6,\ c = 1$ so $\mathbf{t} = 2\mathbf{u} + 6\mathbf{v} + \mathbf{w}$.

28. $\mathbf{A} = \begin{bmatrix} 2 & 4 & 1 & 7 \\ 5 & 1 & 1 & 7 \\ 3 & -1 & 5 & 7 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} = \mathbf{E}$

Thus $a = 1,\ b = 1,\ c = 1$ so $\mathbf{t} = \mathbf{u} + \mathbf{v} + \mathbf{w}$.

29. Given vectors $(0, y, z)$ and $(0, v, w)$ in V, we see that their sum $(0, y+v, z+w)$ and the scalar multiple $c(0, y, z) = (0, cy, cz)$ both have first component 0, and therefore are elements of V.

30. If (x, y, z) and (u, v, w) are in V, then

$$(x+v)+(y+u)+(z+w) = (x+y+z)+(u+v+w) = 0+0 = 0,$$

so their sum $(x+u, y+v, z+w)$ is in V. Similarly,

$$cx+cy+cz = c(x+y+x) = c(0) = 0,$$

so the scalar multiple (cx, cy, cz) is in V.

31. If (x, y, z) and (u, v, w) are in V, then

$$2(x+u) = (2x)+(2u) = (3y)+(3v) = 3(y+v),$$

so their sum $(x+u, y+v, z+w)$ is in V. Similarly,

$$2(cx) = c(2x) = c(3y) = 3(cy),$$

so the scalar multiple (cx, cy, cz) is in V.

32. If (x, y, z) and (u, v, w) are in V, then

$$z+w = (2x+3y)+(2u+3v) = 2(x+u)+3(y+v),$$

so their sum $(x+u, y+v, z+w)$ is in V. Similarly,

$$cz = c(2x+3y) = 2(cx)+3(cy),$$

so the scalar multiple (cx, cy, cz) is in V.

33. $(0,1,0)$ is in V but the sum $(0,1,0)+(0,1,0) = (0,2,0)$ is not in V; thus V is not closed under addition. Alternatively, $2(0,1,0) = (0,2,0)$ is not in V, so V is not closed under multiplication by scalars.

34. $(1,1,1)$ is in V, but

$$2(1,1,1) = (1,1,1)+(1,1,1) = (2,2,2)$$

is not, so V is closed neither under addition of vectors nor under multiplication by scalars.

35. Evidently V is closed under addition of vectors. However, $(0,0,1)$ is in V but $(-1)(0,0,1) = (0,0,-1)$ is not, so V is not closed under multiplication by scalars.

36. $(1,1,1)$ is in V, but

$$2(1,1,1) = (1,1,1)+(1,1,1) = (2,2,2)$$

is not, so V is closed neither under addition of vectors nor under multiplication by scalars.

37. Pick a fixed element $\mathbf{u}$ in the (nonempty) vector space V. Then, with $c = 0$, the scalar multiple $c\mathbf{u} = 0\mathbf{u} = \mathbf{0}$ must be in V. Thus V necessarily contains the zero vector $\mathbf{0}$.

38. Suppose $\mathbf{u}$ and $\mathbf{v}$ are vectors in the subspace V of $\mathbf{R}^3$ and a and b are scalars. Then $a\mathbf{u}$ and $b\mathbf{v}$ are in V because V is closed under multiplication by scalars. But then it follows that the linear combination $a\mathbf{u}+b\mathbf{v}$ is in V because V is closed under addition of vectors.

39. It suffices to show that every vector $\mathbf{v}$ in V is a scalar multiple of the given nonzero vector $\mathbf{u}$ in V. If $\mathbf{u}$ and $\mathbf{v}$ were linearly independent, then — as illustrated in Example 2 of this section — every vector in $\mathbf{R}^2$ could be expressed as a linear combination of $\mathbf{u}$ and $\mathbf{v}$. In this case it would follow that V is all of $\mathbf{R}^2$ (since, by Problem 38, V is closed under taking linear combinations). But we are given that V is a proper subspace of $\mathbf{R}^2$, so we must conclude that $\mathbf{u}$ and $\mathbf{v}$ are linearly dependent vectors. Since $\mathbf{u} \neq 0$, it follows that the arbitrary vector $\mathbf{v}$ in V is a scalar multiple of $\mathbf{u}$, and thus V is precisely the set of all scalar multiples of $\mathbf{u}$. In geometric language, the subspace V is then the straight line through the origin determined by the nonzero vector $\mathbf{u}$.

40. Since the vectors $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ are linearly dependent , there exist scalars p, q, r not all zero such that $p\mathbf{u}+q\mathbf{v}+r\mathbf{w} = \mathbf{0}$. If $r = 0$, then p and q are scalars not both zero such that $p\mathbf{u}+q\mathbf{v} = \mathbf{0}$. But this contradicts the given fact that $\mathbf{u}$ and $\mathbf{v}$ are linearly independent. Hence $r \neq 0$, so we can solve for

$$\mathbf{w} = -\frac{p}{r}\mathbf{u}-\frac{q}{r}\mathbf{v} = a\mathbf{u}+b\mathbf{v},$$

thereby expressing $\mathbf{w}$ as a linear combination of $\mathbf{u}$ and $\mathbf{v}$.

41. If the vectors $\mathbf{u}$ and $\mathbf{v}$ are in the intersection V of the subspaces V_1 and V_2, then their sum $\mathbf{u}+\mathbf{v}$ is in V_1 because both vectors are in V_1, and $\mathbf{u}+\mathbf{v}$ is in V_2 because both are in V_2. Therefore $\mathbf{u}+\mathbf{v}$ is in V, and thus V is closed under addition of vectors. Similarly, the intersection V is closed under multiplication by scalars, and is therefore itself a subspace.

SECTION 4.2

THE VECTOR SPACE R^n AND SUBSPACES

The main objective in this section is for the student to understand what types of subsets of the vector space $\mathbf{R}^n$ of n-tuples of real numbers are subspaces — playing the role in $\mathbf{R}^n$ of lines and planes through the origin in $\mathbf{R}^3$. Our first reason for studying subspaces is the fact that the *solution space* of any homogeneous linear system $\mathbf{Ax} = \mathbf{0}$ is a subspace of $\mathbf{R}^n$.

1. If $\mathbf{x} = (x_1, x_2, 0)$ and $\mathbf{y} = (y_1, y_2, 0)$ are vectors in W, then their sum

$$\mathbf{x}+\mathbf{y} = (x_1, x_2, 0)+(y_1, y_2, 0) = (x_1+y_1, x_2+y_2, 0)$$

and the scalar multiple $c\mathbf{x} = (cx_1, cx_2, 0)$ both have third coordinate zero, and therefore are also elements of W. Hence W is a subspace of $\mathbf{R}^3$.

2. Suppose $\mathbf{x} = (x_1, x_2, x_3)$ and $\mathbf{y} = (y_1, y_2, y_3)$ are vectors in W, so $x_1 = 5x_2$ and $y_1 = 5y_2$. Then their sum $\mathbf{s} = \mathbf{x}+\mathbf{y} = (x_1+y_1, x_2+y_2, x_3+y_3) = (s_1, s_2, s_3)$ satisfies the same condition

$$s_1 = x_1+y_1 = 5x_2+5y_2 = 5(x_2+y_2) = 5s_2,$$

and thus is an element of W. Similarly, the scalar multiple $\mathbf{m} = c\mathbf{x} = (cx_1, cx_2, cx_3) = (m_1, m_2, m_3)$ satisfies the condition $m_1 = cx_1 = c(5x_2) = 5(cx_2) = 5m_2$, and hence is also an element of W. Therefore W is a subspace of $\mathbf{R}^3$.

3. The typical vector in W is of the form $\mathbf{x} = (x_1, 1, x_3)$ with second coordinate 1. But the particular scalar multiple $2\mathbf{x} = (2x_1, 2, 2x_3)$ of such a vector has second coordinate $2 \neq 1$, and thus is not in W. Hence W is not closed under multiplication by scalars, and therefore is not a subspace of $\mathbf{R}^3$. (Since $2\mathbf{x} = \mathbf{x} + \mathbf{x}$, W is not closed under vector addition either.)

4. The typical vector $\mathbf{x} = (x_1, x_2, x_3)$ in W has coordinate sum $x_1 + x_2 + x_3$ equal to 1. But then the particular scalar multiple $2\mathbf{x} = (2x_1, 2x_2, 2x_3)$ of such a vector has coordinate sum

$$2x_1 + 2x_2 + 2x_3 \;=\; 2(x_1 + x_2 + x_3) \;=\; 2(1) \;=\; 2 \;\neq\; 1,$$

and thus is not in W. Hence W is not closed under multiplication by scalars, and therefore is not a subspace of $\mathbf{R}^3$. (Since $2\mathbf{x} = \mathbf{x} + \mathbf{x}$, W is not closed under vector addition either.)

5. Suppose $\mathbf{x} = (x_1, x_2, x_3, x_4)$ and $\mathbf{y} = (y_1, y_2, y_3, y_4)$ are vectors in W, so

$$x_1 + 2x_2 + 3x_3 + 4x_4 \;=\; 0 \quad \text{and} \quad y_1 + 2y_2 + 3y_3 + 4y_4 \;=\; 0.$$

Then their sum $\mathbf{s} \;=\; \mathbf{x} + \mathbf{y} \;=\; (x_1 + y_1, x_2 + y_2, x_3 + y_3, x_4 + y_4) \;=\; (s_1, s_2, s_3, s_4)$ satisfies the same condition

$$\begin{aligned} s_1 + 2s_2 + 3s_3 + 4s_4 \;&=\; (x_1 + y_1) + 2(x_2 + y_2) + 3(x_3 + y_3) + 4(x_4 + y_4) \\ &=\; (x_1 + 2x_2 + 3x_3 + 4x_4) \;+\; (y_1 + 2y_2 + 3y_3 + 4y_4) \;=\; 0 + 0 \;=\; 0, \end{aligned}$$

and thus is an element of W. Similarly, the scalar multiple $\mathbf{m} = c\mathbf{x} = (cx_1, cx_2, cx_3, cx_4) \;=\; (m_1, m_2, m_3, m_4)$ satisfies the condition

$$m_1 + 2m_2 + 3m_3 + 4m_4 \;=\; cx_1 + 2cx_2 + 3cx_3 + 4cx_4 \;=\; c(x_1 + 2x_2 + 3x_3 + 4x_4) \;=\; 0,$$

and hence is also an element of W. Therefore W is a subspace of $\mathbf{R}^4$.

6. Suppose $\mathbf{x} = (x_1, x_2, x_3, x_4)$ and $\mathbf{y} = (y_1, y_2, y_3, y_4)$ are vectors in W, so

$$x_1 = 3x_3, \quad x_2 = 4x_4 \quad \text{and} \quad y_1 = 3y_3, \quad y_2 = 4y_4.$$

Then their sum $\mathbf{s} \;=\; \mathbf{x} + \mathbf{y} \;=\; (x_1 + y_1, x_2 + y_2, x_3 + y_3, x_4 + y_4) \;=\; (s_1, s_2, s_3, s_4)$ satisfies the same conditions

$$\begin{aligned} s_1 \;&=\; x_1 + y_1 \;=\; 3x_3 + 3y_3 \;=\; 3(x_3 + y_3) \;=\; 3s_3, \\ s_2 \;&=\; x_2 + y_2 \;=\; 4x_4 + 4y_4 \;=\; 4(x_4 + y_4) \;=\; 4s_4, \end{aligned}$$

and thus is an element of W. Similarly, the scalar multiple $\mathbf{m} = c\mathbf{x} = (cx_1, cx_2, cx_3, cx_4) = (m_1, m_2, m_3, m_4)$ satisfies the conditions

$$m_1 = cx_1 = c(3x_3) = 3(cx_3) = 3m_3, \quad m_2 = cx_2 = c(4x_4) = 4(cx_4) = 4m_4,$$

and hence is also an element of W. Therefore W is a subspace of $\mathbf{R}^4$.

7. The vectors $\mathbf{x} = (1,1)$ and $\mathbf{y} = (1,-1)$ are in W, but their sum $\mathbf{x}+\mathbf{y} = (2,0)$ is not, because $|2| \neq |0|$. Hence W is not a subspace of $\mathbf{R}^2$.

8. W is simply the zero subspace $\{\mathbf{0}\}$ of $\mathbf{R}^2$.

9. The vector $\mathbf{x} = (1,0)$ is in W, but its scalar multiple $2\mathbf{x} = (2,0)$ is not, because $(2)^2 + (0)^2 = 4 \neq 1$. Hence W is not a subspace of $\mathbf{R}^2$.

10. The vectors $\mathbf{x} = (1,0)$ and $\mathbf{y} = (0,1)$ are in W, but their sum $\mathbf{s} = \mathbf{x}+\mathbf{y} = (1,1)$ is not, because $|1| + |1| = 2 \neq 1$. Hence W is not a subspace of $\mathbf{R}^2$.

11. Suppose $\mathbf{x} = (x_1, x_2, x_3, x_4)$ and $\mathbf{y} = (y_1, y_2, y_3, y_4)$ are vectors in W, so

$$x_1 + x_2 = x_3 + x_4 \quad \text{and} \quad y_1 + y_2 = y_3 + y_4.$$

Then their sum $\mathbf{s} = \mathbf{x}+\mathbf{y} = (x_1 + y_1, x_2 + y_2, x_3 + y_3, x_4 + y_4) = (s_1, s_2, s_3, s_4)$ satisfies the same condition

$$\begin{aligned} s_1 + s_2 &= (x_1 + y_1) + (x_2 + y_2) = (x_1 + x_2) + (y_1 + y_2) \\ &= (x_3 + x_4) + (y_3 + y_4) = (x_3 + y_3) + (x_4 + y_4) = s_3 + s_4 \end{aligned}$$

and thus is an element of W. Similarly, the scalar multiple $\mathbf{m} = c\mathbf{x} = (cx_1, cx_2, cx_3, cx_4) = (m_1, m_2, m_3, m_4)$ satisfies the condition

$$m_1 + m_2 = cx_1 + cx_2 = c(x_1 + x_2) = c(x_3 + x_4) = cx_3 + cx_4 = m_3 + m_4,$$

and hence is also an element of W. Therefore W is a subspace of $\mathbf{R}^4$.

12. The vectors $\mathbf{x} = (1,0,1,0)$ and $\mathbf{y} = (0,2,0,3)$ are in W (because both products are 0 in each case) but their sum $\mathbf{s} = \mathbf{x}+\mathbf{y} = (1,2,1,3)$ is not, because $s_1 s_2 = 2$ but $s_3 s_4 = 3$. Hence W is not a subspace of $\mathbf{R}^4$.

13. The vectors $\mathbf{x} = (1, 0, 1, 0)$ and $\mathbf{y} = (0, 1, 0, 1)$ are in W (because the product of the 4 components is 0 in each case) but their sum $\mathbf{s} = \mathbf{x} + \mathbf{y} = (1, 1, 1, 1)$ is not, because $s_1 s_2 s_3 s_4 = 1 \neq 0$. Hence W is not a subspace of $\mathbf{R}^4$.

14. The vector $\mathbf{x} = (1, 1, 1, 1)$ is in W (because all 4 components are nonzero) but the multiple $0\mathbf{x} = (0, 0, 0, 0)$ is not. Hence W is not a subspace of $\mathbf{R}^4$.

In Problems 15–22, we first reduce the coefficient matrix $\mathbf{A}$ to echelon form $\mathbf{E}$ in order to solve the given homogeneous system $\mathbf{Ax} = \mathbf{0}$.

15. $$\mathbf{A} = \begin{bmatrix} 1 & -4 & 1 & -4 \\ 1 & 2 & 1 & 8 \\ 1 & 1 & 1 & 6 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 & 4 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \mathbf{E}$$

Thus $x_3 = s$ and $x_4 = t$ are free variables. We solve for $x_1 = -s - 4t$ and $x_2 = -2t$, so

$$\begin{aligned} \mathbf{x} &= (x_1, x_2, x_3, x_4) = (-s - 4t, -2t, s, t) \\ &= (-s, 0, s, 0) + (-4t, -2t, 0, t) = s\,\mathbf{u} + t\,\mathbf{v} \end{aligned}$$

where $\mathbf{u} = (-1, 0, 1, 0)$ and $\mathbf{v} = (-4, -2, 0, 1)$.

16. $$\mathbf{A} = \begin{bmatrix} 1 & -4 & -3 & -7 \\ 2 & -1 & 1 & 7 \\ 1 & 2 & 3 & 11 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 & 5 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \mathbf{E}$$

Thus $x_3 = s$ and $x_4 = t$ are free variables. We solve for $x_1 = -s - 5t$ and $x_2 = -s - 3t$, so

$$\begin{aligned} \mathbf{x} &= (x_1, x_2, x_3, x_4) = (-s - 5t, -s - 3t, s, t) \\ &= (-s, -s, s, 0) + (-5t, -3t, 0, t) = s\,\mathbf{u} + t\,\mathbf{v} \end{aligned}$$

where $\mathbf{u} = (-1, -1, 1, 0)$ and $\mathbf{v} = (-5, -3, 0, 1)$.

17. $$\mathbf{A} = \begin{bmatrix} 1 & 3 & 8 & -1 \\ 1 & -3 & -10 & 5 \\ 1 & 4 & 11 & -2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1 & 2 \\ 0 & 1 & 3 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \mathbf{E}$$

Thus $x_3 = s$ and $x_4 = t$ are free variables. We solve for $x_1 = s - 2t$ and $x_2 = -3s + t$, so

$$\begin{aligned}\mathbf{x} &= (x_1,x_2,x_3,x_4) = (-s-5t,-s-3t,s,t)\\ &= (s,-3s,s,0)+(-2t,t,0,t) = s\,\mathbf{u}+t\,\mathbf{v}\end{aligned}$$

where $\mathbf{u} = (1,-3,1,0)$ and $\mathbf{v} = (-2,1,0,1)$.

18. $\mathbf{A} = \begin{bmatrix} 1 & 3 & 2 & 5 & -1 \\ 2 & 7 & 4 & 11 & 2 \\ 2 & 6 & 5 & 12 & -7 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & -2 & -3 \\ 0 & 1 & 0 & 1 & 4 \\ 0 & 0 & 1 & 2 & -5 \end{bmatrix} = \mathbf{E}$

Thus $x_4 = s$ and $x_5 = t$ are free variables. We solve for $x_1 = 2s+3t$, $x_2 = -s-4t$, and $x_3 = -2s+5t$, so

$$\begin{aligned}\mathbf{x} &= (x_1,x_2,x_3,x_4,x_5) = (2s+3t,-s-4t,-2s+5t,s,t)\\ &= (2s,-s,-2s,s,0)+(3t,-4t,5t,0,t) = s\,\mathbf{u}+t\,\mathbf{v}\end{aligned}$$

where $\mathbf{u} = (2,-1,-2,1,0)$ and $\mathbf{v} = (3,-4,5,0,1)$.

19. $\mathbf{A} = \begin{bmatrix} 1 & -3 & -5 & -6 \\ 2 & 1 & 4 & -4 \\ 1 & 3 & 7 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \mathbf{E}$

Thus $x_3 = t$ is a free variable and $x_4 = 0$.. We solve for $x_1 = -t$ and $x_2 = -2t$, so

$$\mathbf{x} = (x_1,x_2,x_3,x_4) = (-t,-2t,t,0) = t\,\mathbf{u}$$

where $\mathbf{u} = (-1,-2,1,0)$.

20. $\mathbf{A} = \begin{bmatrix} 1 & 5 & 1 & -8 \\ 2 & 5 & 0 & -5 \\ 2 & 7 & 1 & -9 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 2 \end{bmatrix} = \mathbf{E}$

Thus $x_4 = t$ is a free variable. We solve for $x_1 = -5t$, $x_2 = 3t$, and $x_4 = -2t$. so

$$\mathbf{x} = (x_1,x_2,x_3,x_4) = (-5t,3t,-2t,t) = t\,\mathbf{u}$$

where $\mathbf{u} = (-5,3,-2,1)$.

21. $\mathbf{A} = \begin{bmatrix} 1 & 7 & 2 & -3 \\ 2 & 7 & 1 & -4 \\ 3 & 5 & -1 & -5 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 4 \end{bmatrix} = \mathbf{E}$

Thus $x_4 = t$ is a free variable. We solve for $x_1 = -3t,\ x_2 = 2t,$ and $x_4 = -4t.$ so

$$\mathbf{x} = (x_1, x_2, x_3, x_4) = (-3t, 2t, -4t, t) = t\,\mathbf{u}$$

where $\mathbf{u} = (-3, 2, -4, 1).$

22. $$\mathbf{A} = \begin{bmatrix} 1 & 3 & 3 & 3 \\ 2 & 7 & 5 & -1 \\ 2 & 7 & 4 & -4 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & 6 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 1 & 3 \end{bmatrix} = \mathbf{E}$$

Thus $x_4 = t$ is a free variable. We solve for $x_1 = -6t,\ x_2 = 4t,$ and $x_4 = -3t.$ so

$$\mathbf{x} = (x_1, x_2, x_3, x_4) = (-6t, 4t, -3t, t) = t\,\mathbf{u}$$

where $\mathbf{u} = (-6, 4, -3, 1).$

23. Let $\mathbf{u}$ be a vector in W. Then $0\mathbf{u}$ is also in W. But $0\mathbf{u} = (0+0)\mathbf{u} = 0\mathbf{u} + 0\mathbf{u}$, so upon subtracting $0\mathbf{u}$ from each side, we see that $0\mathbf{u} = 0$, the zero vector.

24. **(a)** Problem 23 shows that $0\mathbf{u} = 0$ for every vector $\mathbf{u}$.

(b) The fact that $c\mathbf{0} = c(\mathbf{0}+\mathbf{0}) = c\mathbf{0} + c\mathbf{0}$ implies (upon adding $-c\mathbf{0}$ to each side) that $c\mathbf{0} = \mathbf{0}$.

(c) The fact that $\mathbf{u} + (-1)\mathbf{u} = (1 + (-1))\mathbf{u} = 0\mathrm{u} = 0$ means that $(-1)\mathbf{u} = -\mathbf{u}$.

25. If W is a subspace, then it contains the scalar multiples $a\mathbf{u}$ and $b\mathbf{v}$, and hence contains their sum $a\mathbf{u} + b\mathbf{v}$. Conversely, if the subset W is closed under taking linear combinations of pairs of vectors, then it contains $(1)\mathbf{u} + (1)\mathbf{v} = \mathbf{u} + \mathbf{v}$ and $(c)\mathbf{u} + (0)\mathbf{v} = c\mathbf{u}$, and hence is a subspace.

26. The sum of any two scalar multiples of $\mathbf{u}$ is a scalar multiple of $\mathbf{u}$, as is any scalar multiple of a scalar multiple of $\mathbf{u}$.

27. Let $a_1\mathbf{u} + b_1\mathbf{v}$ and $a_2\mathbf{u} + b_2\mathbf{v}$ be two vectors in $W = \{a\mathbf{u} + b\mathbf{v}\}$. Then the sum

$$(a_1\mathbf{u} + b_1\mathbf{v}) + (a_2\mathbf{u} + b_2\mathbf{v}) = (a_1 + a_2)\mathbf{u} + (b_1 + b_2)\mathbf{v}$$

and the scalar multiple $c(a_1\mathbf{u} + b_1\mathbf{v}) = (ca_1)\mathbf{u} + (cb_1)\mathbf{v}$ are again scalar multiples of $\mathbf{u}$ and $\mathbf{v}$, and hence are themselves elements of W. Hence W is a subspace.

28. If $\mathbf{u}$ and $\mathbf{v}$ are vectors in W, then $\mathbf{Au} = k\mathbf{u}$ and $\mathbf{Av} = k\mathbf{v}$. It follows that

$$\mathbf{A}(a\mathbf{u}+b\mathbf{v}) = a(\mathbf{Au}) + b(\mathbf{Av}) = a(k\mathbf{u}) + b(k\mathbf{v}) = k(a\mathbf{u} + b\mathbf{v}),$$

so the linear combination $a\mathbf{u} + b\mathbf{v}$ of $\mathbf{u}$ and $\mathbf{v}$ is also in W. Hence W is a subspace.

29. If $\mathbf{Ax}_0 = \mathbf{b}$ and $\mathbf{y} = \mathbf{x} - \mathbf{x}_0$, then

$$\mathbf{Ay} = \mathbf{A}(\mathbf{x} - \mathbf{x}_0) = \mathbf{Ax} - \mathbf{Ax}_0 = \mathbf{Ax} - \mathbf{b}.$$

Hence it is clear that $\mathbf{Ay} = \mathbf{0}$ if and only if $\mathbf{Ax} = \mathbf{b}$.

30. Let W denote the intersection of the subspaces U and V. If $\mathbf{u}$ and $\mathbf{v}$ are vectors in W, then these two vectors are both in U and in V. Hence the linear combination $a\mathbf{u} + b\mathbf{v}$ is both in U and in V, and hence is in the intersection W, which therefore is a subspace. If U and V are non-coincident planes through the origin if $\mathbf{R}^3$, then their intersection W is a line through the origin.

31. Let $\mathbf{w}_1$ and $\mathbf{w}_2$ be two vectors in the sum $U + V$. Then $\mathbf{w}_i = \mathbf{u}_i + \mathbf{v}_i$ where $\mathbf{u}_i$ is in U and $\mathbf{v}_i$ is in V $(i = 1, 2)$. Then the linear combination

$$a\mathbf{w}_1 + b\mathbf{w}_2 = a(\mathbf{u}_1 + \mathbf{v}_1) + b(\mathbf{u}_2 + \mathbf{v}_2) = (a\mathbf{u}_1 + \mathrm{b}\mathbf{u}_2) + (a\mathbf{v}_1 + \mathrm{b}\mathbf{v}_2)$$

is the sum of the vectors $a\mathbf{u}_1 + \mathrm{b}\mathbf{u}_2$ in U and $a\mathbf{v}_1 + \mathrm{b}\mathbf{v}_2$ in U, and therefore is an element of $U + V$. Thus $U + V$ is a subspace. If U and V are noncollinear lines through the origin in $\mathbf{R}^3$, then $U + V$ is a plane through the origin.

SECTION 4.3

LINEAR COMBINATIONS AND INDEPENDENCE OF VECTORS

In this section we use two types of computational problems as aids in understanding linear independence and dependence. The first of these problems is that of expressing a vector $\mathbf{w}$ as a linear combination of k given vectors $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_k$ (if possible). The second is that of determining whether k given vectors $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_k$ are linearly independent. For vectors in $\mathbf{R}^n$, each of these problems reduces to solving a linear system of n equations in k unknowns. Thus an abstract question of linear independence or dependence becomes a concrete question of whether or not a given linear system has a nontrivial solution.

1. $\mathbf{v}_2 = \frac{3}{2}\mathbf{v}_1$, so the two vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly dependent.

2. Evidently the two vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ are not scalar multiples of one another. Hence they are linearly dependent.

3. The three vectors $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ are linearly dependent, as are any 3 vectors in $\mathbf{R}^2$. The reason is that the vector equation $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = \mathbf{0}$ reduces to a homogeneous linear system of 2 equations in the 3 unknowns c_1, c_2, and c_3, and any such system has a nontrivial solution.

4. The four vectors $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$, and $\mathbf{v}_4$ are linearly dependent, as are any 4 vectors in $\mathbf{R}^3$. The reason is that the vector equation $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 + c_4\mathbf{v}_4 = \mathbf{0}$ reduces to a homogeneous linear system of 3 equations in the 4 unknowns c_1, c_2, c_3, and c_4, and any such system has a nontrivial solution.

5. The equation $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = \mathbf{0}$ yields

$$c_1(1,0,0) + c_2(0,-2,0) + c_3(0,0,3) = (c_1, -2c_2, 3c_3) = (0,0,0),$$

and therefore implies immediately that $c_1 = c_2 = c_3 = 0$. Hence the given vectors $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ are linearly independent.

6. The equation $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = \mathbf{0}$ yields

$$c_1(1,0,0) + c_2(1,1,0) + c_3(1,1,1) = (c_1 + c_2 + c_3, c_2 + c_3, c_3) = (0,0,0).$$

But it is obvious by back-substitution that the homogeneous system

$$\begin{aligned} c_1 + c_2 + c_3 &= 0 \\ c_2 + c_3 &= 0 \\ c_3 &= 0 \end{aligned}$$

has only the trivial solution $c_1 = c_2 = c_3 = 0$. Hence the given vectors $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ are linearly independent.

7. The equation $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = \mathbf{0}$ yields

$$c_1(2,1,0,0) + c_2(3,0,1,0) + c_3(4,0,0,1) = (2c_1 + 3c_2, c_1, c_2, c_3) = (0,0,0,0).$$

Obviously it follows immediately that $c_1 = c_2 = c_3 = 0$. Hence the given vectors $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ are linearly independent.

8. Here inspection of the three given vectors reveals that $\mathbf{v}_3 = \mathbf{v}_1 + \mathbf{v}_2$, so the vectors $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ are linearly dependent.

In Problems 9–16 we first set up the linear system to be solved for the linear combination coefficients $\{c_i\}$, and then show the reduction of its augmented coefficient matrix $\mathbf{A}$ to reduced echelon form $\mathbf{E}$.

9. $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 = \mathbf{w}$

$$\mathbf{A} = \begin{bmatrix} 5 & 3 & 1 \\ 3 & 2 & 0 \\ 4 & 5 & -7 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \end{bmatrix} = \mathbf{E}$$

We see that the system of 3 equations in 2 unknowns has the unique solution $c_1 = 2, c_2 = -3,$ so $\mathbf{w} = 2\mathbf{v}_1 - 3\mathbf{v}_2.$

10. $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 = \mathbf{w}$

$$\mathbf{A} = \begin{bmatrix} -3 & 6 & 3 \\ 1 & -2 & -1 \\ -2 & 3 & -2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 7 \\ 0 & 1 & 4 \\ 0 & 0 & 0 \end{bmatrix} = \mathbf{E}$$

We see that the system of 3 equations in 2 unknowns has the unique solution $c_1 = 7, c_2 = 4,$ so $\mathbf{w} = 7\mathbf{v}_1 + 4\mathbf{v}_2.$

11. $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 = \mathbf{w}$

$$\mathbf{A} = \begin{bmatrix} 7 & 3 & 1 \\ -6 & -3 & 0 \\ 4 & 2 & 0 \\ 5 & 3 & -1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \mathbf{E}$$

We see that the system of 4 equations in 2 unknowns has the unique solution $c_1 = 1, c_2 = -2,$ so $\mathbf{w} = \mathbf{v}_1 - 2\mathbf{v}_2.$

12. $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 = \mathbf{w}$

$$\mathbf{A} = \begin{bmatrix} 7 & -2 & 4 \\ 3 & -2 & -4 \\ -1 & 1 & 3 \\ 9 & -3 & 3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 5 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \mathbf{E}$$

We see that the system of 4 equations in 2 unknowns has the unique solution $c_1 = 2, c_2 = 5,$ so $\mathbf{w} = 2\mathbf{v}_1 + 5\mathbf{v}_2.$

13. $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 = \mathbf{w}$

$$\mathbf{A} = \begin{bmatrix} 1 & 5 & 5 \\ 5 & -3 & 2 \\ -3 & 4 & -2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \mathbf{E}$$

The last row of $\mathbf{E}$ corresponds to the scalar equation $0c_1 + 0c_2 = 1$, so the system of 3 equations in 2 unknowns is inconsistent. This means that $\mathbf{w}$ cannot be expressed as a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$.

14. $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = \mathbf{w}$

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & -1 & -3 \\ 0 & -2 & 1 & 2 \\ 3 & 0 & 1 & -3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \mathbf{E}$$

The last row of $\mathbf{E}$ corresponds to the scalar equation $0c_1 + 0c_2 + 0c_3 = 1$, so the system of 4 equations in 3 unknowns is inconsistent. This means that $\mathbf{w}$ cannot be expressed as a linear combination of $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$.

15. $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = \mathbf{w}$

$$\mathbf{A} = \begin{bmatrix} 2 & 3 & 1 & 4 \\ -1 & 0 & 2 & 5 \\ 4 & 1 & -1 & 6 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 4 \end{bmatrix} = \mathbf{E}$$

We see that the system of 3 equations in 3 unknowns has the unique solution $c_1 = 3, c_2 = -2, c_3 = 4$, so $\mathbf{w} = 3\mathbf{v}_1 - 2\mathbf{v}_2 + 4\mathbf{v}_3$.

16. $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = \mathbf{w}$

$$\mathbf{A} = \begin{bmatrix} 2 & 4 & 1 & 7 \\ 0 & 1 & 3 & 7 \\ 3 & 3 & -1 & 9 \\ 1 & 2 & 3 & 11 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & 6 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \mathbf{E}$$

We see that the system of 4 equations in 3 unknowns has the unique solution $c_1 = 6, c_2 = -2, c_3 = 3$, so $\mathbf{w} = 6\mathbf{v}_1 - 2\mathbf{v}_2 + 3\mathbf{v}_3$.

In Problems 17–22, $\mathbf{A} = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 \end{bmatrix}$ is the coefficient matrix of the homogeneous linear system corresponding to the vector equation $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = \mathbf{0}$. Inspection of the indicated reduced echelon form $\mathbf{E}$ of $\mathbf{A}$ then reveals whether or not a nontrivial solution exists.

17. $$\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & 5 \\ 1 & 4 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \mathbf{E}$$

We see that the system of 3 equations in 3 unknowns has the unique solution $c_1 = c_2 = c_3 = 0$, so the vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are linearly independent.

18. $$\mathbf{A} = \begin{bmatrix} 2 & 4 & -2 \\ 0 & -5 & 1 \\ -3 & -6 & 3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -3/5 \\ 0 & 1 & -1/5 \\ 0 & 0 & 0 \end{bmatrix} = \mathbf{E}$$

We see that the system of 3 equations in 3 unknowns has a 1-dimensional solution space. If we choose $c_3 = 5$ then $c_1 = 3$ and $c_2 = 1$. Therefore $3\mathbf{v}_1 + \mathbf{v}_2 + 5\mathbf{v}_3 = \mathbf{0}$.

19. $$\mathbf{A} = \begin{bmatrix} 2 & 5 & 2 \\ 0 & 4 & -1 \\ 3 & -2 & 1 \\ 0 & 1 & -1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \mathbf{E}$$

We see that the system of 4 equations in 3 unknowns has the unique solution $c_1 = c_2 = c_3 = 0$, so the vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are linearly independent.

20. $$\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \\ -1 & 1 & 4 \\ 1 & 1 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \mathbf{E}$$

We see that the system of 4 equations in 3 unknowns has the unique solution $c_1 = c_2 = c_3 = 0$, so the vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are linearly independent.

21. $$\mathbf{A} = \begin{bmatrix} 3 & 1 & 1 \\ 0 & -1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \mathbf{E}$$

We see that the system of 4 equations in 3 unknowns has a 1-dimensional solution space. If we choose $c_3 = -1$ then $c_1 = 1$ and $c_2 = -2$. Therefore $\mathbf{v}_1 - 2\mathbf{v}_2 - \mathbf{v}_3 = \mathbf{0}$.

22. $$\mathbf{A} = \begin{bmatrix} 3 & 3 & 5 \\ 9 & 0 & 7 \\ 0 & 9 & 5 \\ 5 & -7 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 7/9 \\ 0 & 1 & 5/9 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \mathbf{E}$$

We see that the system of 4 equations in 3 unknowns has a 1-dimensional solution space. If we choose $c_3 = -9$ then $c_1 = 7$ and $c_2 = 5$. Therefore $7\mathbf{v}_1 + 5\mathbf{v}_2 - 9\mathbf{v}_3 = \mathbf{0}$.

23. Because $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly independent, the vector equation

$$c_1\mathbf{u}_1 + c_2\mathbf{u}_2 = c_1(\mathbf{v}_1 + \mathbf{v}_2) + c_2(\mathbf{v}_1 - \mathbf{v}_2) = \mathbf{0}$$

yields the homogeneous linear system

$$\begin{aligned} c_1 + c_2 &= 0 \\ c_1 - c_2 &= 0. \end{aligned}$$

It follows readily that $c_1 = c_2 = 0,$ and therefore that the vectors $\mathbf{u}_1$ and $\mathbf{u}_2$ are linearly independent.

24. Because $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly independent, the vector equation

$$c_1\mathbf{u}_1 + c_2\mathbf{u}_2 = c_1(\mathbf{v}_1 + \mathbf{v}_2) + c_2(2\mathbf{v}_1 + 3\mathbf{v}_2) = \mathbf{0}$$

yields the homogeneous linear system

$$\begin{aligned} c_1 + 2c_2 &= 0 \\ c_1 + 3c_2 &= 0. \end{aligned}$$

Subtraction of the first equation from the second one gives $c_2 = 0,$ and then it follows from the first equation that $c_2 = 0$ also. Therefore the vectors $\mathbf{u}_1$ and $\mathbf{u}_2$ are linearly independent.

25. Because the vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are linearly independent, the vector equation

$$c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + c_3\mathbf{u}_3 = c_1(\mathbf{v}_1) + c_2(\mathbf{v}_1 + 2\mathbf{v}_2) + c_3(\mathbf{v}_1 + 2\mathbf{v}_2 + 3\mathbf{v}_3) = \mathbf{0}$$

yields the homogeneous linear system

$$\begin{aligned} c_1 + c_2 + c_3 &= 0 \\ 2c_2 + 2c_3 &= 0 \\ 3c_3 &= 0. \end{aligned}$$

It follows by back-substitution that $c_1 = c_2 = c_3 = 0,$ and therefore that the vectors $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ are linearly independent.

26. Because the vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are linearly independent, the vector equation

$$c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + c_3\mathbf{u}_3 = c_1(\mathbf{v}_2 + \mathbf{v}_3) + c_2(\mathbf{v}_1 + \mathbf{v}_3) + c_3(\mathbf{v}_1 + \mathbf{v}_2) = \mathbf{0}$$

yields the homogeneous linear system

$$\begin{aligned} c_2 + c_3 &= 0 \\ c_1 \quad + c_3 &= 0 \\ c_1 + c_2 \quad &= 0. \end{aligned}$$

The reduction

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \mathbf{E}$$

then shows that $c_1 = c_2 = c_3 = 0,$ and therefore that the vectors $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ are linearly independent.

27. If the elements of S are $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_k$ with $\mathbf{v}_1 = \mathbf{0},$ then we can take $c_1 = 1$ and $c_2 = \cdots = c_k = 0.$ This choice gives coefficients $c_1, c_2, \cdots, c_k$ not all zero such that $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k = \mathbf{0}.$ This means that the vectors $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_k$ are linearly dependent.

28. Because the set S of vectors $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_k$ is linearly dependent, there exist scalars $c_1, c_2, \cdots, c_k$ not all zero such that $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k = \mathbf{0}.$ If $c_{k+1} = \cdots = c_m = 0,$ then $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_m\mathbf{v}_m = \mathbf{0}$ with the coefficients $c_1, c_2, \cdots, c_m$ not all zero. This means that the vectors $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_m$ comprising T are linearly dependent.

29. If some subset of S were linearly dependent, then Problem 28 would imply immediately that S itself is linearly dependent (contrary to hypothesis).

30. Let W be the subspace of V spanned by the vectors $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_k.$ Because U is a *subspace* containing each of these vectors, it contains every linear combination of $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_k.$ But W consists solely of such linear combinations, so it follows that U contains $W.$

31. If S is contained in span(T), then every vector in S is a linear combination of vectors in $T.$ Hence every vector in span(S) is a linear combination of linear combinations of vectors in $T.$ Therefore every vector in span(S) is a linear combination of vectors in $T,$ and therefore is itself in span(T). Thus span(S) is a subset of span(T).

32. If $\mathbf{u}$ is another vector in S then the k+1 vectors $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_k, \mathbf{u}$ are linearly dependent. Hence there exist scalars $c_1, c_2, \cdots, c_k, c$ not all zero such that

$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k + c\mathbf{u} = \mathbf{0}$. If $c = 0$ then we have a contradiction to the hypothesis that the vectors $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_k$ are linearly independent. Therefore $c \neq 0$, so we can solve for u as a linear combination of the vectors $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_k$.

33. The determinant of the $k \times k$ identity matrix is nonzero, so it follows immediately from Theorem 3 in this section that the vectors $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_k$ are linearly independent.

34. If the vectors $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n$ are linearly independent, then by Theorem 2 the matrix $\mathbf{A} = [\mathbf{v}_1 \quad \mathbf{v}_2 \quad \cdots \quad \mathbf{v}_n]$ is nonsingular. If $\mathbf{B}$ is another nonsingular $n \times n$ matrix, then the product $\mathbf{AB}$ is also nonsingular, and therefore (by Theorem 2) has linearly independent column vectors.

35. Because the vectors $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_k$ are linearly independent, Theorem 3 implies that some $k \times k$ submatrix $\mathbf{A}_0$ of $\mathbf{A}$ has nonzero determinant. Let $\mathbf{A}_0$ consist of the rows $i_1, i_2, \cdots, i_k$ of the matrix $\mathbf{A}$, and let $\mathbf{C}_0$ denote the $k \times k$ submatrix consisting of the same rows of the product matrix $\mathbf{C} = \mathbf{AB}$. Then $\mathbf{C}_0 = \mathbf{A}_0\mathbf{B}$, so $|\mathbf{C}_0| = |\mathbf{A}_0||\mathbf{B}| \neq 0$ because (by hypothesis) the $k \times k$ matrix $\mathbf{B}$ is also nonsingular. Therefore Theorem 3 implies that the column vectors of $\mathbf{AB}$ are linearly independent.

SECTION 4.4

BASES AND DIMENSION FOR VECTOR SPACES

A basis $\{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_k\}$ for a subspace W of $\mathbf{R}^n$ enables up to visualize W as a k-dimensional plane (or "hyperplane") through the origin in $\mathbf{R}^n$. In case W is the solution space of a homogeneous linear system, a basis for W is a maximal linearly independent set of solutions of the system, and every other solution is a linear combination of these particular solutions.

1. The vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly independent (because neither is a scalar multiple of the other) and therefore form a basis for $\mathbf{R}^2$.

2. We note that $\mathbf{v}_2 = 2\mathbf{v}_1$. Consequently the vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are linearly dependent, and therefore do not form a basis for $\mathbf{R}^3$.

3. Any four vectors in $\mathbf{R}^3$ are linearly dependent, so the given vectors do not form a basis for $\mathbf{R}^3$.

4. Any basis for $\mathbf{R}^4$ contains four vectors, so the given vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ do not form a basis for $\mathbf{R}^4$.

5. The three given vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ all lie in the 2-dimensional subspace $x_1 = 0$ of $\mathbf{R}^3$. Therefore they are linearly dependent, and hence do not form a basis for $\mathbf{R}^3$.

6. $\text{Det}\left(\begin{bmatrix}\mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3\end{bmatrix}\right) = -1 \neq 0,$ so the three vectors are linearly independent, and hence do form a basis for $\mathbf{R}^3$.

7. $\text{Det}\left(\begin{bmatrix}\mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3\end{bmatrix}\right) = 1 \neq 0,$ so the three vectors are linearly independent, and hence do form a basis for $\mathbf{R}^3$.

8. $\text{Det}\left(\begin{bmatrix}\mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 & \mathbf{v}_4\end{bmatrix}\right) = 66 \neq 0,$ so the four vectors are linearly independent, and hence do form a basis for $\mathbf{R}^4$.

9. The single equation $x - 2y + 5z = 0$ is already a system in reduced echelon form, with free variables y and z. With $y = s,\; z = t,\; x = 2s - 5t$ we get the solution vector

$$(x, y, z) = (2s - 5t, s, t) = s(2,1,0) + t(-5,0,1).$$

Hence the plane $x - 2y + 5z = 0$ is a 2-dimensional subspace of $\mathbf{R}^3$ with basis consisting of the vectors $\mathbf{v}_1 = (2,1,0)$ and $\mathbf{v}_2 = (-5,0,1)$.

10. The single equation $y - z = 0$ is already a system in reduced echelon form, with free variables x and z. With $x = s,\; y = z = t$ we get the solution vector

$$(x, y, z) = (s, t, t) = s(1,0,0) + t(0,1,1).$$

Hence the plane $y - z = 0$ is a 2-dimensional subspace of $\mathbf{R}^3$ with basis consisting of the vectors $\mathbf{v}_1 = (1,0,0)$ and $\mathbf{v}_2 = (0,1,1)$.

11. The line of intersection of the planes in Problems 9 and 11 is the solution space of the system

$$\begin{aligned} x - 2y + 5z &= 0 \\ y - z &= 0. \end{aligned}$$

This system is in echelon form with free variable $z = t$. With $y = t$ and $x = -3t$ we have the solution vector $(-3t, t, t) = t(-3,1,1)$. Thus the line is a 1-dimensional subspace of $\mathbf{R}^3$ with basis consisting of the vector $\mathbf{v} = (-3,1,1)$.

12. The typical vector in $\mathbf{R}^4$ of the form (a, b, c, d) with $a = b + c + d$ can be written as

$$\mathbf{v} = (b + c + d, b, c, d) = b(1,1,0,0) + c(1,0,1,0) + d(1,0,0,1).$$

Hence the subspace consisting of all such vectors is 3-dimensional with basis consisting of the vectors $\mathbf{v}_1 = (1,1,0,0)$, $\mathbf{v}_2 = (1,0,1,0)$, and $\mathbf{v}_3 = (1,0,0,1)$.

13. The typical vector in $\mathbf{R}^4$ of the form (a,b,c,d) with $a = 3c$ and $b = 4d$ can be written as

$$\mathbf{v} = (3c,4d,c,d) = c(3,0,1,0) + d(0,4,0,1).$$

Hence the subspace consisting of all such vectors is 2-dimensional with basis consisting of the vectors $\mathbf{v}_1 = (3,0,1,0)$ and $\mathbf{v}_2 = (0,4,0,1)$.

14. The typical vector in $\mathbf{R}^4$ of the form (a,b,c,d) with $a = -2b$ and $c = -3d$ can be written as

$$\mathbf{v} = (-2b,b,-3d,d) = b(-2,1,0,0) + d(0,0,-3,1).$$

Hence the subspace consisting of all such vectors is 2-dimensional with basis consisting of the vectors $\mathbf{v}_1 = (-2,1,0,0)$ and $\mathbf{v}_2 = (0,0,-3,1)$.

In Problems 15–26, we show first the reduction of the coefficient matrix $\mathbf{A}$ to echelon form $\mathbf{E}$. Then we write the typical solution vector as a linear combination of basis vectors for the subspace of the given system.

15. $$\mathbf{A} = \begin{bmatrix} 1 & -2 & 3 \\ 2 & -3 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -11 \\ 0 & 1 & -7 \end{bmatrix} = \mathbf{E}$$

With free variable $x_3 = t$ and $x_1 = 11t$, $x_2 = 7t$ we get the solution vector $\mathbf{x} = (11t,7t,t) = t(11,7,1)$. Thus the solution space of the given system is 1-dimensional with basis consisting of the vector $\mathbf{v}_1 = (11,7,1)$.

16. $$\mathbf{A} = \begin{bmatrix} 1 & 3 & 4 \\ 3 & 8 & 7 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -11 \\ 0 & 1 & 5 \end{bmatrix} = \mathbf{E}$$

With free variable $x_3 = t$ and with $x_1 = 11t$, $x_2 = -5t$ we get the solution vector $\mathbf{x} = (11t,-5t,t) = t(11,-5,1)$. Thus the solution space of the given system is 1-dimensional with basis consisting of the vector $\mathbf{v}_1 = (11,-5,1)$.

17. $$\mathbf{A} = \begin{bmatrix} 1 & -3 & 2 & -4 \\ 2 & -5 & 7 & -3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 11 & 11 \\ 0 & 1 & 3 & 5 \end{bmatrix} = \mathbf{E}$$

With free variables $x_3 = s$, $x_4 = t$ and with $x_1 = -11s - 11t$, $x_2 = -3s - 5t$ we get the solution vector

$$\mathbf{x} = (-11s-11t,-3s-5t,s,t) = s(-11,-3,1,0) + t(-11,-5,0,1).$$

Thus the solution space of the given system is 2-dimensional with basis consisting of the vectors $\mathbf{v}_1 = (-11,-3,1,0)$ and $\mathbf{v}_2 = (-11,-5,0,1)$.

18. $\mathbf{A} = \begin{bmatrix} 1 & 3 & 4 & 5 \\ 2 & 6 & 9 & 5 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & 0 & 25 \\ 0 & 0 & 1 & -5 \end{bmatrix} = \mathbf{E}$

With free variables $x_2 = s,\ x_4 = t$ and with $x_1 = -3s - 25t,\ x_3 = 5t$ we get the solution vector

$$\mathbf{x} = (-3s - 25t, s, 5t, t) = s(-3,1,0,0) + t(-25,0,5,1).$$

Thus the solution space of the given system is 2-dimensional with basis consisting of the vectors $\mathbf{v}_1 = (-3,1,0,0)$ and $\mathbf{v}_2 = (-25,0,5,1)$.

19. $\mathbf{A} = \begin{bmatrix} 1 & -3 & -8 & -5 \\ 2 & 1 & -4 & 11 \\ 1 & 3 & 3 & 13 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \mathbf{E}$

With free variables $x_3 = s,\ x_4 = t$ and with $x_1 = 3s - 4t,\ x_2 = -2s - 3t$ we get the solution vector

$$\mathbf{x} = (3s - 4t, -2s - 3t, s, t) = s(3,-2,1,0) + t(-4,-3,0,1).$$

Thus the solution space of the given system is 2-dimensional with basis consisting of the vectors $\mathbf{v}_1 = (3,-2,1,0)$ and $\mathbf{v}_2 = (-4,-3,0,1)$.

20. $\mathbf{A} = \begin{bmatrix} 1 & -3 & -10 & 5 \\ 1 & 4 & 11 & -2 \\ 1 & 3 & 8 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1 & 2 \\ 0 & 1 & 3 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \mathbf{E}$

With free variables $x_3 = s,\ x_4 = t$ and with $x_1 = s - 2t,\ x_2 = -3s + t$ we get the solution vector

$$\mathbf{x} = (s - 2t, -3s + t, s, t) = s(1,-3,1,0) + t(-2,1,0,1).$$

Thus the solution space of the given system is 2-dimensional with basis consisting of the vectors $\mathbf{v}_1 = (1,-3,1,0)$ and $\mathbf{v}_2 = (-2,1,0,1)$.

21. $\mathbf{A} = \begin{bmatrix} 1 & -4 & -3 & -7 \\ 2 & -1 & 1 & 7 \\ 1 & 2 & 3 & 11 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 & 5 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \mathbf{E}$

With free variables $x_3 = s,\ x_4 = t$ and with $x_1 = -s-5t,\ x_2 = -s-3t$ we get the solution vector

$$\mathbf{x} = (-s-5t, -s-3t, s, t) = s(-1,-1,1,0) + t(-5,-3,0,1).$$

Thus the solution space of the given system is 2-dimensional with basis consisting of the vectors $\mathbf{v}_1 = (-1,-1,1,0)$ and $\mathbf{v}_2 = (-5,-3,0,1).$

22. $$\mathbf{A} = \begin{bmatrix} 1 & -2 & -3 & -16 \\ 2 & -4 & 1 & 17 \\ 1 & -2 & 3 & 26 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & 0 & 5 \\ 0 & 0 & 1 & 7 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \mathbf{E}$$

With free variables $x_2 = s,\ x_4 = t$ and with $x_1 = 2s-5t,\ x_3 = -7t$ we get the solution vector

$$\mathbf{x} = (2s-5t, s, -7t, t) = s(2,1,0,0) + t(-5,0,-7,1).$$

Thus the solution space of the given system is 2-dimensional with basis consisting of the vectors $\mathbf{v}_1 = (2,1,0,0)$ and $\mathbf{v}_2 = (-5,0,-7,1).$

23. $$\mathbf{A} = \begin{bmatrix} 1 & 5 & 13 & 14 \\ 2 & 5 & 11 & 12 \\ 2 & 7 & 17 & 19 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \mathbf{E}$$

With free variable $x_3 = s$ and with $x_1 = 2s,\ x_2 = -3s,\ x_4 = 0$ we get the solution vector $\mathbf{x} = (2s, -3s, s, 0) = s(2,-3,1,0).$ Thus the solution space of the given system is 1-dimensional with basis consisting of the vector $\mathbf{v}_1 = (2,-3,1,0).$

24. $$\mathbf{A} = \begin{bmatrix} 1 & 3 & -4 & -8 & 6 \\ 1 & 0 & 2 & 1 & 3 \\ 2 & 7 & -10 & -19 & 13 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 2 & 1 & 3 \\ 0 & 1 & -2 & -3 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} = \mathbf{E}$$

With free variables $x_3 = r,\ x_4 = s,\ x_5 = t$ and with $x_1 = -2r-s-3t,\ x_2 = 2r+3s-t$ we get the solution vector

$$\mathbf{x} = (-2r-s-3t, 2r+3s-t, r, s, t) = r(-2,2,1,0,0) + s(-1,3,0,1,0) + t(-3,-1,0,0,1).$$

Thus the solution space of the given system is 3-dimensional with basis consisting of the vectors $\mathbf{v}_1 = (-2,2,1,0,0),\ \mathbf{v}_2 = (-1,3,0,1,0),$ and $\mathbf{v}_3 = (-3,-1,0,0,1).$

25. $\mathbf{A} = \begin{bmatrix} 1 & 2 & 7 & -9 & 31 \\ 2 & 4 & 7 & -11 & 34 \\ 3 & 6 & 5 & -11 & 29 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 0 & -2 & 3 \\ 0 & 0 & 1 & -1 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} = \mathbf{E}$

With free variables $x_2 = r,\ x_4 = s,\ x_5 = t$ and with $x_1 = -2r + 2s - 3t,\ x_3 = s - 4t$ we get the solution vector

$$\mathbf{x} = (-2r + 2s - 3t, r, s - 4t, s, t) = r(-2,1,0,0,0) + s(2,0,1,1,0) + t(-3,0,-4,0,1).$$

Thus the solution space of the given system is 3-dimensional with basis consisting of the vectors $\mathbf{v}_1 = (-2,1,0,0,0)$, $\mathbf{v}_2 = (2,0,1,1,0)$, and $\mathbf{v}_3 = (-3,0,-4,0,1)$.

26. $\mathbf{A} = \begin{bmatrix} 3 & 1 & -3 & 11 & 10 \\ 5 & 8 & 2 & -2 & 7 \\ 2 & 5 & 0 & -1 & 14 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & 2 & -3 \\ 0 & 1 & 0 & -1 & 4 \\ 0 & 0 & 1 & -2 & -5 \end{bmatrix} = \mathbf{E}$

With free variables $x_4 = s,\ x_5 = t$ and with $x_1 = -2s + 3t,\ x_2 = s - 4t,\ x_3 = 2s + 5t$ we get the solution vector

$$\mathbf{x} = (-2s + 3t, s - 4t, 2s + 5t, s, t) = s(-2,1,2,1,0) + t(3,-4,5,0,1).$$

Thus the solution space of the given system is 2-dimensional with basis consisting of the vectors $\mathbf{v}_1 = (-2,1,2,1,0)$ and $\mathbf{v}_2 = (3,-4,5,0,1)$.

27. If the vectors $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n$ are linearly independent, and $\mathbf{w}$ is another vector in V, then the vectors $\mathbf{w}, \mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n$ are linearly dependent (because no n+1 vectors in the n-dimensional vector space V are linearly independent). Hence there exist scalars $c, c_1, c_2, \cdots, c_n$ not all zero such that

$$c\mathbf{w} + c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n = \mathbf{0}.$$

If $c = 0$ then the coefficients $c_1, c_2, \cdots, c_n$ would not all be zero, and hence this equation would say (contrary to hypothesis) that the vectors $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n$ are linearly dependent. Therefore $c \neq 0$, so we can solve for $\mathbf{w}$ as a linear combination of the vectors $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n$. Thus the linearly independent vectors $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n$ span V, and therefore form a basis for V.

28. If the n vectors in S were not linearly independent, then some one of them would be a linear combination of the others. These remaining n–1 vectors would then span the n-dimensional vector space V, which is impossible. Therefore the spanning set S is also linearly independent, and therefore is a basis for V.

29. Suppose $c\mathbf{v} + c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k = \mathbf{0}$. Then $c = 0$ because, otherwise, we could solve for $\mathbf{v}$ as a linear combination of the vectors $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_k$. But this is impossible, because $\mathbf{v}$ is not in the subspace W spanned by $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_k$. It follows that $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k = \mathbf{0}$, which implies that $c_1 = c_2 = \cdots = c_k = 0$ also, because the vectors $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_k$ are linearly independent. Hence we have shown that the $k+1$ vectors $\mathbf{v}, \mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_k$ are linearly independent.

30. Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_k\}$ be a linearly independent set of $k < n$ vectors in V. If the vector $\mathbf{v}_{k+1}$ in V is not in $W = \text{span}(S)$, then Problem 29 implies that the $k+1$ vectors $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_k, \mathbf{v}_{k+1}$ are linearly independent. Continuing in this fashion, we can add one vector at a time until we have n linearly independent vectors in V, which then form a basis for V that contains the original basis S.

31. If $\mathbf{v}_{k+1}$ is a linear combination of the vectors $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_k$, then obviously every linear combination of the vectors $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_k, \mathbf{v}_{k+1}$ is also a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_k$. But the former set of $k+1$ vectors spans V, so the latter set of k vectors also spans V.

32. If the spanning set S for V is not linearly independent, then some vector in S is a linear combination of the others. But Problem 31 says that when we remove this dependent vector from S, the resulting set of one fewer vectors still spans V. Continuing in this fashion, we remove one vector at a time from S until we wind up with a spanning set for V that is also a linearly independent set, and therefor forms a basis for V that is contained by the original spanning set S.

33. If S is a maximal linearly independent set in V, the we see immediately that every other vector in V is a linear combination of the vectors in S. Thus S also spans V, and is therefore a basis for V.

34. If the minimal spanning set S for V were not linearly independent, then (by Problem 28) some vector S would be a linear combination of the others. Then the set obtained from the minimal spanning set S by deleting this dependent vector would be a *smaller* spanning set for S (which is impossible). Hence the spanning set S is also a linearly independent set, and therefore is a basis for V.

35. Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n\}$ be a uniquely spanning set for V. Then the fact, that

$$\mathbf{0} = 0\,\mathbf{v}_1 + 0\,\mathbf{v}_2 + \cdots + 0\,\mathbf{v}_n$$

is the unique expression of the zero vector $\mathbf{0}$ as a linear combination of the vectors in S, means that S is a linearly independent set of vectors. Hence S is a basis for V.

36. If $a_1, a_2, \cdots, a_k$ are scalars, then the linear combination $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k$ — of the column vectors of the matrix in Eq. (12) having the $k \times k$ identity matrix as its "bottom" $k \times k$ submatrix — is a vector of the form $(*, *, \cdots, *, a_1, a_2, \cdots, a_k)$. Hence this linear combination can equal the zero vector only if $a_1 = a_2 = \cdots = a_k = 0$. Thus the vectors $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_k$ are linearly independent.

SECTION 4.5

ROW AND COLUMN SPACES

Conventional wisdom (at a certain level) has it that a homogeneous linear system $\mathbf{Ax} = \mathbf{0}$ of m equations in $n > m$ unknowns ought to have $n - m$ independent solutions. In Section 4.5 of the text we use row and column spaces to show that this "conventional wisdom" is valid under the condition that the m equations are irredundant — meaning that the rank of the coefficient matrix **A** is m (so its m row vectors are linearly independent).

In each of Problems 1–12 we give the *reduced* echelon form **E** of the matrix **A**, a basis for the row space of **A**, and a basis for the column space of **A**.

1. $$\mathbf{E} = \begin{bmatrix} 1 & 0 & 11 \\ 0 & 1 & -4 \\ 0 & 0 & 0 \end{bmatrix}$$

Row basis: The first and second row vectors of **E**.
Column basis: The first and second column vectors of **A**.

2. $$\mathbf{E} = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \end{bmatrix}$$

Row basis: The first and second row vectors of **E**.
Column basis: The first and second column vectors of **A**.

3. $$\mathbf{E} = \begin{bmatrix} 1 & 0 & 1 & 5 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Row basis: The first and second row vectors of **E**.
Column basis: The first and second column vectors of **A**.

4. $\mathbf{E} = \begin{bmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 0 \end{bmatrix}$

Row basis: The three row vectors of **E**.
Column basis: The first three column vectors of **A**.

5. $\mathbf{E} = \begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Row basis: The three row vectors of **E**.
Column basis: The first, second, and fourth column vectors of **A**.

6. $\mathbf{E} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Row basis: The three row vectors of **E**.
Column basis: The first, second, and fourth column vectors of **A**.

7. $\mathbf{E} = \begin{bmatrix} 1 & 0 & -3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Row basis: The first two row vectors of **E**.
Column basis: The first two column vectors of **A**.

8. $\mathbf{E} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Row basis: The first three row vectors of **E**.
Column basis: The first, second, and fourth column vectors of **A**.

9. $\mathbf{E} = \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Row basis: The first three row vectors of **E**.
Column basis: The first three column vectors of **A**.

10. $\mathbf{E} = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

Row basis: The first three row vectors of **E**.
Column basis: The first, second, and fourth column vectors of **A**.

11. $\mathbf{E} = \begin{bmatrix} 1 & 0 & 2 & 1 & 0 \\ 0 & 1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

Row basis: The first three row vectors of **E**.
Column basis: The first, second, and fifth column vectors of **A**.

12. **E** is the same reduced echelon matrix as in Problem 11.
Row basis: The first three row vectors of **E**.
Column basis: The first, second, and fifth column vectors of **A**.

In each of Problems 13–16 we give the *reduced* echelon form **E** of the matrix having the given vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots$ as its column vectors.

13. $\mathbf{E} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

Linearly independent: $\mathbf{v}_1$ and $\mathbf{v}_2$

14. $\mathbf{E} = \begin{bmatrix} 1 & 0 & \frac{1}{5} & 2 \\ 0 & 1 & \frac{2}{5} & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Linearly independent: $\mathbf{v}_1$ and $\mathbf{v}_2$

15. $\mathbf{E} = \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Linearly independent: $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_4$

16. $\mathbf{E} = \begin{bmatrix} 1 & 0 & 2 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$

Linearly independent: $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_4$, and $\mathbf{v}_5$

In each of Problems 17–20 the matrix $\mathbf{E}$ is the *reduced* echelon matrix of the matrix $\mathbf{A} = [\mathbf{v}_1 \ \cdots \ \mathbf{v}_k \ \mathbf{e}_1 \ \cdots \ \mathbf{e}_n]$.

17. $\mathbf{E} = \begin{bmatrix} 1 & 0 & -3 & 0 & 2 \\ 0 & 1 & 2 & 0 & -1 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix}$

Basis vectors: $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{e}_2$

18. $\mathbf{E} = \begin{bmatrix} 1 & 0 & \frac{1}{5} & 0 & -\frac{2}{5} \\ 0 & 1 & \frac{1}{5} & 0 & \frac{3}{5} \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix}$

Basis vectors: $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{e}_2$

19. $\mathbf{E} = \begin{bmatrix} 1 & 0 & 3 & 0 & 0 & -2 \\ 0 & 1 & -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix}$

Basis vectors: $\mathbf{v}_1, \mathbf{v}_2, \mathbf{e}_2, \mathbf{e}_3$

20. $\mathbf{E} = \begin{bmatrix} 1 & 0 & 0 & -\frac{5}{2} & 0 & 2 \\ 0 & 1 & 0 & \frac{3}{2} & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix}$

Basis vectors: $\mathbf{v}_1, \mathbf{v}_2, \mathbf{e}_1, \mathbf{e}_3$

In each of Problems 21–24 the matrix $\mathbf{E}$ is the *reduced* echelon form of the transpose $\mathbf{A}^T$ of the coefficient matrix $\mathbf{A}$.

21. $\mathbf{E} = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

The first and second equations are irredundant.

22. $\mathbf{E} = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

The first and second equations are irredundant.

23. $\mathbf{E} = \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

The first, second, and fourth equations are irredundant.

24. $\mathbf{E} = \begin{bmatrix} 1 & 0 & 1 & 2 & 0 \\ 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$

The first, second, and fifth equations are irredundant.

25. The row vectors of $\mathbf{A}$ are the column vectors of its transpose matrix $\mathbf{A}^T$, so

$$\operatorname{rank}(\mathbf{A}) = \text{row rank of } \mathbf{A} = \text{column rank of } \mathbf{A}^T = \operatorname{rank}(\mathbf{A}^T).$$

26. The rank of the $n \times n$ matrix $\mathbf{A}$ is n if and only if its column vectors are linearly independent, in which case $\det(\mathbf{A}) \neq 0$ by Theorem 2 in Section 4.3, so it follows by Theorem 2 in Section 3.6 that $\mathbf{A}$ is invertible.

27. The rank of the 3×5 matrix $\mathbf{A}$ is 3, so its column vectors $\mathbf{a}_1, \mathbf{a}_1, \ldots, \mathbf{a}_5$ span $\mathbf{R}^3$. Therefore any given vector $\mathbf{b}$ in $\mathbf{R}^3$ can be expressed as a linear combination $\mathbf{b} = x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_5\mathbf{a}_5$ of the column vectors of $\mathbf{A}$. The column vector $\mathbf{x}$ whose elements are the coefficients in this linear combination is then a solution of the equation $\mathbf{Ax} = \mathbf{b}$.

28. The rank of the 5×3 matrix $\mathbf{A}$ is 3, so its three column vectors $\mathbf{a}_1, \mathbf{a}_1, \mathbf{a}_3$ are linearly independent. Therefore any given vector $\mathbf{b}$ in $\mathbf{R}^5$ can be expressed *in at most one way* as a linear combination $\mathbf{b} = x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + x_3\mathbf{a}_3$ of the column vectors of $\mathbf{A}$. This means that the equation $\mathbf{Ax} = \mathbf{b}$ has at most one solution $\mathbf{x} = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}^T$.

29. The rank of the $m \times n$ matrix $\mathbf{A}$ is at most $m < n$, and therefore is less than the number n of its column vectors. Hence the column vectors $\mathbf{a}_1, \mathbf{a}_1, \ldots, \mathbf{a}_n$ of $\mathbf{A}$ are linearly dependent, so there exists a linear combination $y_1\mathbf{a}_1 + y_2\mathbf{a}_2 + \cdots + y_n\mathbf{a}_n = \mathbf{0}$ with not all the coefficients being zero. If $\mathbf{x} = \begin{bmatrix} x_1 & x_2 & \ldots & x_n \end{bmatrix}^T$ is one solution of the equation $\mathbf{Ax} = \mathbf{b}$, then $\mathbf{A}(\mathbf{x} + \mathbf{y}) = \mathbf{Ax} + \mathbf{Ay} = \mathbf{b} + \mathbf{0} = \mathbf{b}$, so $\mathbf{x} + \mathbf{y}$ is a second different solution. Thus solutions of the equation are not unique.

30. The rank of the $m \times n$ matrix $\mathbf{A}$ is at most $n < m$, and therefore is less than the number m of its row vectors. Hence the dimension of the column space of $\mathbf{A}$ is less than m, so this column space is a *proper* subspace of $\mathbf{R}^m$. Hence there exists a vector $\mathbf{b}$ in $\mathbf{R}^m$ that is not a linear combination of the column vectors of $\mathbf{A}$. This means that the equation $\mathbf{Ax} = \mathbf{b}$ has no solution.

31. The rank of the $m \times n$ matrix $\mathbf{A}$ is m if and only if $\mathbf{A}$ has m linearly independent column vectors — in which case these m linearly independent column vectors constitute a basis for $\mathbf{R}^m$. Hence the rank of $\mathbf{A}$ is m if and only if its column vectors

$\mathbf{a}_1, \mathbf{a}_1, \dots, \mathbf{a}_n$ span $\mathbf{R}^m$ — in which case every vector $\mathbf{b}$ in $\mathbf{R}^m$ can be expressed as a linear combination $\mathbf{b} = x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n$, so the equation $\mathbf{Ax} = \mathbf{b}$ has the solution $\mathbf{x} = \begin{bmatrix} x_1 & x_2 & \dots & x_n \end{bmatrix}^T$.

32. The rank of the $m \times n$ matrix $\mathbf{A}$ is n if and only if the n column vectors $\mathbf{a}_1, \mathbf{a}_1, \dots, \mathbf{a}_n$ of $\mathbf{A}$ are linearly independent — in which a vector $\mathbf{b}$ in $\mathbf{R}^m$ can be expressed in at most one way as a linear combination $\mathbf{b} = x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n$. This means that the equation $\mathbf{Ax} = \mathbf{b}$ has at most one solution $\mathbf{x} = \begin{bmatrix} x_1 & x_2 & \dots & x_n \end{bmatrix}^T$.

33. Suppose that some linear combination of the k pivot column vectors $\mathbf{p}_1, \mathbf{p}_1, \dots, \mathbf{p}_k$ in (8) equals the zero vector. Denote by $c_1, c_1, \dots, c_k$ the coefficients in this linear combination. Then the first k scalar components of the equation $c_1\mathbf{p}_1 + c_2\mathbf{p}_2 + c_3\mathbf{p}_3 + \cdots + c_k\mathbf{p}_k = \mathbf{0}$ yield the $k \times k$ upper-triangular system

$$\begin{aligned}
c_1 d_1 + c_2 p_{21} + c_3 p_{31} + \cdots + c_k p_{k1} &= 0 \\
c_2 d_2 + c_3 p_{32} + \cdots + c_k p_{k2} &= 0 \\
c_3 d_3 + \cdots + c_k p_{k3} &= 0 \\
&\vdots \\
c_k d_k &= 0
\end{aligned}$$

where p_{ij} (for $i > j$) denotes the jth element of the vector $\mathbf{p}_i$ and $p_{ii} = d_i$. Because the leading entries $d_1, d_1, \dots, d_k$ are all nonzero, it follows by back-substitution that $c_1 = c_2 = \cdots = c_k = 0$. Therefore the column vectors are linearly independent.

34. If no row interchanges are involved, then (for any k) the space spanned by the first k row vectors of $\mathbf{A}$ is never changed in the process of reducing $\mathbf{A}$ to the echelon matrix $\mathbf{E}$; this follows immediately from the proof of Theorem 2 in this section. Hence the first r row vectors of $\mathbf{A}$ span the r-dimensional space Row($\mathbf{A}$), and therefore are linearly independent.

35. Look at the r row vectors of the matrix $\mathbf{A}$ that are determined by its largest nonsingular $r \times r$ submatrix. Then Theorem 3 in Section 4.3 says that these r row vectors are linearly independent, whereas any $r+1$ row vectors of $\mathbf{A}$ are linearly dependent.

SECTION 4.6

ORTHOGONAL VECTORS IN $\mathbf{R}^n$

The generalization in this section, of the dot product to vectors in $\mathbf{R}^n$, enables us to flesh out the algebra of vectors in $\mathbf{R}^n$ with the Euclidean geometry of angles and distance. We can now refer to the vector space $\mathbf{R}^n$ (provided with the dot product) as n-dimensional *Euclidean space*.

1. $\mathbf{v}_1 \cdot \mathbf{v}_2 = (2)(3)+(1)(-6)+(2)(1)+(1)(-2) = 6-6+2-2=0$
$\mathbf{v}_1 \cdot \mathbf{v}_3 = (2)(3)+(1)(-1)+(2)(-5)+(1)(5) = 6-1-10+5=0$
$\mathbf{v}_2 \cdot \mathbf{v}_3 = (3)(3)+(-6)(-1)+(1)(-5)+(-2)(5) = 9+6-5-10=0$

Yes, the three vectors are mutually orthogonal.

2. $\mathbf{v}_1 \cdot \mathbf{v}_2 = (3)(6)+(-2)(3)+(3)(4)+(-4)(6) = 18-6+12-24=0$
$\mathbf{v}_1 \cdot \mathbf{v}_3 = (3)(17)+(-2)(-12)+(3)(-21)+(-4)(3) = 51+24-63-12=0$
$\mathbf{v}_2 \cdot \mathbf{v}_3 = (6)(17)+(3)(-12)+(4)(-21)+(6)(3) = 102-36-84+18=0$

Yes, the three vectors are mutually orthogonal.

3. $\mathbf{v}_1 \cdot \mathbf{v}_2 = 15-10-4-1=0,\ \mathbf{v}_1 \cdot \mathbf{v}_3 = 15+0-32+17=0,\ \mathbf{v}_2 \cdot \mathbf{v}_3 = 9+0+8-17=0$
Yes, the three vectors are mutually orthogonal.

4. $\mathbf{v}_1 \cdot \mathbf{v}_2 = 3+4+9-12-4=0,\ \mathbf{v}_1 \cdot \mathbf{v}_3 = 6+4-12-2+4=0,$
$\mathbf{v}_2 \cdot \mathbf{v}_3 = 18+4-12+6-16=0$
Yes, the three vectors are mutually orthogonal.

In each of Problems 5–8 we write $\mathbf{u} = \overrightarrow{CB},\ \mathbf{v} = \overrightarrow{CA},$ and $\mathbf{w} = \overrightarrow{AB}.$ Then we calculate $a = |\mathbf{u}|,$ $b = |\mathbf{v}|,$ and $c = |\mathbf{w}|$ so as to verify that $a^2 + b^2 = c^2.$

5. $\mathbf{u} = (1,1,2,-1),\ \mathbf{v} = (1,-1,1,2),\ \mathbf{w} = (0,2,1,-3);\quad a^2 = 7,\ b^2 = 7,\ c^2 = 14$

6. $\mathbf{u} = (3,-1,2,2),\ \mathbf{v} = (2,2,-3,1),\ \mathbf{w} = (1,-3,5,1);\quad a^2 = 18,\ b^2 = 18,\ c^2 = 36$

7. $\mathbf{u} = (2,1,-2,1,3),\ \mathbf{v} = (3,2,2,2,-2),\ \mathbf{w} = (-1,-1,-4,-1,5);\quad a^2 = 19,\ b^2 = 25,\ c^2 = 44$

8. $\mathbf{u} = (3,2,4,5,7),\ \mathbf{v} = (7,5,-5,2,-3),\ \mathbf{w} = (-4,-3,9,3,10);\quad a^2 = 103,\ b^2 = 112,\ c^2 = 215$

The computations in Problems 5–8 show that in each triangle ΔABC the angle at C is a right angle. The angles at the vertices A and B are then determined by the relations

$$\cos\angle A = \frac{\overrightarrow{AB}\cdot\overrightarrow{AC}}{\left|\overrightarrow{AB}\right|\left|\overrightarrow{AC}\right|} = -\frac{\mathbf{v}\cdot\mathbf{w}}{|\mathbf{v}|\,|\mathbf{w}|} \quad \text{and} \quad \cos\angle B = \frac{\overrightarrow{BA}\cdot\overrightarrow{BC}}{\left|\overrightarrow{BA}\right|\left|\overrightarrow{BC}\right|} = +\frac{\mathbf{u}\cdot\mathbf{w}}{|\mathbf{u}|\,|\mathbf{w}|}.$$

The fact that $\angle A + \angle B = 90^\circ$ then serves as a check on our numerical computations.

9. $\angle A = \cos^{-1}\left(-\frac{-7}{\sqrt{7}\sqrt{14}}\right) = \cos^{-1}\left(\frac{1}{\sqrt{2}}\right) = 45^\circ$, $\angle B = \cos^{-1}\left(+\frac{7}{\sqrt{7}\sqrt{14}}\right) = \cos^{-1}\left(\frac{1}{\sqrt{2}}\right) = 45^\circ$

10. $\angle A = \cos^{-1}\left(-\frac{-18}{\sqrt{18}\sqrt{36}}\right) = \cos^{-1}\left(\frac{1}{\sqrt{2}}\right) = 45^\circ$,

$\angle B = \cos^{-1}\left(+\frac{18}{\sqrt{18}\sqrt{36}}\right) = \cos^{-1}\left(\frac{1}{\sqrt{2}}\right) = 45^\circ$

11. $\angle A = \cos^{-1}\left(-\frac{-25}{\sqrt{25}\sqrt{44}}\right) = \cos^{-1}\left(\sqrt{\frac{25}{44}}\right) = 41.08^\circ$,

$\angle B = \cos^{-1}\left(\frac{19}{\sqrt{19}\sqrt{44}}\right) = \cos^{-1}\left(\sqrt{\frac{19}{44}}\right) = 48.92^\circ$

12. $\angle A = \cos^{-1}\left(-\frac{-112}{\sqrt{112}\sqrt{215}}\right) = \cos^{-1}\left(\sqrt{\frac{112}{215}}\right) = 43.80^\circ$,

$\angle B = \cos^{-1}\left(\frac{103}{\sqrt{103}\sqrt{215}}\right) = \cos^{-1}\left(\sqrt{\frac{103}{215}}\right) = 46.20^\circ$

In each of Problems 13–22, we denote by $\mathbf{A}$ the matrix having the given vectors as its row vectors, and by $\mathbf{E}$ the reduced echelon form of $\mathbf{A}$. From $\mathbf{E}$ we find the general solution of the homogeneous system $\mathbf{Ax} = \mathbf{0}$ in terms of parameters $s, t, \ldots$. We then get basis vectors $\mathbf{u}_1, \mathbf{u}_2, \ldots$ for the orthogonal complement $V^\perp$ by setting each parameter in turn equal to 1 (and the others then equal to 0).

13. $\mathbf{A} = \mathbf{E} = \begin{bmatrix} 1 & -2 & 3 \end{bmatrix}$; $x_2 = s,\ x_3 = t,\ x_1 = 2s - 3t$

$\mathbf{u}_1 = (2,1,0),\ \ \mathbf{u}_2 = (-3,0,1)$

14. $\mathbf{A} = \mathbf{E} = \begin{bmatrix} 1 & 5 & -3 \end{bmatrix}$; $x_2 = s,\ x_3 = t,\ x_1 = -5s + 3t$

$\mathbf{u}_1 = (-5,1,0),\ \ \mathbf{u}_2 = (3,0,1)$

15. $\mathbf{A} = \mathbf{E} = \begin{bmatrix} 1 & -2 & -3 & 5 \end{bmatrix}$; $x_2 = r,\ x_3 = s,\ x_4 = t,\ x_1 = 2r + 3s - 5t$

$\mathbf{u}_1 = (2,1,0,0),\ \ \mathbf{u}_2 = (3,0,1,0),\ \ \mathbf{u}_3 = (-5,0,0,1)$

16. $\mathbf{A} = \mathbf{E} = \begin{bmatrix}1 & 7 & -6 & -9\end{bmatrix}$; $x_2 = r,\ x_3 = s,\ x_4 = t,\ x_1 = -7r + 6s + 9t$

$\mathbf{u}_1 = (-7,1,0,0),\quad \mathbf{u}_2 = (6,0,1,0),\quad \mathbf{u}_3 = (9,0,0,1)$

17. $\mathbf{E} = \begin{bmatrix}1 & 0 & -7 & 19 \\ 0 & 1 & 3 & -5\end{bmatrix}$

$x_3 = s,\ x_4 = t,\ x_2 = -3s + 5t,\ x_1 = 7s - 19t$

$\mathbf{u}_1 = (7,-3,1,0),\quad \mathbf{u}_2 = (-19,5,0,1)$

18. $\mathbf{E} = \begin{bmatrix}1 & 0 & 12 & -16 \\ 0 & 1 & 3 & -7\end{bmatrix}$

$x_3 = s,\ x_4 = t,\ x_2 = -3s + 7t,\ x_1 = -12s + 16t$

$\mathbf{u}_1 = (-12,-3,1,0),\quad \mathbf{u}_2 = (16,7,0,1)$

19. $\mathbf{E} = \begin{bmatrix}1 & 0 & 13 & -4 & 11 \\ 0 & 1 & -4 & 3 & -4\end{bmatrix}$

$x_3 = r,\ x_4 = s,\ x_5 = t,\ x_2 = 4r - 3s + 4t,\ x_1 = -13r + 4s - 11t$

$\mathbf{u}_1 = (-13,4,1,0,0),\quad \mathbf{u}_2 = (4,-3,0,1,0),\quad \mathbf{u}_3 = (-11,4,0,0,1)$

20. $\mathbf{E} = \begin{bmatrix}1 & 0 & 5 & 12 & 19 \\ 0 & 1 & -1 & -4 & -7\end{bmatrix}$

$x_3 = r,\ x_4 = s,\ x_5 = t,\ x_2 = r + 4s + 7t,\ x_1 = -5r - 12s - 19t$

$\mathbf{u}_1 = (-5,1,1,0,0),\quad \mathbf{u}_2 = (-12,4,0,1,0),\quad \mathbf{u}_3 = (-19,7,0,0,1)$

21. $\mathbf{E} = \begin{bmatrix}1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1\end{bmatrix}$

$x_3 = s,\ x_5 = t,\ x_4 = -t,\ x_2 = -s - t,\ x_1 = -s$

$\mathbf{u}_1 = (-1,-1,1,0,0),\quad \mathbf{u}_2 = (0,-1,0,-1,1)$

22. $\mathbf{E} = \begin{bmatrix} 1 & 0 & 2 & -1 & 0 \\ 0 & 1 & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$

$x_3 = s,\ x_4 = t,\ x_5 = 0,\ x_2 = s - 2t,\ x_1 = -2s + t$

$\mathbf{u}_1 = (-2,1,1,0,0), \quad \mathbf{u}_2 = (1,-2,0,1,0)$

23. (a) $\quad |\mathbf{u}+\mathbf{v}|^2 + |\mathbf{u}-\mathbf{v}|^2 = (\mathbf{u}\cdot\mathbf{u} + 2\mathbf{u}\cdot\mathbf{v} + \mathbf{v}\cdot\mathbf{v}) + (\mathbf{u}\cdot\mathbf{u} - 2\mathbf{u}\cdot\mathbf{v} + \mathbf{v}\cdot\mathbf{v})$

$$= 2|\mathbf{u}|^2 + |\mathbf{v}|^2 = 2\mathbf{u}\cdot\mathbf{u} + 2\mathbf{u}\cdot\mathbf{u}$$

(b) $\quad |\mathbf{u}+\mathbf{v}|^2 - |\mathbf{u}-\mathbf{v}|^2 = (\mathbf{u}\cdot\mathbf{u} + 2\mathbf{u}\cdot\mathbf{v} + \mathbf{v}\cdot\mathbf{v}) - (\mathbf{u}\cdot\mathbf{u} - 2\mathbf{u}\cdot\mathbf{v} + \mathbf{v}\cdot\mathbf{v}) = 4\mathbf{u}\cdot\mathbf{v}$

24. Equation (15) in the text says that the given formula holds for $k = 2$ vectors. Assume inductively that it holds for $k = n-1$ vectors. Then

$$\begin{aligned} |\mathbf{v}_1 + \mathbf{v}_2 + \cdots \mathbf{v}_{n-1} + \mathbf{v}_n|^2 &= |\mathbf{v}_1 + \mathbf{v}_2 + \cdots \mathbf{v}_{n-1}|^2 + |\mathbf{v}_n|^2 \\ &= \left(|\mathbf{v}_1|^2 + |\mathbf{v}_2|^2 + \cdots + |\mathbf{v}_{n-1}|^2\right) + |\mathbf{v}_n|^2 \end{aligned}$$

as desired. The case $k = 2$ is used for the first equality here, and the case $k = n-1$ for the second one.

25. Suppose, for instance, that $A = \mathbf{e}_1 = (1,0,0,0,0)$, $B = \mathbf{e}_3 = (0,0,1,0,0)$, and $C = \mathbf{e}_5 = (0,0,0,0,1)$ in $\mathbf{R}^5$. Then $\overrightarrow{AB} = \mathbf{e}_3 - \mathbf{e}_1 = (-1,0,1,0,0)$ and $\overrightarrow{AC} = \mathbf{e}_5 - \mathbf{e}_1 = (0,0,1,0,0,-1)$. Then $\overrightarrow{AB}\cdot\overrightarrow{AC} = 1$ while $|\overrightarrow{AB}| = |\overrightarrow{AC}| = \sqrt{2}$. It follows that $\cos\angle A = 1/(\sqrt{2})(\sqrt{2}) = \frac{1}{2}$, so $\angle A = 60°$. Similarly, $\angle B = \angle C = 60°$, so we see that ΔABC is an equilateral triangle.

26. Because $\mathbf{u}\cdot\mathbf{v} = |\mathbf{u}||\mathbf{v}|\cos\theta$, it follows that $\mathbf{u}\cdot\mathbf{v} = |\mathbf{u}||\mathbf{v}|$ if and only if $\cos\theta = 1$, in which case $\theta = 0$ so the two vectors are collinear.

27. If the $\mathbf{u}$ lines both in the subspace V and in its orthogonal complement $V^\perp$, then the vector $\mathbf{u}$ is orthogonal to itself. Hence $\mathbf{u}\cdot u = |\mathbf{u}|^2 = 0$, so it follows that $\mathbf{u} = \mathbf{0}$.

28. If W is the orthogonal complement of V, then every vector in V is orthogonal to every vector in W. Hence V is contained in $W^\perp$. But it follows from Equation (18) in this section that the two subspaces V and $W^\perp$ have the same dimension. Because one contains the other, they must therefore be the same subspace, so $W^\perp = V$ as desired.

29. If $\mathbf{u}$ is orthogonal to each vector in the set S of vectors, then it follows easily (using the dot product) that $\mathbf{u}$ is orthogonal to every linear combination of vectors in S. Therefore $\mathbf{u}$ is orthogonal to $V = \text{Span}(S)$.

30. If $\mathbf{u} \cdot \mathbf{v} = 0$ and $\mathbf{u} + \mathbf{v} = \mathbf{0}$ then

$$0 = \mathbf{u} \cdot (\mathbf{u} + \mathbf{v}) = \mathbf{u} \cdot \mathbf{u} + \mathbf{u} \cdot \mathbf{v} = \mathbf{u} \cdot \mathbf{u},$$

so it follows that $\mathbf{u} = \mathbf{0}$, and then $\mathbf{u} + \mathbf{v} = \mathbf{0}$ implies that $\mathbf{v} = \mathbf{0}$ also.

31. We want to show that any linear combination of vectors $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_p$ of vectors in S is orthogonal to every linear combination of vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_q$ in T. But if each $\mathbf{u}_i$ is orthogonal to each $\mathbf{v}_j$, so $\mathbf{u}_i \cdot \mathbf{v}_j = 0$, then it follows that

$$\left(a_1\mathbf{u}_1 + a_2\mathbf{u}_2 + \cdots + a_p\mathbf{u}_p\right) \cdot \left(b_1\mathbf{v}_1 + b_2\mathbf{v}_2 + \cdots + b_q\mathbf{v}_q\right) = \sum_{i=1}^{p}\sum_{j=1}^{q} a_i b_j \mathbf{u}_i \cdot \mathbf{v}_j = 0,$$

so we see that the two linear combinations are orthogonal, as desired.

32. Suppose that the linear combination $a_1\mathbf{u}_1 + a_2\mathbf{u}_2 + b_1\mathbf{v}_1 + b_2\mathbf{v}_2 = \mathbf{0}$, and we want to deduce that all four coefficients a_1, a_2, b_1, b_2 must necessarily be zero. For this purpose, write

$$\mathbf{u} = a_1\mathbf{u}_1 + a_2\mathbf{u}_2 \qquad \text{and} \qquad \mathbf{v} = b_1\mathbf{v}_1 + b_2\mathbf{v}_2.$$

Then the vectors $\mathbf{u}$ and $\mathbf{v}$ are orthogonal by Problem 31, so by Problem 30 the fact that $\mathbf{u} + \mathbf{v} = \mathbf{0}$ implies that

$$\mathbf{u} = a_1\mathbf{u}_1 + a_2\mathbf{u}_2 = \mathbf{0} \qquad \text{and} \qquad \mathbf{v} = b_1\mathbf{v}_1 + b_2\mathbf{v}_2 = \mathbf{0}.$$

Now the assumed linear independence of $\{\mathbf{u}_1, \mathbf{u}_2\}$ implies that $a_1 = a_2 = 0$, and the assumed linear independence of $\{\mathbf{v}_1, \mathbf{v}_2\}$ implies that $b_1 = b_2 = 0$. Thus we conclude that all four coefficients are zero, as desired.

33. This is the same as Problem 32, except with

$$\mathbf{u} = a_1\mathbf{u}_1 + a_2\mathbf{u}_2 + \cdots + a_k\mathbf{u}_k \qquad \text{and} \qquad \mathbf{v} = b_1\mathbf{v}_1 + b_2\mathbf{v}_2 + \cdots + b_m\mathbf{v}_m.$$

34. It follows immediately from Problem 33 and from Equation (18) in the text that the union of a basis for the subspace V and a basis for its orthogonal complement $V^{\perp}$ is a linearly independent set of n vectors, and is therefore a basis for the n-dimensional vector space $\mathbf{R}^n$.

35. This is one of the fundamental theorems of linear algebra. The nonhomogeneous system

$$\mathbf{Ax} = \mathbf{b}$$

is consistent if and only if the vector $\mathbf{b}$ is in the subspace $\mathrm{Col}(\mathbf{A}) = \mathrm{Row}(\mathbf{A}^T)$. But $\mathbf{b}$ is in $\mathrm{Row}(\mathbf{A}^T)$ if and only if $\mathbf{b}$ is orthogonal to the orthogonal complement of $\mathrm{Row}(\mathbf{A}^T)$. But $\mathrm{Row}(\mathbf{A}^T)^{\perp} = \mathrm{Null}(\mathbf{A}^T)$, which is the solution space of the homogeneous system

$$\mathbf{A}^T\mathbf{y} = \mathbf{0}.$$

Thus we have proved (as desired) that the nonhomogeneous system $\mathbf{Ax} = \mathbf{b}$ has a solution if and only if the constant vector $\mathbf{b}$ is orthogonal to every solution $\mathbf{y}$ of the nonhomogeneous system $\mathbf{A}^T\mathbf{y} = \mathbf{0}$.

SECTION 4.7

GENERAL VECTOR SPACES

In each of Problems 1–12, a certain subset of a vector space is described. This subset is a subspace of the vector space if and only if it is closed under the formation of linear combinations of its elements. Recall also that every subspace of a vector space must contain the zero vector.

1. It is a subspace of M_{33}, because any linear combination of diagonal 3×3 matrices — with only zeros off the principal diagonal — obviously is again a diagonal matrix.

2. The square matrix $\mathbf{A}$ is symmetric if and only if $\mathbf{A}^T = \mathbf{A}$. If $\mathbf{A}$ and $\mathbf{B}$ are symmetric 3×3 matrices, then $(c\mathbf{A}+d\mathbf{B})^T = c\mathbf{A}^T + d\mathbf{B}^T = c\mathbf{A}+d\mathbf{B}$, so the linear combination $c\mathbf{A}+d\mathbf{B}$ is also symmetric. Thus the set of all such matrices is a subspace.

3. The set of all nonsingular 3×3 matrices does not contain the zero matrix, so it is not a subspace.

4. The set of all singular 3×3 matrices is not a subspace, because the sum

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

of singular matrices is not singular.

5. The set of all functions $f:\mathbf{R}\to\mathbf{R}$ with $f(0)=0$ is a vector space, because if $f(0)=g(0)=0$ then $(af+bg)(0) = af(0)+bg(0) = a\cdot 0 + b\cdot 0 = 0.$

6. The set of all functions $f: \mathbf{R} \to \mathbf{R}$ with $f(0) \neq 0$ is not a vector space, because it does not contain the zero function $f(0) \equiv 0$.

7. The set of all functions $f: \mathbf{R} \to \mathbf{R}$ with $f(0) = 0$ and $f(1) = 1$ is not a vector space. For instance, if $g = 2f$ then $g(1) = 2f(1) = 2 \cdot 1 = 2 \neq 1$, so g is not such a function. Also, this set does not contain the zero function.

8. A function $f: \mathbf{R} \to \mathbf{R}$ such that $f(-x) = -f(x)$ is called an **odd function**. Any linear combination $af + bg$ of odd functions is again odd, because

$$(af+bg)(-x) \;=\; af(-x)+bg(-x) \;=\; -af(x)-bg(x) \;=\; -(af+bg)(x).$$

Thus the set of all odd functions is a vector space.

For Problems 9–12, let us call a polynomial of the form $a_0 + a_1 x + a_2 x^2 + a_3 x^3$ a "degree at most 3" polynomial.

9. The set of all degree at most 3 polynomials with nonzero leading coefficient $a_3 \neq 0$ is not a vector space, because it does not contain the zero polynomial (with all coefficients zero).

10. The set of all degree at most 3 polynomials not containing x or x^2 terms is a vector space, because any linear combination of such polynomials obviously is such a polynomial.

11. The set of all degree at most 3 polynomials with coefficient sum zero is a vector space, because any linear combination of such polynomials obviously is such a polynomial.

12. If the degree at most 3 polynomials f and g have all-integer coefficients, the linear combination $a\,f + b\,g$ may have non-integer coefficient, because a and b need not be integers. Hence the set of all degree at most 3 polynomials having all-integer coefficients is not a vector space.

13. The functions $\sin x$ and $\cos x$ are linearly independent, because neither is a scalar multiple of the other. (This follows, for instance, from the facts that $\sin(0) = 0$, $\cos(0) = 1$ and $\sin(\pi/2) = 1$, $\cos(\pi/2) = 0$, noting that any scalar multiple of a function with a zero value must have the value 0 at the same point.)

14. The functions e^x and xe^x are linearly independent, since obviously neither is a scalar multiple of the other (their ratios $xe^x / e^x = x$ and e^x / xe^x neither being constants).

15. If

$$c_1(1+x)+c_2(1-x)+c_3(1-x^2) \;=\; (c_1+c_2+c_3)+(c_1-c_2)x-c_3x^2 \;=\; 0,$$

then

$$c_1+c_2+c_3 \;=\; c_1-c_2 \;=\; c_3 \;=\; 0.$$

It follows easily that $c_1=c_2=c_3=0$, so we conclude that the functions $(1+x)$, $(1-x)$, and $(1-x^2)$ are linearly independent.

16. $(-1)\cdot(1+x)+(1)\cdot(x+x^2)+(1)\cdot(1-x^2) \;=\; 0$, so the three given polynomials are linearly dependent.

17. $\cos 2x \;=\; \cos^2 x-\sin^2 x$ according to a well-known trigonometric identity. Thus these three trigonometric functions are linearly dependent.

18. If

$$c_1(2\cos x+3\sin x)+c_2(4\cos x+5\sin x) \;=\; (2c_1+4c_2)\cos x+(3c_1+5c_2)\sin x \;=\; 0$$

then the fact that $\sin x$ and $\cos x$ are linearly independent (Problem 13) implies that

$2c_1+4c_2 \;=\; 3c_1+5c_2 \;=\; 0$. It follows readily that $c_1=c_2=0$, so we conclude that the two original linear combinations of $\sin x$ and $\cos x$ are linearly independent.

19. Multiplication by $(x-2)(x-3)$ yields

$$x-5 \;=\; A(x-3)+B(x-2) \;=\; (A+B)x-(3A+2B).$$

Hence $A+B=1$ and $3A+2B=5$, and it follows readily that $A=3$ and $B=-2$.

20. Multiplication by $x(x^2-1)$ yields

$$2 \;=\; A(x^2-1)+Bx(x+1)+Cx(x-1) \;=\; -A+(B-C)x+(A+B+C)x^2.$$

Hence $-A=2$, $B-C=0$ and $A+B+C=0$. It follows readily that $A=-2$ and $B=C=1$.

21. Multiplication by $x(x^2+4)$ yields

$$8 \;=\; A(x^2+4)+Bx^2+Cx \;=\; 4A+Cx+(A+B)x^2.$$

Hence $4A=8$, $C=0$ and $A+B=0$. It follows readily that $A=2$ and $B=-2$.

22. Multiplication by $(x+1)(x+2)(x+3)$ yields

$$\begin{aligned} 2x &= A(x+2)(x+3)+B(x+1)(x+3)+C(x+1)(x+2) \\ &= (A+B+C)x^2+(5A+4B+3C)x+(6A+3B+2C). \end{aligned}$$

Hence

$$\begin{aligned} A+\ \ B+\ \ C &= 0 \\ 5A+4B+3C &= 2 \\ 6A+3B+2C &= 0, \end{aligned}$$

and we solve these three equations for $A=-1$, $B=4$, and $C=-3$.

23. If $y'''(x)=0$ then

$$\begin{aligned} y''(x) &= \int y'''(x)\,dx = \int (0)\,dx = A, \\ y'(x) &= \int y''(x)\,dx = \int A\,dx = Ax+B, \text{ and} \\ y(x) &= \int y'(x)\,dx = \int (Ax+B)\,dx = \tfrac{1}{2}Ax^2+Bx+C, \end{aligned}$$

where A, B, and C are arbitrary constants of integration. It follows that the function $y(x)$ is a solution of the differential equation $y'''(x)=0$ if and only if it is a quadratic (at most 2nd degree) polynomial. Thus the solution space is 3-dimensional with basis $\{1, x, x^2\}$.

24. If $y^{(4)}(x)=0$ then

$$\begin{aligned} y'''(x) &= \int y^{(4)}(x)\,dx = \int (0)\,dx = A, \\ y''(x) &= \int y'''(x)\,dx = \int A\,dx = Ax+B, \\ y'(x) &= \int y''(x)\,dx = \int (Ax+B)\,dx = \tfrac{1}{2}Ax^2+Bx+C, \text{ and} \\ y(x) &= \int y'(x)\,dx = \int (\tfrac{1}{2}Ax^2+Bx+C)\,dx = \tfrac{1}{6}Ax^2+\tfrac{1}{2}Bx+Cx+D. \end{aligned}$$

where A, B, C, and D are arbitrary constants of integration. It follows that the function $y(x)$ is a solution of the differential equation $y^{(4)}(x)=0$ if and only if it is a cubic (at most 3rd degree) polynomial. Thus the solution space is 4-dimensional with basis $\{1, x, x^2, x^3\}$.

25. If $y(x)$ is any solution of the second-order differential equation $y''-5y'=0$ and $v(x)=y'(x)$, then $v(x)$ is a solution of the first-order differential equation $v'(x)=5v(x)$ with the familiar exponential solution $v(x) = Ce^{5x}$. Therefore

$$y(x) = \int y'(x)\,dx = \int v(x)\,dx = \int Ce^{5x}\,dx = \tfrac{1}{5}Ce^{5x} + D.$$

We therefore see that the solution space of the equation $y'' - 5y' = 0$ is 2-dimensional with basis $\{1, e^{5x}\}$.

26. If $y(x)$ is any solution of the second-order differential equation $y'' + 10y' = 0$ and $v(x) = y'(x)$, then $v(x)$ is a solution of the first-order differential equation $v'(x) = -10v(x)$ with the familiar exponential solution $v(x) = Ce^{-10x}$. Therefore

$$y(x) = \int y'(x)\,dx = \int v(x)\,dx = \int Ce^{-10x}\,dx = -\tfrac{1}{10}Ce^{10x} + D.$$

We therefore see that the solution space of the equation $y'' + 10y' = 0$ is 2-dimensional with basis $\{1, e^{-10x}\}$.

27. If we take the positive sign in Eq. (20) of the text, then we have $v^2 = y^2 + a^2$ where $v(x) = y'(x)$. Then

$$\left(\frac{dy}{dx}\right)^2 = y^2 + a^2, \quad \text{so} \quad \frac{dx}{dy} = \frac{1}{\sqrt{y^2 + a^2}}$$

(taking the positive square root as in the text). Then

$$\begin{aligned} x &= \int \frac{dy}{\sqrt{y^2 + a^2}} = \int \frac{a\,du}{\sqrt{a^2u^2 + a^2}} \qquad (y = au) \\ &= \int \frac{du}{\sqrt{u^2 + 1}} = \sinh^{-1} u + b = \sinh^{-1} \frac{y}{a} + b. \end{aligned}$$

It follows that

$$\begin{aligned} y(x) &= a\sinh(x - b) = a\left(\sinh x \cosh b - \cosh x \sinh b\right) \\ &= A\cosh x + B\sinh x. \end{aligned}$$

28. We start with the second-order differential equation $y'' + y = 0$ and substitute $v(x) = y'(x)$, so

$$y'' = \frac{dv}{dx} = \frac{dv}{dy}\frac{dy}{dx} = v\frac{dv}{dy} = -y$$

as in Example 9 of the text. Then $v\,dv = -y\,dy$, and integration gives

$$\tfrac{1}{2}v^2 = -\tfrac{1}{2}y^2 + C, \quad \text{so} \quad v^2 = a^2 - y^2$$

(taking for illustration a positive value for the arbitrary constant C). Then

$$\left(\frac{dy}{dx}\right)^2 = a^2 - y^2, \quad \text{so} \quad \frac{dx}{dy} = \frac{1}{\sqrt{a^2 - y^2}}$$

(taking the positive square root). Then

$$\begin{aligned} x &= \int \frac{dy}{\sqrt{a^2 - y^2}} = \int \frac{a\,du}{\sqrt{a^2 - a^2u^2}} \qquad (y = au) \\ &= \int \frac{du}{\sqrt{1-u^2}} = \sin^{-1} u + b = \sin^{-1}\frac{y}{a} + b. \end{aligned}$$

It follows that

$$\begin{aligned} y(x) &= a\sin(x-b) = a\left(\sin x\cos b - \cos x \sin b\right) \\ &= A\cos x + B\sin x. \end{aligned}$$

Thus the general solution of the 2nd-order differential equation $y'' + y = 0$ is a linear combination of $\cos x$ and $\sin x$. It follows that the solution space is 2-dimensional with basis $\{\cos x, \sin x\}$.

29. **(a)** The verification in a component-wise manner that V is a vector space is the same as the verification that $\mathbf{R}^n$ is a vector space, except with vectors having infinitely many components rather than finitely many components. It boils down to the fact that a linear combination of infinite sequences of real numbers is itself such a sequence,

$$a \cdot \{x_n\}_1^\infty + b \cdot \{y_n\}_1^\infty = \{ax_n + by_n\}_1^\infty.$$

(b) If $\mathbf{e}_n = \{0, \cdots, 0, 1, 0, 0, \cdots\}$ is the indicated infinite sequence with 1 in the nth position, then the fact that

$$c_1\mathbf{e}_1 + c_2\mathbf{e}_2 + \cdots + c_k\mathbf{e}_k = \{c_1, c_2, \cdots, c_k, 0, 0, \cdots\}$$

evidently implies that any finite set $\mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_k$ of these vectors is linearly independent. Thus V contains "arbitrarily large" sets of linearly independent vectors, and therefore is infinite-dimensional.

30. **(a)** If $x_n = x_{n-1} + x_{n-2}$, $y_n = y_{n-1} + y_{n-2}$ and $z_n = ax_n + by_n$ for each n, then

$$\begin{aligned} z_n &= a(x_{n-1} + x_{n-2}) + b(y_{n-1} + y_{n-2}) \\ &= (ax_{n-1} + by_{n-1}) + (ax_{n-2} + by_{n-2}) \ = \ z_{n-1} + z_{n-2}. \end{aligned}$$

Thus W is a subspace of V.

(b) Let $\mathbf{v}_1 = \{1,0,1,1,2,3,5,\cdots\}$ be the element with $x_1 = 1$ and $x_2 = 0$, and let $\mathbf{v}_2 = \{0,1,1,2,3,5,\cdots\}$ be the element with $x_1 = 0$ and $x_2 = 1$. Then $\mathbf{v}_1$ and $\mathbf{v}_2$ form a basis for W.

31. **(a)** If $z_1 = a_1 + ib_1$ and $z_2 = a_2 + ib_2$, then direct computation shows that

$$T(c_1z_1 + c_2z_2) \ = \ c_1T(z_1) + c_2T(z_2) \ = \begin{bmatrix} c_1a_1 + c_2a_2 & -c_1b_1 - b_2a_2 \\ c_1b_1 + b_2a_2 & c_1a_1 + c_2a_2 \end{bmatrix}.$$

(b) If $z_1 = a_1 + ib_1$ and $z_2 = a_2 + ib_2$, then $z_1z_2 \ = \ (a_1a_2 - b_1b_2) + i(a_1b_2 + a_2b_1)$ and direct computation shows that

$$T(z_1z_2) \ = \ T(z_1)T(z_2) \ = \begin{bmatrix} a_1a_2 - b_1b_2 & -a_1b_2 - a_2b_1 \\ a_1b_2 + a_2b_1 & a_1a_2 - b_1b_2 \end{bmatrix}.$$

(b) If $z = a + ib$ then

$$\frac{1}{z} \ = \ \frac{1}{a+bi} \cdot \frac{a-bi}{a-bi} \ = \ \frac{a-bi}{a^2+b^2}.$$

Therefore

$$T(z^{-1}) \ = \ \frac{1}{a^2+b^2}\begin{bmatrix} a & b \\ -b & a \end{bmatrix} = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}^{-1} \ = \ T(z)^{-1}.$$

CHAPTER 5

HIGHER-ORDER LINEAR DIFFERENTIAL EQUATIONS

SECTION 5.1

INTRODUCTION: SECOND-ORDER LINEAR EQUATIONS

In this section the central ideas of the theory of linear differential equations are introduced and illustrated concretely in the context of **second-order** equations. These key concepts include superposition of solutions (Theorem 1), existence and uniqueness of solutions (Theorem 2), linear independence, the Wronskian (Theorem 3), and general solutions (Theorem 4). This discussion of second-order equations serves as preparation for the treatment of *n*th order linear equations in Section 5.2. Although the concepts in this section may seem somewhat abstract to students, the problems set is quite tangible and largely computational.

In each of Problems 1–16 the verification that y_1 and y_2 satisfy the given differential equation is a routine matter. As in Example 2, we then impose the given initial conditions on the general solution $y = c_1y_1 + c_2y_2$. This yields two linear equations that determine the values of the constants c_1 and c_2.

1. Imposition of the initial conditions $y(0) = 0,\ y'(0) = 5$ on the general solution $y(x) = c_1e^x + c_2e^{-x}$ yields the two equations $c_1 + c_2 = 0,\ c_1 - c_2 = 0$ with solution $c_1 = 5/2,\ c_2 = -5/2$. Hence the desired particular solution is $y(x) = 5(e^x - e^{-x})/2$.

2. Imposition of the initial conditions $y(0) = -1,\ y'(0) = 15$ on the general solution $y(x) = c_1e^{3x} + c_2e^{-3x}$ yields the two equations $c_1 + c_2 = -1,\ 3c_1 - 3c_2 = 15$ with solution $c_1 = 2,\ c_2 = 3$. Hence the desired particular solution is $y(x) = 2e^{3x} - 3e^{-3x}$.

3. Imposition of the initial conditions $y(0) = 3,\ y'(0) = 8$ on the general solution $y(x) = c_1\cos 2x + c_2\sin 2x$ yields the two equations $c_1 = 3,\ 2c_2 = 8$ with solution $c_1 = 3,\ c_2 = 4$. Hence the desired particular solution is $y(x) = 3\cos 2x + 4\sin 2x$.

4. Imposition of the initial conditions $y(0) = 10,\ y'(0) = -10$ on the general solution $y(x) = c_1\cos 5x + c_2\sin 5x$ yields the two equations $c_1 = 10,\ 5c_2 = -10$ with solution $c_1 = 3,\ c_2 = 4$. Hence the desired particular solution is $y(x) = 10\cos 5x - 2\sin 5x$.

5. Imposition of the initial conditions $y(0)=1,\ y'(0)=0$ on the general solution $y(x) = c_1e^x + c_2e^{2x}$ yields the two equations $c_1 + c_2 = 1,\ c_1 + 2c_2 = 0$ with solution $c_1 = 2,\ c_2 = -1$. Hence the desired particular solution is $y(x) = 2e^x - e^{2x}$.

6. Imposition of the initial conditions $y(0)=7,\ y'(0)=-1$ on the general solution $y(x) = c_1e^{2x} + c_2e^{-3x}$ yields the two equations $c_1 + c_2 = 7,\ 2c_1 - 3c_2 = -1$ with solution $c_1 = 4,\ c_2 = 3$. Hence the desired particular solution is $y(x) = 4e^{2x} + 3e^{-3x}$.

7. Imposition of the initial conditions $y(0)=-2,\ y'(0)=8$ on the general solution $y(x) = c_1 + c_2e^{-x}$ yields the two equations $c_1 + c_2 = -2,\ -c_2 = 8$ with solution $c_1 = 6,\ c_2 = -8$. Hence the desired particular solution is $y(x) = 6 - 8e^{-x}$.

8. Imposition of the initial conditions $y(0)=4,\ y'(0)=-2$ on the general solution $y(x) = c_1 + c_2e^{3x}$ yields the two equations $c_1 + c_2 = 4,\ 3c_2 = -2$ with solution $c_1 = 14/3,\ c_2 = 2/3$. Hence the desired particular solution is $y(x) = (14 - 2e^{3x})/3$.

9. Imposition of the initial conditions $y(0)=2,\ y'(0)=-1$ on the general solution $y(x) = c_1e^{-x} + c_2x\,e^{-x}$ yields the two equations $c_1 = 2,\ -c_1 + c_2 = -1$ with solution $c_1 = 2,\ c_2 = 1$. Hence the desired particular solution is $y(x) = 2e^{-x} + xe^{-x}$.

10. Imposition of the initial conditions $y(0)=3,\ y'(0)=13$ on the general solution $y(x) = c_1e^{5x} + c_2x\,e^{5x}$ yields the two equations $c_1 = 3,\ 5c_1 + c_2 = 13$ with solution $c_1 = 3,\ c_2 = -2$. Hence the desired particular solution is $y(x) = 3e^{5x} - 2xe^{5x}$.

11. Imposition of the initial conditions $y(0)=0,\ y'(0)=5$ on the general solution $y(x) = c_1e^x\cos x + c_2e^x\sin x$ yields the two equations $c_1 = 0,\ c_1 + c_2 = 5$ with solution $c_1 = 0,\ c_2 = 5$. Hence the desired particular solution is $y(x) = 5e^x \sin x$.

12. Imposition of the initial conditions $y(0)=2,\ y'(0)=0$ on the general solution $y(x) = c_1e^{-3x}\cos 2x + c_2e^{-3x}\sin 2x$ yields the two equations $c_1 = 2,\ -3c_1 + 2c_2 = 5$ with solution $c_1 = 2,\ c_2 = 3$. Hence the desired particular solution is $y(x) = e^{-3x}(2\cos 2x + 3\sin 2x)$.

13. Imposition of the initial conditions $y(1)=3,\ y'(1)=1$ on the general solution $y(x) = c_1x + c_2x^2$ yields the two equations $c_1 + c_2 = 3,\ c_1 + 2c_2 = 1$ with solution $c_1 = 5,\ c_2 = -2$. Hence the desired particular solution is $y(x) = 5x - 2x^2$.

14. Imposition of the initial conditions $y(2)=10,\ y'(2)=15$ on the general solution $y(x) = c_1x^2+c_2x^{-3}$ yields the two equations $4c_1+c_2/8=10,\ 4c_1-3c_2/16=15$ with solution $c_1=3,\ c_2=-16$. Hence the desired particular solution is $y(x) = 3x^2 - 16/x^3$.

15. Imposition of the initial conditions $y(1)=7,\ y'(1)=2$ on the general solution $y(x) = c_1x+c_2x\ln x$ yields the two equations $c_1=7,\ c_1+c_2=2$ with solution $c_1=7,\ c_2=-5$. Hence the desired particular solution is $y(x) = 7x - 5x\ln x$.

16. Imposition of the initial conditions $y(1)=2,\ y'(1)=3$ on the general solution $y(x) = c_1\cos(\ln x)+c_2\sin(\ln x)$ yields the two equations $c_1=2,\ c_2=3$. Hence the desired particular solution is $y(x) = 2\cos(\ln x) + 3\sin(\ln x)$.

17. If $y=c/x$ then $y'+y^2 = -c/x^2+c^2/x^2 = c(c-1)/x^2 \neq 0$ unless either $c=0$ or $c=1$.

18. If $y=cx^3$ then $yy'' = cx^3\cdot 6cx = 6c^2x^4 \neq 6x^4$ unless $c^2=1$.

19. If $y=1+\sqrt{x}$ then $yy''+(y')^2 = (1+\sqrt{x})(-x^{-3/2}/4)+(x^{-1/2}/2)^2 = -x^{-3/2}/4 \neq 0$.

20. Linearly dependent, because

$$f(x) = \pi = \pi(\cos^2 x + \sin^2 x) = \pi g(x)$$

21. Linearly independent, because $x^3=+x^2|x|$ if $x>0$, whereas $x^3=-x^2|x|$ if $x<0$.

22. Linearly independent, because $1+x=c(1+|x|)$ would require that $c = 1$ with $x = 0$, but $c = 0$ with $x = -1$. Thus there is no such constant c.

23. Linearly independent, because $f(x) = +g(x)$ if $x>0$, whereas $f(x) = -g(x)$ if $x<0$.

24. Linearly dependent, because $g(x) = 2f(x)$.

25. $f(x) = e^x\sin x$ and $g(x) = e^x\cos x$ are linearly independent, because $f(x) = k\,g(x)$ would imply that $\sin x = k\cos x$, whereas $\sin x$ and $\cos x$ are linearly independent.

26. To see that $f(x)$ and $g(x)$ are linearly independent, assume that $f(x) = c\,g(x)$, and then substitute both $x = 0$ and $x = \pi/2$.

27. Let $L[y] = y''+py'+qy$. Then $L[y_c] = 0$ and $L[y_p] = f$, so

$$L[y_c+y_p] = L[y_c]+[y_p] = 0+f = f.$$

28. If $y(x) = 1 + c_1\cos x + c_2\sin x$ then $y'(x) = -c_1\sin x + c_2\cos x$, so the initial conditions $y(0) = y'(0) = -1$ yield $c_1 = -2, c_2 = -1$. Hence $y = 1 - 2\cos x - \sin x$.

29. There is no contradiction because if the given differential equation is divided by x^2 to get the form in Equation (8) in the text, then the resulting functions $p(x) = -4/x$ and $q(x) = 6/x^2$ are not continuous at $x = 0$.

30. **(a)** $y_2 = x^3$ and $y_2 = |x^3|$ are linearly independent because $x^3 = c|x^3|$ would require that $c = 1$ with $x = 1$, but $c = -1$ with $x = -1$.

(b) The fact that $W(y_1, y_2) = 0$ everywhere does not contradict Theorem 3, because when the given equation is written in the required form

$$y'' - (3/x)y' + (3/x^2)y = 0,$$

the coefficient functions $p(x) = -3/x$ and $q(x) = 3/x^2$ are not continuous at $x = 0$.

31. $W(y_1, y_2) = -2x$ vanishes at $x = 0$, whereas if y_1 and y_2 were (linearly independent) solutions of an equation $y'' + py' + qy = 0$ with p and q both continuous on an open interval I containing $x = 0$, then Theorem 3 would imply that $W \neq 0$ on I.

32. **(a)** $W = y_1y_2' - y_1'y_2$, so

$$\begin{aligned} AW' &= A(y_1'y_2' + y_1y_2'' - y_1''y_2 - y_1'y_2') \\ &= y_1(Ay_2'') - y_2(Ay_1'') \\ &= y_1(-By_2' - Cy_2) - y_2(-By_1' - Cy_1) \\ &= -B(y_1y_2' - y_1'y_2) \end{aligned}$$

and thus $AW' = -BW$.

(b) Just separate the variables.

(c) Because the exponential factor is never zero.

In Problems 33–42 we give the characteristic equation, its roots, and the corresponding general solution.

33. $r^2 - 3r + 2 = 0$; $r = 1, 2$; $y(x) = c_1e^x + c_2e^{2x}$

34. $r^2 + 2r - 15 = 0$; $r = 3, -5$; $y(x) = c_1e^{-5x} + c_2e^{3x}$

35. $r^2 + 5r = 0$; $r = 0, -5$; $y(x) = c_1 + c_2e^{-5x}$

36. $2r^2+3r = 0; \quad r = 0, -3/2; \quad y(x) = c_1 + c_2e^{-3x/2}$

37. $2r^2-r-2 = 0; \quad r = 1, -1/2; \quad y(x) = c_1e^{-x/2} + c_2e^{x}$

38. $4r^2+8r+3 = 0; \quad r = -1/2, -3/2; \quad y(x) = c_1e^{-x/2} + c_2e^{-3x/2}$

39. $4r^2+4r+1 = 0; \quad r = -1/2, -1/2; \quad y(x) = (c_1 + c_2x)e^{-x/2}$

40. $9r^2-12r+4 = 0; \quad r = -2/3, -2/3; \quad y(x) = (c_1 + c_2x)e^{2x/3}$

41. $6r^2-7r-20 = 0; \quad r = -4/3, 5/2; \quad y(x) = c_1e^{-4x/3} + c_2e^{5x/2}$

42. $35r^2-r-12 = 0; \quad r = -4/7, 3/5; \quad y(x) = c_1e^{-4x/7} + c_2e^{3x/5}$

In Problems 43–48 we first write and simplify the equation with the indicated characteristic roots, and then write the corresponding differential equation.

43. $(r-0)(r+10) = r^2+10r = 0; \quad y''+10y' = 0$

44. $(r-10)(r+10) = r^2-100 = 0; \quad y''-100y = 0$

45. $(r+10)(r+10) = r^2+20r+100 = 0; \quad y''+20y'+100y = 0$

46. $(r-10)(r-100) = r^2-110r+1000 = 0; \quad y''-110y'+1000y = 0$

47. $(r-0)(r-0) = r^2 = 0; \quad y'' = 0$

48. $(r-1-\sqrt{2})(r-1+\sqrt{2}) = r^2-2r-1 = 0; \quad y''-2y'-y = 0$

49. The solution curve with $y(0)=1,\ y'(0)=6$ is $y(x) = 8e^{-x}-7e^{-2x}$. We find that $y'(x)=0$ when $x=\ln(7/4)$ so $e^{-x}=4/7$ and $e^{-2x}=16/49$. It follows that $y(\ln(7/4)) = 16/7$, so the high point on the curve is $(\ln(7/4)), 16/7) \approx (0.56, 2.29)$, which looks consistent with Fig. 3.1.6.

50. The two solution curves with $y(0)=a$ and $y(0)=b$ (as well as $y'(0)=1$) are

$$y = (2a+1)e^{-x}-(a+1)e^{-2x},$$
$$y = (2b+1)e^{-x}-(b+1)e^{-2x}.$$

Subtraction and then division by $a - b$ gives $2e^{-x} = e^{-2x}$, so it follows that $x = -\ln 2$. Now substitution in either formula gives $y = -2$, so the common point of intersection is $(-\ln 2, -2)$.

51. **(a)** The substitution $v = \ln x$ gives

$$y' = \frac{dy}{dx} = \frac{dy}{dv}\frac{dv}{dx} = \frac{1}{x}\frac{dy}{dv}$$

Then another differentiation using the chain rule gives

$$\begin{aligned} y'' &= \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{1}{x}\cdot\frac{dy}{dv}\right) \\ &= -\frac{1}{x^2}\cdot\frac{dy}{dv}+\frac{1}{x}\cdot\frac{d}{dv}\left(\frac{dy}{dv}\right)\frac{dv}{dx} = -\frac{1}{x^2}\cdot\frac{dy}{dv}+\frac{1}{x^2}\cdot\frac{d^2y}{dv^2}. \end{aligned}$$

Substitution of these expressions for y' and y'' into Eq. (21) in the text then yields immediately the desired Eq. (22):

$$a\frac{d^2y}{dv^2}+(b-a)\frac{dy}{dv}+c\,y = 0.$$

(b) If the roots r_1 and r_2 of the characteristic equation of Eq. (22) are real and distinct, then a general solution of the original Euler equation is

$$y(x) = c_1e^{r_1v}+c_2e^{r_2v} = c_1\left(e^v\right)^{r_1}+c_2\left(e^v\right)^{r_2} = c_1x^{r_1}+c_2x^{r_2}.$$

52. The substitution $v = \ln x$ yields the converted equation $d^2y/dv^2 - y = 0$ whose characteristic equation $r^2 - 1 = 0$ has roots $r_1 = 1$ and $r_2 = -1$. Because $e^v = x$, the corresponding general solution is

$$y = c_1e^v + c_2e^{-v} = c_1x+\frac{c_2}{x}.$$

53. The substitution $v = \ln x$ yields the converted equation $d^2y/dv^2 + dy/dv - 12y = 0$ whose characteristic equation $r^2 + r - 12 = 0$ has roots $r_1 = -4$ and $r_2 = 3$. Because $e^v = x$, the corresponding general solution is

$$y = c_1e^{-4v} + c_2e^{3v} = c_1x^{-4} + c_2x^3.$$

54. The substitution $v = \ln x$ yields the converted equation $4\,d^2y/dv^2 + 4\,dy/dv - 3y = 0$ whose characteristic equation $4r^2 + 4r - 3 = 0$ has roots $r_1 = -3/2$ and $r_2 = 1/2$. Because $e^v = x$, the corresponding general solution is

$$y = c_1 e^{-3v/2} + c_2 e^{v/2} = c_1 x^{-3/2} + c_2 x^{1/2}.$$

55. The substitution $v = \ln x$ yields the converted equation $d^2y/dv^2 = 0$ whose characteristic equation $r^2 = 0$ has repeated roots $r_1, r_2 = 0$. Because $v = \ln x$, the corresponding general solution is

$$y = c_1 + c_2 v = c_1 + c_2 \ln x.$$

56. The substitution $v = \ln x$ yields the converted equation $d^2y/dv^2 - 4\,dy/dv + 4y = 0$ whose characteristic equation $r^2 - 4r + 4 = 0$ has roots $r_1, r_2 = 2$. Because $e^v = x$, the corresponding general solution is

$$y = c_1 e^{2v} + c_2 v\, e^{2v} = x^2(c_1 + c_2 \ln v).$$

SECTION 5.2

GENERAL SOLUTIONS OF LINEAR EQUATIONS

Students should check each of Theorems 1 through 4 in this section to see that, in the case $n = 2$, it reduces to the corresponding theorem in Section 5.1. Similarly, the computational problems for this section largely parallel those for the previous section. By the end of Section 5.2 students should understand that, although we do not prove the existence-uniqueness theorem now, it provides the basis for everything we do with linear differential equations.

The linear combinations listed in Problems 1–6 were discovered "by inspection" — that is, by trial and error.

1. $(5/2)(2x) + (-8/3)(3x^2) + (-1)(5x - 8x^2) = 0$

2. $(-4)(5) + (5)(2 - 3x^2) + (1)(10 + 15x^2) = 0$

3. $(1)(0) + (0)(\sin x) + (0)(e^x) = 0$

4. $(1)(17) + (-17/2)(2 \sin^2 x) + (-17/3)(3 \cos^2 x) = 0$, because $\sin^2 x + \cos^2 x = 1$.

5. $(1)(17) + (-34)(\cos^2 x) + (17)(\cos 2x) = 0$, because $2\cos^2 x = 1 + \cos 2x$.

6. $(-1)(e^x) + (1)(\cosh x) + (1)(\sinh x) = 0$, because $\cosh x = (e^x + e^{-x})/2$ and $\sinh x = (e^x - e^{-x})/2$.

7. $W = \begin{vmatrix} 1 & x & x^2 \\ 0 & 1 & 2x \\ 0 & 0 & 2 \end{vmatrix} = 2$ is nonzero everywhere.

8. $W = \begin{vmatrix} e^x & e^{2x} & e^{3x} \\ e^x & 2e^{2x} & 3e^{3x} \\ e^x & 4e^{2x} & 9e^{3x} \end{vmatrix} = 2e^{6x}$ is never zero.

9. $W = e^x(\cos^2 x + \sin^2 x) = e^x \neq 0$

10. $W = x^{-7}e^x(x+1)(x+4)$ is nonzero for $x > 0$.

11. $W = x^3 e^{2x}$ is nonzero if $x \neq 0$.

12. $W = x^{-2}[2\cos^2(\ln x) + 2\sin^2(\ln x)] = 2x^{-2}$ is nonzero for $x > 0$.

In each of Problems 13 - 20 we first form the general solution

$$y(x) = c_1y_1(x) + c_2y_2(x) + c_3y_3(x),$$

then calculate $y'(x)$ and $y''(x)$, and finally impose the given initial conditions to determine the values of the coefficients c_1, c_2, c_3.

13. Imposition of the initial conditions $y(0) = 1,\ y'(0) = 2,\ y''(0) = 0$ on the general solution $y(x) = c_1e^x + c_2e^{-x} + c_3e^{-2x}$ yields the three equations

$$c_1 + c_2 + c_3 = 1, \quad c_1 - c_2 - 2c_3 = 2, \quad c_1 + c_2 + 4c_3 = 0$$

with solution $c_1 = 4/3,\ c_2 = 0,\ c_3 = -1/3$. Hence the desired particular solution is given by $y(x) = (4e^x - e^{-2x})/3$.

14. Imposition of the initial conditions $y(0) = 0,\ y'(0) = 0,\ y''(0) = 3$ on the general solution $y(x) = c_1e^x + c_2e^{2x} + c_3e^{3x}$ yields the three equations

$$c_1 + c_2 + c_3 = 1, \quad c_1 + 2c_2 + 3c_3 = 2, \quad c_1 + 4c_2 + 9c_3 = 0$$

with solution $c_1 = 3/2,\ c_2 = -3,\ c_3 = 3/2$. Hence the desired particular solution is given by $y(x) = (3e^x - 6e^{2x} + 3e^{3x})/2$.

15. Imposition of the initial conditions $y(0) = 2,\ y'(0) = 0,\ y''(0) = 0$ on the general solution $y(x) = c_1e^x + c_2x\,e^x + c_3x^2e^{3x}$ yields the three equations

$$c_1 = 2, \quad c_1 + c_2 = 0, \quad c_1 + 2c_2 + 2c_3 = 0$$

with solution $c_1 = 2,\ c_2 = -2,\ c_3 = 1$. Hence the desired particular solution is given by $y(x) = (2 - 2x + x^2)e^x$.

16. Imposition of the initial conditions $y(0) = 1,\ y'(0) = 4,\ y''(0) = 0$ on the general solution $y(x) = c_1 e^x + c_2 e^{2x} + c_3 x e^{2x}$ yields the three equations

$$c_1 + c_2 = 1, \quad c_1 + 2c_2 + c_3 = 4, \quad c_1 + 4c_2 + 4c_3 = 0$$

with solution $c_1 = -12,\ c_2 = 13,\ c_3 = -10$. Hence the desired particular solution is given by $y(x) = -12e^x + 13e^{2x} - 10xe^{2x}$.

17. Imposition of the initial conditions $y(0) = 3,\ y'(0) = -1,\ y''(0) = 2$ on the general solution $y(x) = c_1 + c_2 \cos 3x + c_3 \sin 3x$ yields the three equations

$$c_1 + c_2 = 3, \quad 3c_3 = -1, \quad -9c_2 = 2$$

with solution $c_1 = 29/9,\ c_2 = -2/9,\ c_3 = -1/3$. Hence the desired particular solution is given by $y(x) = (29 - 2\cos 3x - 3\sin 3x)/9$.

18. Imposition of the initial conditions $y(0) = 1,\ y'(0) = 0,\ y''(0) = 0$ on the general solution $y(x) = e^x\left(c_1 + c_2 \cos x + c_3 \sin x\right)$ yields the three equations

$$c_1 + c_2 = 1, \quad c_1 + c_2 + c_3 = 0, \quad c_1 + 2c_3 = 0$$

with solution $c_1 = 2,\ c_2 = -1,\ c_3 = -1$. Hence the desired particular solution is given by $y(x) = e^x(2 - \cos x - \sin x)$.

19. Imposition of the initial conditions $y(1) = 6,\ y'(1) = 14,\ y''(1) = 22$ on the general solution $y(x) = c_1 x + c_2 x^2 + c_3 x^3$ yields the three equations

$$c_1 + c_2 + c_3 = 6, \quad c_1 + 2c_2 + 3c_3 = 14, \quad 2c_2 + 6c_3 = 22$$

with solution $c_1 = 1,\ c_2 = 2,\ c_3 = 3$. Hence the desired particular solution is given by $y(x) = x + 2x^2 + 3x^3$.

20. Imposition of the initial conditions $y(1) = 1,\ y'(1) = 5,\ y''(1) = -11$ on the general solution $y(x) = c_1 x + c_2 x^{-2} + c_3 x^{-2} \ln x$ yields the three equations

$$c_1 + c_2 = 1, \quad c_1 - 2c_2 + c_3 = 5, \quad 6c_2 - 5c_3 = -11$$

with solution $c_1 = 2,\ c_2 = -1,\ c_3 = 1$. Hence the desired particular solution is given by $y(x) = 2x - x^{-2} + x^{-2}\ln x$.

In each of Problems 21-24 we first form the general solution

$$y(x) = y_c(x) + y_p(x) = c_1y_1(x) + c_2y_2(x) + y_p(x),$$

then calculate $y'(x)$, and finally impose the given initial conditions to determine the values of the coefficients c_1 and c_2.

21. Imposition of the initial conditions $y(0) = 2,\ y'(0) = -2$ on the general solution $y(x) = c_1 \cos x + c_2 \sin x + 3x$ yields the two equations $c_1 = 2,\ c_2 + 3 = -2$ with solution $c_1 = 2,\ c_2 = -5$. Hence the desired particular solution is given by $y(x) = 2 \cos x - 5 \sin x + 3x$.

22. Imposition of the initial conditions $y(0) = 0,\ y'(0) = 10$ on the general solution $y(x) = c_1e^{2x} + c_2e^{-2x} - 3$ yields the two equations $c_1 + c_2 - 3 = 0,\ 2c_1 - 2c_2 = 10$ with solution $c_1 = 4,\ c_2 = -1$. Hence the desired particular solution is given by $y(x) = 4e^{2x} - e^{-2x} - 3$.

23. Imposition of the initial conditions $y(0) = 3,\ y'(0) = 11$ on the general solution $y(x) = c_1e^{-x} + c_2e^{3x} - 2$ yields the two equations $c_1 + c_2 - 2 = 3,\ -c_1 + 3c_2 = 11$ with solution $c_1 = 1,\ c_2 = 4$. Hence the desired particular solution is given by $y(x) = e^{-x} + 4e^{3x} - 2$.

24. Imposition of the initial conditions $y(0) = 4,\ y'(0) = 8$ on the general solution $y(x) = c_1e^x \cos x + c_2e^x \cos x + x + 1$ yields the two equations $c_1 + 1 = 4,\ c_1 + c_2 + 1 = 8$ with solution $c_1 = 3,\ c_2 = 4$. Hence the desired particular solution is given by $y(x) = e^x(3 \cos x + 4 \sin x) + x + 1$.

25. $L[y] = L[y_1 + y_2] = L[y_1] + L[y_2] = f + g$

26. **(a)** $y_1 = 2$ and $y_2 = 3x$ **(b)** $y = y_1 + y_2 = 2 + 3x$

27. The equations

$$c_1 + c_2x + c_3x^2 = 0, \qquad c_2 + 2c_3x + 0, \qquad 2c_3 = 0$$

(the latter two obtained by successive differentiation of the first one) evidently imply — by substituting $x = 0$ — that $c_1 = c_2 = c_3 = 0$.

28. If you differentiate the equation $c_0 + c_1x + c_2x^2 + \cdots + c_nx^n = 0$ repeatedly, n times in succession, the result is the system

$$\begin{aligned} c_0 + c_1x + c_2x^2 + \cdots + c_nx^n &= 0 \\ c_1 + 2c_2x + \cdots + nc_nx^{n-1} &= 0 \\ &\vdots \\ (n-1)!c_{n-1} + n!c_nx &= 0 \\ n!c_n &= 0 \end{aligned}$$

of $n+1$ equations in the $n+1$ coefficients $c_0, c_1, c_2, \cdots, c_n$. Upon substitution of $x = 0$, the $(k+1)$st of these equations reduces to $k!c_k = 0$, so it follows that all these coefficients must vanish.

29. If $c_0e^{rx} + c_1xe^{rx} + \cdots + c_nx^ne^{rx} = 0$, then division by e^{rx} yields

$$c_0 + c_1x + \cdots + c_nx^n = 0,$$

so the result of Problem 28 applies.

30. When the equation $x^2y'' - 2xy' + 2y = 0$ is rewritten in standard form

$$y'' + (-2/x)y' + (2/x^2)y = 0,$$

the coefficient functions $p_1(x) = -2/x$ and $p_2(x) = 2/x^2$ are not continuous at $x = 0$. Thus the hypotheses of Theorem 3 are not satisfied.

31. **(a)** Substitution of $x = a$ in the differential equation gives $y''(a) = -p\,y'(a) - q(a)$.

(b) If $y(0) = 1$ and $y'(0) = 0$, then the equation $y'' - 2y' - 5y = 0$ implies that $y''(0) = 2y'(0) + 5y(0) = 5$.

32. Let the functions $y_1, y_2, \cdots, y_n$ be chosen as indicated. Then evaluation at $x = a$ of the $(k-1)$st derivative of the equation $c_1y_1 + c_2y_2 + \cdots c_ny_n = 0$ yields $c_k = 0$. Thus $c_1 = c_2 = \cdots = c_n = 0$, so the functions are linearly independent.

33. This follows from the fact that

$$\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} = (b-a)(c-b)(c-a).$$

34. $W(f_1, f_2, \cdots, f_n) = V \exp(r_i x)$, and neither V nor $\exp(r_i x)$ vanishes.

36. If $y = vy_1$ then substitution of the derivatives

$$y' = vy_1' + v'y_1, \qquad y'' = vy_1'' + 2v'y_1' + v''y_1$$

in the differential equation $y'' + py' + qy = 0$ gives

$$\begin{aligned} [vy_1'' + 2v'y_1' + v''y_1] + p[vy_1' + v'y_1] + q[vy_1] &= 0, \\ v[y_1'' + py_1' + qy_1] + v''y_1 + 2v'y_1' + pv'y_1 &= 0. \end{aligned}$$

But the terms within brackets vanish because y_1 is a solution, and this leaves the equation

$$y_1 v'' + (2y_1' + py_1)v' = 0$$

that we can solve by writing

$$\begin{aligned} \frac{v''}{v'} &= -2\frac{y_1'}{y_1} - p \quad \Rightarrow \quad \ln v' = -2\ln y_1' - \int p(x)\,dx + \ln C, \\ v'(x) &= \frac{C}{y_1^2} e^{-\int p(x)dx} \quad \Rightarrow \quad v(x) = C\int \frac{e^{-\int p(x)dx}}{y_1^2}dx + K. \end{aligned}$$

With $C = 1$ and $K = 0$ this gives the second solution

$$y_2(x) = y_1(x)\int \frac{e^{-\int p(x)dx}}{y_1^2}dx.$$

37. When we substitute $y = vx^3$ in the given differential equation and simplify, we get the separable equation $xv'' + v' = 0$ that we solve by writing

$$\begin{aligned} \frac{v''}{v'} &= -\frac{1}{x} \quad \Rightarrow \quad \ln v' = -\ln x + \ln A, \\ v' &= \frac{A}{x} \quad \Rightarrow \quad v(x) = A\ln x + B. \end{aligned}$$

With $A = 1$ and $B = 0$ we get $v(x) = \ln x$ and hence $y_2(x) = x^3 \ln x$.

38. When we substitute $y = vx^3$ in the given differential equation and simplify, we get the separable equation $xv'' + 7v' = 0$ that we solve by writing

$$\begin{aligned} \frac{v''}{v'} &= -\frac{7}{x} &\Rightarrow\quad \ln v' &= -7\ln x + \ln A, \\ v' &= \frac{A}{x^7} &\Rightarrow\quad v(x) &= -\frac{A}{6x^6} + B. \end{aligned}$$

With $A = -6$ and $B = 0$ we get $v(x) = 1/x^6$ and hence $y_2(x) = 1/x^3$.

39. When we substitute $y = v e^{x/2}$ in the given differential equation and simplify, we eventually get the simple equation $v'' = 0$ with general solution $v(x) = Ax + B$. With $A = 1$ and $B = 0$ we get $v(x) = x$ and hence $y_2(x) = x e^{x/2}$.

40. When we substitute $y = vx$ in the given differential equation and simplify, we get the separable equation $v'' - v' = 0$ that we solve by writing

$$\begin{aligned} \frac{v''}{v'} &= 1 &\Rightarrow\quad \ln v' &= x + \ln A, \\ v' &= Ae^x &\Rightarrow\quad v(x) &= Ae^x + B. \end{aligned}$$

With $A = 1$ and $B = 0$ we get $v(x) = e^x$ and hence $y_2(x) = xe^x$.

41. When we substitute $y = v e^x$ in the given differential equation and simplify, we get the separable equation $(1+x)v'' + x v' = 0$ that we solve by writing

$$\begin{aligned} \frac{v''}{v'} &= -\frac{x}{1+x} = -1 + \frac{1}{1+x} &\Rightarrow\quad \ln v' &= -x + \ln(1+x) + \ln A, \\ v' &= A(1+x)e^{-x} &\Rightarrow\quad v(x) &= A\int(1+x)e^{-x}\,dx = -A(2+x)e^{-x} + B. \end{aligned}$$

With $A = -1$ and $B = 0$ we get $v(x) = (2+x)e^{-x}$ and hence $y_2(x) = 2 + x$.

42. When we substitute $y = v x$ in the given differential equation and simplify, we get the separable equation $x(x^2 - 1)v'' = 2v'$ that we solve by writing

$$\frac{v''}{v'} = \frac{2}{x(x^2-1)} = -\frac{2}{x}+\frac{1}{1+x}-\frac{1}{1-x},$$
$$\ln v' = -2\ln x + \ln(1+x) + \ln(1-x) + \ln A,$$
$$v' = \frac{A(1-x^2)}{x^2} = A\left(\frac{1}{x^2}-1\right) \Rightarrow \quad v(x) = A\left(-\frac{1}{x}-x\right)+B.$$

With $A=-1$ and $B=0$ we get $v(x)=x+1/x$ and hence $y_2(x)=x^2+1$.

43. When we substitute $y = vx$ in the given differential equation and simplify, we get the separable equation $x(x^2-1)v'' = (2-4x^2)v'$ that we solve by writing

$$\frac{v''}{v'} = \frac{2-4x^2}{x(x^2-1)} = -\frac{2}{x}-\frac{1}{1+x}+\frac{1}{1-x},$$
$$\ln v' = -2\ln x - \ln(1+x) - \ln(1-x) + \ln A,$$
$$v' = \frac{A}{x^2(1-x^2)} = A\left(\frac{1}{x^2}+\frac{1}{2(1+x)}+\frac{1}{2(1-x)}\right),$$
$$v(x) = A\left(-\frac{1}{x}+\frac{1}{2}\ln(1+x)-\frac{1}{2}\ln(1-x)\right)+B.$$

With $A=-1$ and $B=0$ we get

$$v(x)=\frac{1}{x}-\frac{1}{2}\ln(1+x)+\frac{1}{2}\ln(1-x) \quad \Rightarrow \quad y_2(x) = 1-\frac{x}{2}\ln\frac{1+x}{1-x}.$$

44. When we substitute $y = vx^{-1/2}\cos x$ in the given differential equation and simplify, we eventually get the separable equation $(\cos x)v'' = 2(\sin x)v'$ that we solve by writing

$$\frac{v''}{v'} = \frac{2\sin x}{\cos x} \quad \Rightarrow \quad \ln v' = -2\ln|\cos x| + \ln A = \ln\sec^2 x + \ln A,$$
$$v' = A\sec^2 x \quad \Rightarrow \quad v(x) = A\tan x + B.$$

With $A=1$ and $B=0$ we get $v(x) = \tan x$ and hence

$$y_2(x) = (\tan x)(x^{-1/2}\cos x) = x^{-1/2}\sin x.$$

SECTION 5.3

HOMOGENEOUS EQUATIONS WITH CONSTANT COEFFICIENTS

This is a purely computational section devoted to the single most widely applicable type of higher order differential equations — linear ones with constant coefficients. In Problems 1–20, we write first the characteristic equation and its list of roots, then the corresponding general solution of the given differential equation. Explanatory comments are included only when the solution of the characteristic equation is not routine.

1. $r^2 - 4 = (r-2)(r+2) = 0; \quad r = -2, 2; \quad y(x) = c_1 e^{2x} + c_2 e^{-2x}$

2. $2r^2 - 3r = r(2r-3) = 0; \quad r = 0, 3/2; \quad y(x) = c_1 + c_2 e^{3x/2}$

3. $r^2 + 3r - 10 = (r+5)(r-2) = 0; \quad r = -5, 2; \quad y(x) = c_1 e^{2x} + c_2 e^{-5x}$

4. $2r^2 - 7r + 3 = (2r-1)(r-3) = 0; \quad r = 1/2, 3; \quad y(x) = c_1 e^{x/2} + c_2 e^{3x}$

5. $r^2 + 6r + 9 = (r+3)^2 = 0; \quad r = -3, -3; \quad y(x) = c_1 e^{-3x} + c_2 x e^{-3x}$

6. $r^2 + 5r + 5 = 0; \quad r = \left(-5 \pm \sqrt{5}\right)/2$
$y(x) = e^{-5x/2}[c_1 \exp(x\sqrt{5}/2) + c_2 \exp(-x\sqrt{5}/2)]$

7. $4r^2 - 12r + 9 = (2r-3)^2 = 0; \quad r = -3/2, -3/2; \quad y(x) = c_1 e^{3x/2} + c_2 x e^{3x/2}$

8. $r^2 - 6r + 13 = 0; \quad r = \left(6 \pm \sqrt{-16}\right)/2 = 3 \pm 2i; \quad y(x) = e^{3x}(c_1 \cos 2x + c_2 \sin 2x)$

9. $r^2 + 8r + 25 = 0; \quad r = \left(-8 \pm \sqrt{-36}\right)/2 = -4 \pm 3i; \quad y(x) = e^{-4x}(c_1 \cos 3x + c_2 \sin 3x)$

10. $5r^4 + 3r^3 = r^3(5r+3) = 0; \quad r = 0, 0, 0, -3/5; \quad y(x) = c_1 + c_2 x + c_3 x^2 + c_4 e^{-3x/5}$

11. $r^4 - 8r^3 + 16r^2 = r^2(r-4)^2 = 0; \quad r = 0, 0, 4, 4; \quad y(x) = c_1 + c_2 x + c_3 e^{4x} + c_4 x e^{4x}$

12. $r^4 - 3r^3 + 3r^2 - r = r(r-1)^3 = 0; \quad r = 0, 1, 1, 1; \quad y(x) = c_1 + c_2 e^x + c_3 x e^x + c_4 x^2 e^x$

13. $9r^3 + 12r^2 + 4r = r(3r+2)^2 = 0; \quad r = 0, -2/3, -2/3$
$y(x) = c_1 + c_2 e^{-2x/3} + c_3 x e^{-2x/3}$

14. $r^4 + 3r^2 - 4 = \left(r^2 - 1\right)\left(r^2 + 4\right) = 0; \quad r = -1, 1, \pm 2i$

$y(x) = c_1e^x + c_2e^{-x} + c_3\cos 2x + c_4\sin 2x$

15. $4r^4 - 8r^2 + 16 = \left(r^2 - 4\right)^2 = (r-2)^2(r+2)^2 = 0; \quad r = 2, 2, -2, -2$

$y(x) = c_1e^{2x} + c_2xe^{2x} + c_3e^{-2x} + c_4xe^{-2x}$

16. $r^4 + 18r^2 + 81 = \left(r^2 + 9\right)^2 = 0; \quad r = \pm 3i, \pm 3i$

$y(x) = (c_1 + c_2x)\cos 3x + (c_3 + c_4x)\sin 3x$

17. $6r^4 + 11r^2 + 4 = (2r^2+1)(3r^2+4) = 0; \quad r = \pm i/\sqrt{2}, \pm 2i/\sqrt{3},$

$y(x) = c_1\cos(x/\sqrt{2}\,) + c_2\sin(x/\sqrt{2}\,) + c_3\cos(2x/\sqrt{3}\,) + c_4\sin(2x/\sqrt{3}\,)$

18. $r^4 - 16 = \left(r^2 - 4\right)\left(r^2 + 4\right) = 0; \quad r = -2, 2, \pm 2i$

$y(x) = c_1e^{2x} + c_2e^{-2x} + c_3\cos 2x + c_4\sin 2x$

19. $r^3 + r^2 - r - 1 = r\left(r^2 - 1\right) + \left(r^2 - 1\right) = \left(r-1\right)\left(r+1\right)^2 = 0; \quad r = 1, -1, -1;$

$y(x) = c_1e^x + c_2e^{-x} + c_3xe^{-x}$

20. $r^4 + 2r^3 + 3r^2 + 2r + 1 = (r^2 + r + 1)^2 = 0; \quad \left(-1 \pm \sqrt{3}\,i\right)/2, \ \left(-1 \pm \sqrt{3}\,i\right)/2$

$y = e^{-x/2}(c_1 + c_2x)\cos(x\sqrt{3}/2) + e^{-x/2}(c_3 + c_4x)\sin(x\sqrt{3}/2)$

21. Imposition of the initial conditions $y(0) = 7,\ y'(0) = 11$ on the general solution $y(x) = c_1e^x + c_2e^{3x}$ yields the two equations $c_1 + c_2 = 7,\ c_1 + 3c_2 = 11$ with solution $c_1 = 5,\ c_2 = 2$. Hence the desired particular solution is $y(x) = 5e^x + 2e^{3x}$.

22. Imposition of the initial conditions $y(0) = 3,\ y'(0) = 4$ on the general solution $y(x) = e^{-x/3}\left[c_1\cos\left(x/\sqrt{3}\right) + c_2\sin\left(x/\sqrt{3}\right)\right]$ yields the two equations $c_1 = 3,\ -c_1/3 + c_2/\sqrt{3} = 4$ with solution $c_1 = 3,\ c_2 = 5\sqrt{3}$. Hence the desired particular solution is $y(x) = e^{-x/3}\left[3\cos\left(x/\sqrt{3}\right) + 5\sqrt{3}\sin\left(x/\sqrt{3}\right)\right]$.

23. Imposition of the initial conditions $y(0) = 3,\ y'(0) = 1$ on the general solution $y(x) = e^{3x}\left(c_1\cos 4x + c_2\sin 4x\right)$ yields the two equations $c_1 = 3,\ 3c_1 + 4c_2 = 1$ with solution $c_1 = 3,\ c_2 = -2$. Hence the desired particular solution is $y(x) = e^{3x}(3\cos 4x - 2\sin 4x)$.

24. Imposition of the initial conditions $y(0)=1,\ y'(0)=-1,\ y''(0)=3$ on the general solution $y(x) = c_1+c_2e^{2x}+c_3e^{-x/2}$ yields the three equations

$$c_1+c_2+c_3=1, \quad 2c_2-c_3/2=-1, \quad 4c_2+c_3/4=3$$

with solution $c_1=-7/2,\ c_2=1/2,\ c_3=4$. Hence the desired particular solution is $y(x) = (-7+e^{2x}+8e^{-x/2})/2$.

25. Imposition of the initial conditions $y(0)=-1,\ y'(0)=0,\ y''(0)=1$ on the general solution $y(x) = c_1+c_2x+c_3e^{-2x/3}$ yields the three equations

$$c_1+c_3=-1, \quad c_2-2c_3/3=0, \quad 4c_3/9=1$$

with solution $c_1=-13/4,\ c_2=3/2,\ c_3=9/4$. Hence the desired particular solution is $y(x) = (-13+6x+9e^{-2x/3})/4$.

26. Imposition of the initial conditions $y(0)=1,\ y'(0)=-1,\ y''(0)=3$ on the general solution $y(x) = c_1+c_2e^{-5x}+c_3x\,e^{-5x}$ yields the three equations

$$c_1+c_2=3, \quad -5c_2+c_3=4, \quad 25c_2-10c_3=5$$

with solution $c_1=24/5,\ c_2=-9/5,\ c_3=-5$. Hence the desired particular solution is $y(x) = (24-9e^{-5x}-25xe^{-5x})/5$.

27. First we spot the root $r = 1$. Then long division of the polynomial r^3+3r^2-4 by $r-1$ yields the quadratic factor $r^2+4r+4 = (r+2)^2$ with roots $r = -2, -2$. Hence the general solution is $y(x) = c_1e^{x}+c_2e^{-2x}+c_3xe^{-2x}$.

28. First we spot the root $r = 2$. Then long division of the polynomial $2r^3-r^2-5r-2$ by the factor $r-2$ yields the quadratic factor $2r^2+3r+1 = (2r+1)(r+1)$ with roots $r = -1, -1/2$. Hence the general solution is $y(x) = c_1e^{2x}+c_2e^{-x}+c_3e^{-x/2}$.

29. First we spot the root $r = -3$. Then long division of the polynomial r^3+27 by $r+3$ yields the quadratic factor r^2-3r+9 with roots $r=3\left(1\pm i\sqrt{3}\right)/2$. Hence the general solution is $y(x) = c_1e^{-3x}+e^{3x/2}[c_2\cos(3x\sqrt{3}/2)+c_3\sin(3x\sqrt{3}/2)]$.

30. First we spot the root $r = -1$. Then long division of the polynomial

$$r^4-r^3+r^2-3r-6$$

by $r + 1$ yields the cubic factor $r^3 - 2r^2 + 3r - 6$. Next we spot the root $r = 2$, and another long division yields the quadratic factor $r^2 + 3$ with roots $r = \pm i\sqrt{3}$. Hence the general solution is $y(x) = c_1e^{-x} + c_2e^{2x} + c_3\cos x\sqrt{3} + c_4\sin x\sqrt{3}$.

31. The characteristic equation $r^3 + 3r^2 + 4r - 8 = 0$ has the evident root $r = 1$, and long division then yields the quadratic factor $r^2 + 4r + 8 = (r + 2)^2 + 4$ corresponding to the complex conjugate roots $-2 \pm 2\,i$. Hence the general solution is

$$y(x) = c_1e^x + e^{-2x}(c_2\cos 2x + c_3\sin 2x).$$

32. The characteristic equation $r^4 + r^3 - 3r^2 - 5r - 2 = 0$ has root $r = 2$ that is readily found by trial and error, and long division then yields the factorization

$$(r - 2)(r + 1)^3 = 0.$$

Thus we obtain the general solution $y(x) = c_1e^{2x} + (c_2 + c_3x + c_4x^2)e^{-x}$.

33. Knowing that $y = e^{3x}$ is one solution, we divide the characteristic polynomial $r^3 + 3r^2 - 54$ by $r - 3$ and get the quadratic factor

$$r^2 + 6r + 18 = (r + 3)^2 + 9.$$

Hence the general solution is $y(x) = c_1e^{3x} + e^{-3x}(c_2\cos 3x + c_3\sin 3x)$.

34. Knowing that $y = e^{2x/3}$ is one solution, we divide the characteristic polynomial $3r^3 - 2r^2 + 12r - 8$ by $3r - 2$ and get the quadratic factor $r^2 + 4$. Hence the general solution is

$$y(x) = c_1e^{2x/3} + c_2\cos 2x + c_3\sin 2x.$$

35. The fact that $y = \cos 2x$ is one solution tells us that $r^2 + 4$ is a factor of the characteristic polynomial

$$6r^4 + 5r^3 + 25r^2 + 20r + 4.$$

Then long division yields the quadratic factor $6r^2 + 5r + 1 = (3r + 1)(2r + 1)$ with roots $r = -1/2, -1/3$. Hence the general solution is

$$y(x) = c_1e^{-x/2} + c_2e^{-x/3} + c_3\cos 2x + c_4\sin 2x$$

36. The fact that $y = e^{-x}\sin x$ is one solution tells us that $(r + 1)^2 + 1 = r^2 + 2r + 2$ is a factor of the characteristic polynomial

$$9r^3 + 11r^2 + 4r - 14.$$

Then long division yields the linear factor $9r - 7$. Hence the general solution is

$$y(x) = c_1 e^{7x/9} + e^{-x}(c_2 \cos x + c_3 \sin x).$$

37. The characteristic equation is $r^4 - r^3 = r^3(r-1) = 0$, so the general solution is $y(x) = A + Bx + Cx^2 + De^x$. Imposition of the given initial conditions yields the equations

$$A + D = 18, \quad B + D = 12, \quad 2C + D = 13, \quad D = 7$$

with solution $A = 11,\ B = 5,\ C = 3,\ D = 7.$ Hence the desired particular solution is

$$y(x) = 11 + 5x + 3x^2 + 7e^x.$$

38. Given that $r = 5$ is one characteristic root, we divide $(r-5)$ into the characteristic polynomial $r^3 - 5r^2 + 100r - 500$ and get the remaining factor $r^2 + 100$. Thus the general solution is

$$y(x) = Ae^{5x} + B\cos 10x + C\sin 10x.$$

Imposition of the given initial conditions yields the equations

$$A + B = 0, \quad 5A + 10C = 10, \quad 25A - 100B = 250$$

with solution $A = 2,\ B = -2,\ C = 0.$ Hence the desired particular solution is $y(x) = 2e^{5x} - 2\cos 10x$.

39. $(r-2)^3 = r^3 - 6r^2 + 12r - 8$, so the differential equation is

$$y''' - 6y'' + 12y' - 8y = 0.$$

40. $(r-2)(r^2+4) = r^3 - 2r^2 + 4r - 8$, so the differential equation is

$$y''' - 2y'' + 4y' - 8y = 0.$$

41. $(r^2+4)(r^2-4) = r^4 - 16$, so the differential equation is $y^{(4)} - 16y = 0$.

42. $(r^2+4)^3 = r^6 + 12r^4 + 48r^2 + 64$, so the differential equation is

$$y^{(6)} + 12y^{(4)} + 48y'' + 64y = 0.$$

44. **(a)** $x = i, -2i$ **(b)** $x = -i, 3i$

45. The characteristic polynomial is the quadratic polynomial of Problem 44(b). Hence the general solution is

$$y(x) = c_1 e^{-ix} + c_2 e^{3ix} = c_1(\cos x - i\sin x) + c_2(\cos 3x + i\sin 3x).$$

46. The characteristic polynomial is $r^2 - ir + 6 = (r+2i)(r-3i)$ so the general solution is

$$y(x) = c_1 e^{3ix} + c_2 e^{-2ix} = c_1(\cos 3x + i\sin 3x) + c_2(\cos 2x - i\sin 2x).$$

47. The characteristic roots are $r = \pm\sqrt{-2+2i\sqrt{3}} = \pm(1+i\sqrt{3})$ so the general solution is

$$y(x) = c_1 e^{(1+i\sqrt{3})x} + c_2 e^{-(1+i\sqrt{3})x} = c_1 e^{x}\left(\cos\sqrt{3}\,x + i\sin\sqrt{3}\,x\right) + c_2 e^{-x}\left(\cos\sqrt{3}\,x - i\sin\sqrt{3}\,x\right)$$

48. The general solution is $y(x) = Ae^{x} + Be^{\alpha x} + Ce^{\beta x}$ where $\alpha = (-1 + i\sqrt{3})/2$ and $\beta = (-1 - i\sqrt{3})/2$. Imposition of the given initial conditions yields the equations

$$\begin{aligned} A + B + C &= 1 \\ A + \alpha B + \beta C &= 0 \\ A + \alpha^2 B + \beta^2 C &= 0 \end{aligned}$$

that we solve for $A = B = C = 1/3$. Thus the desired particular solution is given by $y(x) = \frac{1}{3}\left(e^{x} + e^{(-1+i\sqrt{3})x/2} + e^{(-1-\sqrt{3})x/2}\right)$, which (using Euler's relation) reduces to the given real-valued solution.

49. The general solution is $y = Ae^{2x} + Be^{-x} + C\cos x + D\sin x$. Imposition of the given initial conditions yields the equations

$$\begin{aligned} A + B + C &= 0 \\ 2A - B + D &= 0 \\ 4A + B - C &= 0 \\ 8A - B - D &= 30 \end{aligned}$$

that we solve for $A = 2$, $B = -5$, $C = 3$, and $D = -9$. Thus

$$y(x) = 2e^{2x} - 5e^{-x} + 3\cos x - 9\sin x.$$

50. If $x > 0$ then the differential equation is $y'' + y = 0$ with general solution $y = A\cos x + B\sin x$.. But if $x < 0$ it is $y'' - y = 0$ with general solution $y = C\cosh x + D\sin x$. To satisfy the initial conditions $y_1(0) = 1,\ y_1'(0) = 0$ we choose

$A=C=1$ and $B=D=0$. But to satisfy the initial conditions $y_2(0)=0,\ y_2'(0)=1$ we choose $A=C=0$ and $B=D=1$. The corresponding solutions are defined by

$$y_1(x) = \begin{cases} \cos x & \text{if } x \geq 0, \\ \cosh x & \text{if } x \leq 0; \end{cases} \qquad y_2(x) = \begin{cases} \sin x & \text{if } x \geq 0, \\ \sinh x & \text{if } x \leq 0. \end{cases}$$

51. In the solution of Problem 51 in Section 3.1 we showed that the substitution $v = \ln x$ gives

$$y' = \frac{dy}{dx} = \frac{1}{x}\frac{dy}{dv} \quad \text{and} \quad y'' = \frac{d^2y}{dx^2} = -\frac{1}{x^2}\cdot\frac{dy}{dv}+\frac{1}{x^2}\cdot\frac{d^2y}{dv^2}.$$

A further differentiation using the chain rule gives

$$y''' = \frac{d^3y}{dx^3} = \frac{2}{x^3}\cdot\frac{dy}{dv}-\frac{3}{x^3}\cdot\frac{d^2y}{dv^2}+\frac{1}{x^3}\cdot\frac{d^3y}{dv^3}.$$

Substitution of these expressions for $y',\ y''$, and y''' into the third-order Euler equation $ax^3y'''+bx^2y''+cx\,y'+d\,y=0$ and collection of coefficients quickly yields the desired constant-coefficient equation

$$a\frac{d^3y}{dv^3}+(b-3a)\frac{d^2y}{dv^2}+(c-b+2a)\frac{dy}{dv}+d\,y = 0.$$

In Problems 52 through 58 we list first the transformed constant-coefficient equation, then its characteristic equation and roots, and finally the corresponding general solution with $v = \ln x$ and $e^v = x$.

52. $\dfrac{d^2y}{dv^2}+9y = 0; \quad r^2+9=0; \quad r = \pm 3i$

$y(x) = c_1\cos(3v)+c_2\sin(3v) = c_1\cos(3\ln x)+c_2\sin(3\ln x)$

53. $\dfrac{d^2y}{dv^2}+6\dfrac{dy}{dv}+25y = 0; \quad r^2+6r+25=0; \quad r=-3\pm 4i$

$y(x) = e^{-3v}\left[c_1\cos(4v)+c_2\sin(4v)\right] = x^{-3}\left[c_1\cos(4\ln x)+c_2\sin(4\ln x)\right]$

54. $\dfrac{d^3y}{dv^3}+3\dfrac{d^2y}{dv^2} = 0; \quad r^3+3r^2=0; \quad r=0,\ 0,-3$

$y(x) = c_1+c_2v+c_3e^{-3v} = c_1+c_2\ln x+c_3x^{-3}$

55. $\dfrac{d^3y}{dv^3}-4\dfrac{d^2y}{dv^2}+4\dfrac{dy}{dv} = 0; \quad r^3-4r^2+4r=0; \quad r=0,\ 2,\ 2$

$y(x) = c_1+c_2e^{2v}+c_3ve^{2v} = c_1+x^2\left(c_2+c_3\ln x\right)$

56. $\dfrac{d^3y}{dv^3} = 0; \quad r^3 = 0; \quad r = 0,\ 0,\ 0$

$y(x) = c_1 + c_2 v + c_3 v^2 = c_1 + c_2 \ln x + c_3 (\ln x)^2$

57. $\dfrac{d^3y}{dv^3} - 5\dfrac{d^2y}{dv^2} + 5\dfrac{dy}{dv} = 0; \quad r^3 - 4r^2 + 4r = 0; \quad r = 0,\ 3 \pm \sqrt{3}$

$y(x) = c_1 + c_2 e^{(3-\sqrt{3})v} + c_3 v e^{(3+\sqrt{3})v} = c_1 + x^3\left(c_2 x^{-\sqrt{3}} + c_3 x^{+\sqrt{3}}\right)$

58. $\dfrac{d^3y}{dv^3} + 3\dfrac{d^2y}{dv^2} + 3\dfrac{dy}{dv} + y = 0; \quad r^3 + 3r^2 + 3r + 1 = 0; \quad r = -1, -1, -1$

$y(x) = c_1 e^{-v} + c_2 v e^{-v} + c_3 v^2 e^{-v} = x^{-1}\left[c_1 + c_2 \ln x + c_3 (\ln x)^2\right]$

SECTION 5.4

Mechanical Vibrations

In this section we discuss four types of free motion of a mass on a spring — undamped, underdamped, critically damped, and overdamped. However, the undamped and underdamped cases — in which actual oscillations occur — are emphasized because they are both the most interesting and the most important cases for applications.

1. Frequency: $\omega_0 = \sqrt{k/m} = \sqrt{16/4} = 2 \text{ rad/sec} = 1/\pi \text{ Hz}$

Period: $P = 2\pi/\omega_0 = 2\pi/2 = \pi \text{ sec}$

2. Frequency $\omega_0 = \sqrt{k/m} = \sqrt{48/0.75} = 8 \text{ rad/sec} = 4/\pi \text{ Hz}$

Period: $P = 2\pi/\omega_0 = 2\pi/8 = \pi/4 \text{ sec}$

3. The spring constant is $k = 15$ N/0.20 m $= 75$ N/m. The solution of $3x'' + 75x = 0$ with $x(0) = 0$ and $x'(0) = -10$ is $x(t) = -2\sin 5t$. Thus the amplitude is 2 m; the frequency is $\omega_0 = \sqrt{k/m} = \sqrt{75/3} = 5 \text{ rad/sec} = 2.5/\pi \text{ Hz}$; and the period is $2\pi/5$ sec.

4. **(a)** With $m = 1/4$ kg and $k = (9 \text{ N})/(0.25 \text{ m}) = 36$ N/m we find that $\omega_0 = 12$ rad/sec. The solution of $x'' + 144x = 0$ with $x(0) = 1$ and $x'(0) = -5$ is

$$\begin{aligned} x(t) &= \cos 12t - (5/12)\sin 12t \\ &= (13/12)[(12/13)\cos 12t - (5/13)\sin 12t] \\ x(t) &= (13/12)\cos(12t - \alpha) \end{aligned}$$

where $\alpha = 2\pi - \tan^{-1}(5/12) \approx 5.8884$.

(b) $C = 13/12 \approx 1.0833$ m and $T = 2\pi/12 \approx 0.5236$ sec.

5. The gravitational acceleration at distance R from the center of the earth is $g = GM/R^2$. According to Equation (6) in the text the (circular) frequency ω of a pendulum is given by $\omega^2 = g/L = GM/R^2L$, so its period is $p = 2\pi/\omega = 2\pi R\sqrt{L/GM}$.

6. If the pendulum in the clock executes n cycles per day (86400 sec) at Paris, then its period is $p_1 = 86400/n$ sec. At the equatorial location it takes 24 hr 2 min 40 sec = 86560 sec for the same number of cycles, so its period there is $p_2 = 86560/n$ sec. Now let $R_1 = 3956$ mi be the Earth′s "radius" at Paris, and R_2 its "radius" at the equator. Then substitution in the equation $p_1/p_2 = R_1/R_2$ of Problem 5 (with $L_1 = L_2$) yields $R_2 = 3963.33$ mi. Thus this (rather simplistic) calculation gives 7.33 mi as the thickness of the Earth's equatorial bulge.

7. The period equation $p = 3960\sqrt{100.10} = (3960 + x)\sqrt{100}$ yields $x \approx 1.9795$ mi $\approx$ 10,450 ft for the altitude of the mountain.

8. Let n be the number of cycles required for a correct clock with unknown pendulum length L_1 and period p_1 to register 24 hrs = 86400 sec, so $np_1 = 86400$. The given clock with length $L_2 = 30$ in and period p_2 loses 10 min = 600 sec per day, so $np_2 = 87000$. Then the formula of Problem 5 yields

$$\sqrt{\frac{L_1}{L_2}} = \frac{p_1}{p_2} = \frac{np_1}{np_2} = \frac{86400}{87000},$$

so $L_1 = (30)(86400/87000)^2 \approx 29.59$ in.

10. The $F = ma$ equation $\rho\pi r^2hx'' = \rho\pi r^2hg - \pi r^2xg$ simplifies to

$$x'' + (g/\rho h)x = g.$$

The solution of this equation with $x(0) = x'(0) = 0$ is

$$x(t) = \rho h(1 - \cos\omega_0 t)$$

where $\omega_0 = \sqrt{g/\rho h}$. With the given numerical values of ρ, h, and g, the amplitude of oscillation is $\rho h = 100$ cm and the period is $p = 2\pi\sqrt{\rho h/g} \approx 2.01$ sec.

11. The fact that the buoy weighs 100 lb means that $mg = 100$ so $m = 100/32$ slugs. The weight of water is 62.4 lb/ft^3, so the $F = ma$ equation of Problem 10 is

$$(100/32)x'' = 100 - 62.4\pi r^2 x.$$

It follows that the buoy's circular frequency ω is given by

$$\omega^2 = (32)(62.4\pi)r^2/100.$$

But the fact that the buoy′s period is $p = 2.5$ sec means that $\omega = 2\pi/2.5$. Equating these two results yields $r \approx 0.3173$ ft ≈ 3.8 in.

12. **(a)** Substitution of $M_r = (r/R)^3 M$ in $F_r = -GM_r m/r^2$ yields

$$F_r = -(GMm/R^3)r.$$

(b) Because $GM/R^3 = g/R$, the equation $mr'' = F_r$ yields the differential equation

$$r'' + (g/R)r = 0.$$

(c) The solution of this equation with $r(0) = R$ and $r'(0) = 0$ is $r(t) = R\cos\omega_0 t$ where $\omega_0 = \sqrt{g/R}$. Hence, with $g = 32.2$ ft/sec^2 and $R = (3960)(5280)$ ft, we find that the period of the particle′s simple harmonic motion is

$$p = 2\pi/\omega_0 = 2\pi\sqrt{R/g} \approx 5063.10 \text{ sec} \approx 84.38 \text{ min}.$$

13. **(a)** The characteristic equation $10r^2 + 9r + 2 = (5r+2)(2r+1) = 0$ has roots $r = -2/5, -1/2$. When we impose the initial conditions $x(0) = 0,\ x'(0) = 5$ on the general solution $x(t) = c_1 e^{-2t/5} + c_2 e^{-t/2}$ we get the particular solution $x(t) = 50\left(e^{-2t/5} - e^{-t/2}\right)$.

(b) The derivative $x'(t) = 25e^{-t/2} - 20e^{-2t/5} = 5e^{-2t/5}\left(5e^{-t/10} - 4\right) = 0$ when $t = 10\ln(5/4) \approx 2.23144$. Hence the mass's farthest distance to the right is given by $x(10\ln(5/4)) = 512/125 = 4.096$.

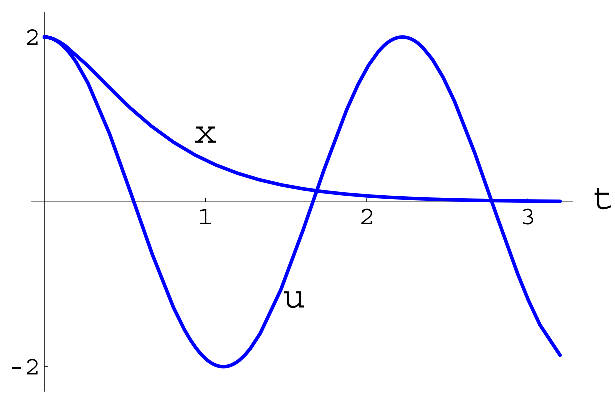
14. **(a)** The characteristic equation $25r^2 + 10r + 226 = (5r+1)^2 + 15^2 = 0$ has roots $r = (-1 \pm 15i)/5 = -1/5 \pm 3i$. When we impose the initial conditions $x(0) = 20,\ x'(0) = 41$ on the general solution $x(t) = e^{-t/5}\left(A\cos 3t + B\sin 3t\right)$ we get $A = 20,\ B = 15$. The corresponding particular solution is given by

$x(t) = e^{-t/5}(20\cos 3t + 15\sin 3t) = 25e^{-t/5}\cos(3t-\alpha)$ where
$\alpha = \tan^{-1}(3/4) \approx 0.6435$.

(b) Thus the oscillations are "bounded" by the curves $x = \pm 25e^{-t/5}$ and the pseudoperiod of oscillation is $T = 2\pi/3$ (because $\omega = 3$).

15. **With damping** The characteristic equation $(1/2)r^2 + 3r + 4 = 0$ has roots $r = -2, -4$. When we impose the initial conditions $x(0) = 2,\ x'(0) = 0$ on the general solution $x(t) = c_1e^{-2t} + c_2e^{-4t}$ we get the particular solution $x(t) = 4e^{-2t} - 2e^{-4t}$ that describes overdamped motion.

Without damping The characteristic equation $(1/2)r^2 + 4 = 0$ has roots $r = \pm 2i\sqrt{2}$. When we impose the initial conditions $x(0) = 2,\ x'(0) = 0$ on the general solution $u(t) = A\cos(2\sqrt{2}\,t) + B\sin(2\sqrt{2}\,t)$ we get the particular solution $u(t) = 2\cos(2\sqrt{2}\,t)$. The graphs of $x(t)$ and $u(t)$ are shown in the following figure.

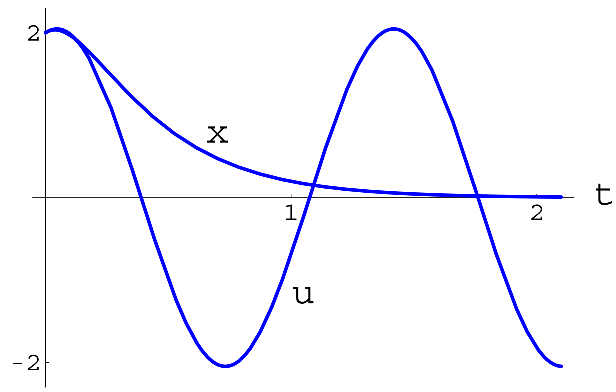


16. **With damping** The characteristic equation $3r^2 + 30r + 63 = 0$ has roots $r = -3, -7$. When we impose the initial conditions $x(0) = 2,\ x'(0) = 2$ on the general solution $x(t) = c_1e^{-3t} + c_2e^{-7t}$ we get the particular solution $x(t) = 4e^{-3t} - 2e^{-7t}$ that describes overdamped motion.

Without damping The characteristic equation $3r^2 + 63 = 0$ has roots $r = \pm i\sqrt{21}$. When we impose the initial conditions $x(0) = 2,\ x'(0) = 2$ on the general solution $u(t) = A\cos(\sqrt{21}\,t) + B\sin(\sqrt{21}\,t)$ we get the particular solution

$$u(t) = 2\cos(\sqrt{21}\,t) + \frac{2}{\sqrt{21}}\sin(\sqrt{21}\,t) \approx 2\sqrt{\frac{22}{21}}\cos(\sqrt{21}\,t - 0.2149).$$

The graphs of $x(t)$ and $u(t)$ are shown in the figure at the top of the next page.

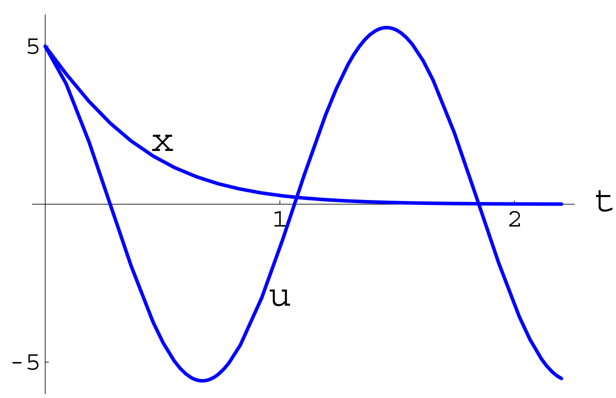


17. **With damping** The characteristic equation $r^2 + 8r + 16 = 0$ has roots $r = -4, -4$. When we impose the initial conditions $x(0) = 5,\ x'(0) = -10$ on the general solution $x(t) = (c_1 + c_2 t)e^{-4t}$ we get the particular solution $x(t) = 5e^{-4t}(2t + 1)$ that describes critically damped motion.

Without damping The characteristic equation $r^2 + 16 = 0$ has roots $r = \pm 4i$. When we impose the initial conditions $x(0) = 5,\ x'(0) = -10$ on the general solution $u(t) = A\cos(4t) + B\sin(4t)$ we get the particular solution

$$u(t) = 5\cos(4t) + \frac{5}{2}\sin(4t) \approx \frac{5}{2}\sqrt{5}\cos(4t - 5.8195).$$

The graphs of $x(t)$ and $u(t)$ are shown in the following figure.

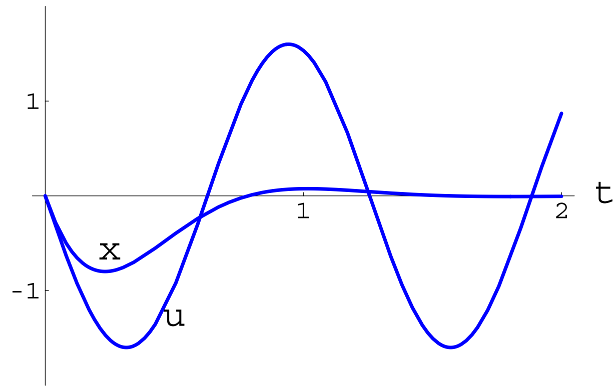


18. **With damping** The characteristic equation $2r^2 + 12r + 50 = 0$ has roots $r = -3 \pm 4i$. When we impose the initial conditions $x(0) = 0,\ x'(0) = -8$ on the general solution $x(t) = e^{-3t}\left(A\cos 4t + B\sin 4t\right)$ we get the particular solution $x(t) = -2e^{-3t}\sin 4t \ = \ 2e^{-3t}\cos(4t - 3\pi/2)$ that describes underdamped motion.

Without damping The characteristic equation $2r^2 + 50 = 0$ has roots $r = \pm 5i$. When we impose the initial conditions $x(0) = 0,\ x'(0) = -8$ on the general solution $u(t) = A\cos(5t) + B\sin(5t)$ we get the particular solution

$$u(t) = -\frac{8}{5}\sin(5t) = \frac{8}{5}\cos\left(5t - \frac{3\pi}{2}\right).$$

The graphs of $x(t)$ and $u(t)$ are shown in the following figure.



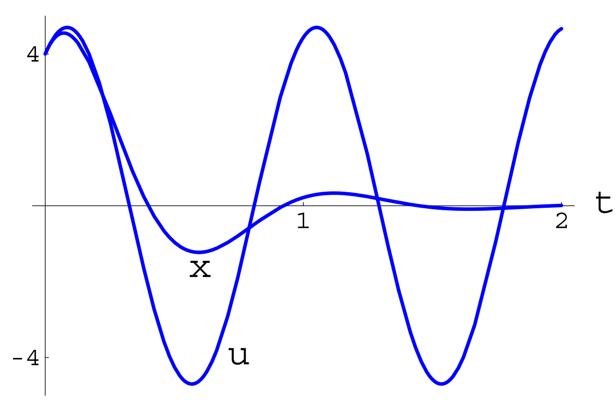
19. The characteristic equation $4r^2 + 20r + 169 = 0$ has roots $r = -5/2 \pm 6i$. When we impose the initial conditions $x(0) = 4,\ x'(0) = 16$ on the general solution $x(t) = e^{-5t/2}\left(A\cos 6t + B\sin 6t\right)$ we get the particular solution

$$x(t) \;=\; e^{-5t/2}\left[\, 4\cos 6t + \frac{13}{3}\sin 6t \,\right] \;\approx\; \frac{1}{3}\sqrt{313}\; e^{-5t/2}\cos(6t - 0.8254)$$

that describes underdamped motion.

Without damping The characteristic equation $4r^2 + 169 = 0$ has roots $r = \pm 13i/2$. When we impose the initial conditions $x(0) = 4,\ x'(0) = 16$ on the general solution $u(t) = A\cos(13t/2) + B\sin(13t/2)$ we get the particular solution

$$u(t) \;=\; 4\cos\left(\frac{13t}{2}\right) + \frac{32}{13}\sin\left(\frac{13t}{2}\right) \;\approx\; \frac{4}{13}\sqrt{233}\cos\left(\frac{13}{2}t - 0.5517\right).$$



20. **With damping** The characteristic equation $2r^2 + 16r + 40 = 0$ has roots $r = -4 \pm 2i$. When we impose the initial conditions $x(0) = 5,\ x'(0) = 4$ on the general solution $x(t) = e^{-4t}\left(A\cos 2t + B\sin 2t\right)$ we get the particular solution

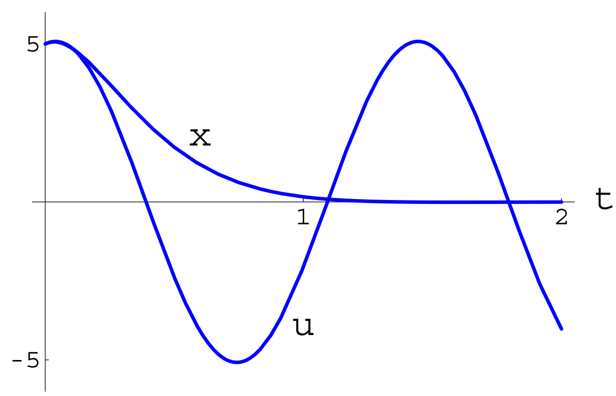
$$x(t) \;=\; e^{-4t}(5\cos 2t + 12\sin 2t) \;\approx\; 13\, e^{-4t}\cos(2t - 1.1760)$$

that describes underdamped motion.

Without damping The characteristic equation $2r^2 + 40 = 0$ has roots $r = \pm 2\sqrt{5}\, i$. When we impose the initial conditions $x(0) = 5,\ x'(0) = 4$ on the general solution $u(t) = A\cos(2\sqrt{5}\, t) + B\sin(2\sqrt{5}\, t)$ we get the particular solution

$$u(t) \;=\; 5\cos\left(2\sqrt{5}\, t\right) + \frac{2}{\sqrt{5}}\sin\left(2\sqrt{5}\, t\right) \;\approx\; \sqrt{\frac{129}{5}}\cos\left(2\sqrt{5}\, t - 0.1770\right).$$

The graphs of $x(t)$ and $u(t)$ are shown in the following figure.



21. **With damping** The characteristic equation $r^2 + 10r + 125 = 0$ has roots $r = -5 \pm 10i$. When we impose the initial conditions $x(0) = 6,\ x'(0) = 50$ on the general solution $x(t) = e^{-5t}\left(A\cos 10t + B\sin 10t\right)$ we get the particular solution

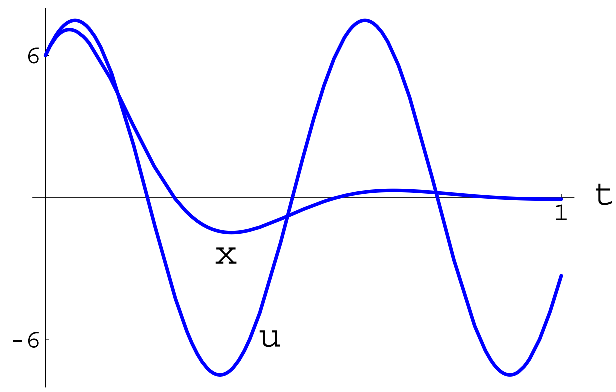
$$x(t) \;=\; e^{-5t}(6\cos 10t + 8\sin 10t) \;\approx\; 10\, e^{-5t}\cos(10t - 0.9273)$$

that describes underdamped motion.

Without damping The characteristic equation $r^2 + 125 = 0$ has roots $r = \pm 5\sqrt{5}\, i$. When we impose the initial conditions $x(0) = 6,\ x'(0) = 50$ on the general solution $u(t) = A\cos(5\sqrt{5}\, t) + B\sin(5\sqrt{5}\, t)$ we get the particular solution

$$u(t) \;=\; 6\cos\left(5\sqrt{5}\, t\right) + 2\sqrt{5}\sin\left(5\sqrt{5}\, t\right) \;\approx\; 2\sqrt{14}\cos\left(5\sqrt{5}\, t - 0.6405\right).$$

The graphs of $x(t)$ and $u(t)$ are shown in the figure at the top of the next page.



22. **(a)** With $m = 12/32 = 3/8$ slug, $c = 3$ lb-sec/ft, and $k = 24$ lb/ft, the differential equation is equivalent to $3x'' + 24x' + 192x = 0$. The characteristic equation $3r^2 + 24r + 192 = 0$ has roots $r = -4 \pm 4\sqrt{3}\,i$. When we impose the initial conditions $x(0) = 1,\ x'(0) = 0$ on the general solution $x(t) = e^{-4t}\left(A\cos 4t\sqrt{3} + B\sin 4t\sqrt{3}\right)$ we get the particular solution

$$x(t) = e^{-4t}[\cos 4t\sqrt{3} + (1/\sqrt{3})\sin 4t\sqrt{3}]$$

$$= (2/\sqrt{3})e^{-4t}[(\sqrt{3}/2)\cos 4t\sqrt{3} + (1/2)\sin 4t\sqrt{3}]$$

$$x(t) = (2/\sqrt{3})e^{-4t}\cos(4t\sqrt{3} - \pi/6).$$

(b) The time-varying amplitude is $2/\sqrt{3} \approx 1.15$ ft; the frequency is $4\sqrt{3} \approx 6.93$ rad/sec; and the phase angle is $\pi/6$.

23. **(a)** With $m = 100$ slugs we get $\omega = \sqrt{k/100}$. But we are given that

$$\omega = (80 \text{ cycles/min})(2\pi)(1 \text{ min}/60 \text{ sec}) = 8\pi/3,$$

and equating the two values yields $k \approx 7018$ lb/ft.

(b) With $\omega_1 = 2\pi(78/60)\ \text{sec}^{-1}$, Equation (21) in the text yields $c \approx 372.31$ lb/(ft/sec). Hence $p = c/2m \approx 1.8615$. Finally $e^{-pt} = 0.01$ gives $t \approx 2.47$ sec.

30. In the underdamped case we have

$$x(t) = e^{-pt}[A\cos\omega_1 t + B\sin\omega_1 t],$$

$$x'(t) = -pe^{pt}[A\cos\omega_1 t + B\sin\omega_1 t] + e^{-pt}[-A\omega_1\sin\omega_1 t + B\omega_1\cos\omega_1 t].$$

The conditions $x(0) = x_0$, $x'(0) = v_0$ yield the equations $A = x_0$ and $-pA + B\omega_1 = v_0$, whence $B = (v_0 + px_0)/\omega_1$.

31. The binomial series

$$(1+x)^{\alpha} = 1+\alpha x+\frac{\alpha(\alpha-1)}{2!}x^2+\frac{\alpha(\alpha-1)(\alpha-2)}{3!}x^3+\cdots$$

converges if $|x|<1$. (See, for instance, Section 11.8 of Edwards and Penney, *Calculus*, 6th edition, Prentice Hall, 2002.) With $\alpha=1/2$ and $x=-c^2/4mk$ in Eq. (21) of Section 5.4 in this text, the binomial series gives

$$\omega_1 = \sqrt{\omega_0^2-p^2} = \sqrt{\frac{k}{m}-\frac{c^2}{4m^2}} = \sqrt{\frac{k}{m}}\sqrt{1-\frac{c^2}{4mk}}$$
$$= \sqrt{\frac{k}{m}}\left(1-\frac{c^2}{8mk}-\frac{c^4}{128m^2k^2}-\cdots\right) \approx \omega_0\left(1-\frac{c^2}{8mk}\right).$$

32. If $x(t) = Ce^{-pt}\cos(\omega_1 t - \alpha)$ then

$$x'(t) = -pCe^{-pt}\cos(\omega_1 t - \alpha) + C\omega_1 e^{-pt}\sin(\omega_1 t - \alpha) = 0$$

yields $\tan(\omega_1 t - \alpha) = -p/\omega_1$.

33. If $x_1 = x(t_1)$ and $x_2 = x(t_2)$ are two successive local maxima, then $\omega_1 t_2 = \omega_1 t_1 + 2\pi$ so

$$x_1 = C\exp(-pt_1)\cos(\omega_1 t_1 - \alpha),$$
$$x_2 = C\exp(-pt_2)\cos(\omega_1 t_2 - \alpha) = C\exp(-pt_2)\cos(\omega_1 t_1 - \alpha).$$

Hence $x_1/x_2 = \exp[-p(t_1 - t_2)]$, and therefore

$$\ln(x_1/x_2) = -p(t_1 - t_2) = 2\pi p/\omega_1.$$

34. With $t_1 = 0.34$ and $t_2 = 1.17$ we first use the equation $\omega_1 t_2 = \omega_1 t_1 + 2\pi$ from Problem 32 to calculate $\omega_1 = 2\pi/(0.83) \approx 7.57$ rad/sec. Next, with $x_1 = 6.73$ and $x_2 = 1.46$, the result of Problem 33 yields

$$p = (1/0.83)\ln(6.73/1.46) \approx 1.84.$$

Then Equation (16) in this section gives

$$c = 2mp = 2(100/32)(1.84) \approx 11.51 \text{ lb-sec/ft},$$

and finally Equation (21) yields

$$k = (4m^2\omega_1^2 + c^2)/4m \approx 189.68 \text{ lb/ft}.$$

35. The characteristic equation $r^2+2r+1=0$ has roots $r=-1,-1$. When we impose the initial conditions $x(0)=0,\ x'(1)=0$ on the general solution $x(t)=\left(c_1+c_2t\right)e^{-t}$ we get the particular solution $x_1(t) \;=\; t\,e^{-t}$.

36. The characteristic equation $r^2+2r+(1-10^{-2n})=0$ has roots $r=-1\pm10^{-n}$. When we impose the initial conditions $x(0)=0,\ x'(1)=0$ on the general solution

$$x(t) \;=\; c_1\exp\left[\left(-1+10^{-n}\right)t\right]+c_1\exp\left[\left(-1-10^{-n}\right)t\right]$$

we get the equations

$$c_1+c_2 = 0, \quad \left(-1+10^{-n}\right)c_1+\left(-1-10^{-n}\right)c_2 \;=\; 1$$

with solution $c_1=2^{n-1}5^n,\ c_2=2^{n-1}5^n$. This gives the particular solution

$$x_2(t) \;=\; 10^n e^{-t}\left(\frac{\exp(10^{-n}t)-\exp(-10^{-n}t)}{2}\right) \;=\; 10^n e^{-t}\sinh(10^{-n}t).$$

37. The characteristic equation $r^2+2r+(1+10^{-2n})=0$ has roots $r=-1\pm10^{-n}i$. When we impose the initial conditions $x(0)=0,\ x'(1)=0$ on the general solution

$$x(t) \;=\; e^{-t}\left[A\cos\left(10^{-n}t\right)+B\sin\left(10^{-n}t\right)\right]$$

we get the equations $c_1 = 0, \quad -c_1+10^{-n}c_2 \;=\; 1$ with solution $c_1=0,\ c_2=10^n$. This gives the particular solution $x_3(t) \;=\; 10^n e^{-t}\sin(10^{-n}t)$.

38. $$\lim_{n\to\infty} x_2(t) \;=\; \lim_{n\to\infty} 10^n e^{-t}\sinh(10^{-n}t) \;=\; t\,e^{-t}\cdot\lim_{n\to\infty}\frac{\sinh(10^{-n}t)}{10^{-n}t} \;=\; t\,e^{-t} \quad \text{and}$$

$$\lim_{n\to\infty} x_3(t) \;=\; \lim_{n\to\infty} 10^n e^{-t}\sin(10^{-n}t) \;=\; t\,e^{-t}\cdot\lim_{n\to\infty}\frac{\sin(10^{-n}t)}{10^{-n}t} \;=\; t\,e^{-t},$$

using the fact that $\lim_{\theta\to0}(\sin\theta)/\theta \;=\; \lim_{\theta\to0}(\sinh\theta)/\theta \;=\; 0$ (by L'Hôpital's rule, for instance).

SECTION 5.5

NONHOMOGENEOUS EQUATIONS AND THE METHOD OF UNDETERMINED COEFFICIENTS

The method of undetermined coefficients is based on "educated guessing". If we can guess correctly the **form** of a particular solution of a nonhomogeneous linear equation with constant coefficients, then we can determine the particular solution explicitly by substitution in the given differential equation. It is pointed out at the end of Section 5.5 that this simple approach is not always successful — in which case the method of variation of parameters is available if a complementary function is known. However, undetermined coefficients *does* turn out to work well with a surprisingly large number of the nonhomogeneous linear differential equations that arise in elementary scientific applications.

In each of Problems 1–20 we give first the form of the trial solution y_{trial}, then the equations in the coefficients we get when we substitute y_{trial} into the differential equation and collect like terms, and finally the resulting particular solution y_p.

1. $y_{\text{trial}} = Ae^{3x}$; $25A = 1$; $y_p = (1/25)e^{3x}$

2. $y_{\text{trial}} = A + Bx$; $-2A - B = 4,\ -2B = 3$; $y_p = -(5 + 6x)/4$

3. $y_{\text{trial}} = A\cos 3x + B\sin 3x$; $-15A - 3B = 0,\ 3A - 15B = 2$;
$y_p = (\cos 3x - 5\sin 3x)/39$

4. $y_{\text{trial}} = Ae^x + Bxe^x$; $9A + 12B = 0,\ 9B = 3$; $y_p = (-4e^x + 3xe^x)/9$

5. First we substitute $\sin^2 x = (1 - \cos 2x)/2$ on the right-hand side of the differential equation. Then:

$y_{\text{trial}} = A + B\cos 2x + C\sin 2x$; $A = 1/2,\ -3B + 2C = -1/2,\ -2B - 3C = 0$;
$y_p = (13 + 3\cos 2x - 2\sin 2x)/26$

6. $y_{\text{trial}} = A + Bx + Cx^2$; $7A + 4B + 4C = 0,\ 7B + 8C = 0,\ 7C = 1$;
$y_p = (4 - 56x + 49x^2)/343$

7. First we substitute $\sinh x = (e^x - e^{-x})/2$ on the right-hand side of the differential equation. Then:

$y_{\text{trial}} = Ae^x + Be^{-x}$; $-3A = 1/2,\ -3B = -1/2$; $y_p = (e^{-x} - e^x)/6 = -(1/3)\sinh x$

8. First we note that $\cosh 2x$ is part of the complementary function

$y_c = c_1 \cosh 2x + c_2 \sinh 2x.$. Then:

$y_{trial} = x(A\cosh 2x + B\sinh 2x); \quad 4A = 0,\ 4B = 1; \quad y_p = (1/4)x \sinh 2x$

9. First we note that e^x is part of the complementary function $y_c = c_1e^x + c_2e^{-3x}$. Then:

$y_{trial} = A + x(B + Cx)e^x; \quad -3A = 1,\ 4B + 2C = 0,\ 8C = 1;$
$y_p = -(1/3) + (2x^2 - x)e^x/16.$

10. First we note the duplication with the complementary function $y_c = c_1 \cos 3x + c_2 \sin 3x$. Then:

$y_{trial} = x(A\cos 3x + B\sin 3x); \quad 6B = 2,\ -6A = 3; \quad y_p = (2x \sin 3x - 3x \cos 3x)/6$

11. First we note the duplication with the complementary function

$y_c = c_1x + c_2 \cos 2x + c_3 \sin 2x$. Then:

$y_{trial} = x(A + Bx); \quad 4A = -1,\ 8B = 3; \quad y_p = (3x^2 - 2x)/8$

12. First we note the duplication with the complementary function

$y_c = c_1x + c_2 \cos x + c_3 \sin x$. Then:

$y_{trial} = Ax + x(B\cos x + C\sin x); \quad A = 2,\ -2B = 0,\ -2C = -1;$
$y_p = 2x + (1/2)x \sin x$

13. $y_{trial} = e^x(A\cos x + B\sin x); \quad 7A + 4B = 0,\ -4A + 7B = 1;$
$y_p = e^x(7 \sin x - 4 \cos x)/65$

14. First we note the duplication with the complementary function

$y_c = (c_1 + c_2x)e^{-x} + (c_3 + c_4x)e^x$. Then:

$y_{trial} = x^2(A + Bx)e^x; \quad 8A + 24B = 0,\ 24B = 1; \quad y_p = (-3x^2e^x + x^3e^x)/24$

15. This is something of a trick problem. We cannot solve the characteristic equation $r^5 + 5r^4 - 1 = 0$ to find the complementary function, but we can see that it contains no constant term (why?). Hence the trial solution $y_{trial} = A$ leads immediately to the particular solution $y_p = -17$.

16. $y_{trial} = A + (B + Cx + Dx^2)e^{3x};$
$9A = 5,\ 18B + 6C + 2D = 0,\ 18C + 12D = 0,\ 18D = 2;$

$y_p = (45 + e^{3x} - 6xe^{3x} + 9x^2e^{3x})/81$

17. First we note the duplication with the complementary function $y_c = c_1 \cos x + c_2 \sin x$. Then:

$y_{\text{trial}} = x[(A+Bx)\cos x + (C+Dx)\sin x];$
$2B + 2C = 0,\ 4D = 1, -2A + 2D = 1, -4B = 0;$
$y_p = (x^2 \sin x - x\cos x)/4$

18. First we note the duplication with the complementary function $y_c = c_1e^{-x} + c_2e^{x} + c_3e^{-2x} + c_4e^{2x}$. Then:

$y_{\text{trial}} = x(Ae^x) + x(B + Cx)\,e^{2x};\quad -6A = 1,\ 12B + 38C = 0,\ 24C = -1;$
$y_p = -(24xe^x - 19xe^{2x} + 6x^2e^{2x})/144$

19. First we note the duplication with the part $c_1 + c_2x$ of the complementary function (which corresponds to the factor r^2 of the characteristic polynomial). Then:

$y_{\text{trial}} = x^2(A + Bx + Cx^2);\qquad 4A + 12B = -1,\ 12B + 48C = 0,\ 24C = 3;$
$y_p = (10x^2 - 4x^3 + x^4)/8$

20. First we note that the characteristic polynomial $r^3 - r$ has the zero $r = 1$ corresponding to the duplicating part e^x of the complementary function. Then:

$y_{\text{trial}} = A + x\left(Be^x\right);\quad -A = 7,\ 3B = 1;\quad y_p = -7 + (1/3)xe^x$

In Problems 21–30 we list first the complementary function y_c, then the initially proposed trial function y_i, and finally the actual trial function y_p in which duplication with the complementary function has been eliminated.

21. $y_c = e^x\left(c_1 \cos x + c_2 \sin x\right);$
$y_i = e^x\left(A\cos x + B\sin x\right)$
$y_p = x\cdot e^x\left(A\cos x + B\sin x\right)$

22. $y_c = \left(c_1 + c_2x + c_3x^2\right) + \left(c_4e^x\right) + \left(c_5e^{-x}\right);$
$y_i = \left(A + Bx + Cx^2\right) + \left(De^x\right)$
$y_p = x^3\cdot\left(A + Bx + Cx^2\right) + x\cdot\left(De^x\right)$

23. $y_c = c_1\cos x + c_2 \sin x;$
$y_i = (A+Bx)\cos 2x + (C+Dx)\sin 2x$
$y_p = x\cdot[(A+Bx)\cos 2x + (C+Dx)\sin 2x]$

24. $$y_c = c_1 + c_2 e^{-3x} + c_3 e^{4x};$$
$$y_i = (A + Bx) + (C + Dx)e^{-3x}$$
$$y_p = x \cdot (A + Bx) + x \cdot (C + Dx)e^{-3x}$$

25. $$y_c = c_1 e^{-x} + c_2 e^{-2x};$$
$$y_i = (A + Bx)e^{-x} + (C + Dx)e^{-2x}$$
$$y_p = x \cdot (A + Bx)e^{-x} + x \cdot (C + Dx)e^{-2x}$$

26. $$y_c = e^{3x}\left(c_1 \cos 2x + c_2 \sin 2x\right);$$
$$y_i = (A + Bx)e^{3x} \cos 2x + (C + Dx)e^{3x} \sin 2x$$
$$y_p = x \cdot \left[(A + Bx)e^{3x} \cos 2x + (C + Dx)e^{3x} \sin 2x\right]$$

27. $$y_c = \left(c_1 \cos x + c_2 \sin x\right) + \left(c_3 \cos 2x + c_3 \sin 2x\right)$$
$$y_i = \left(A \cos x + B \sin x\right) + \left(C \cos 2x + D \sin 2x\right)$$
$$y_p = x \cdot \left[\left(A \cos x + B \sin x\right) + \left(C \cos 2x + D \sin 2x\right)\right]$$

28. $$y_c = \left(c_1 + c_2 x\right) + \left(c_3 \cos 3x + c_3 \sin 3x\right)$$
$$y_i = \left(A + Bx + Cx^2\right) \cos 3x + \left(D + Ex + Fx^2\right) \sin 3x$$
$$y_p = x \cdot \left[\left(A + Bx + Cx^2\right) \cos 3x + \left(D + Ex + Fx^2\right) \sin 3x\right]$$

29. $$y_c = \left(c_1 + c_2 x + c_3 x^2\right) e^{x} + c_4 e^{2x} + c_5 e^{-2x};$$
$$y_i = \left(A + Bx\right) e^{x} + C\, e^{2x} + D\, e^{-2x}$$
$$y_p = x^3 \cdot \left(A + Bx\right) e^{x} + x \cdot \left(C\, e^{2x}\right) + x \cdot \left(D\, e^{-2x}\right)$$

30. $$y_c = \left(c_1 + c_2 x\right) e^{-x} + \left(c_3 + c_4 x\right) e^{x}$$
$$y_i = y_p = \left(A + Bx + Cx^2\right) \cos x + \left(D + Ex + Fx^2\right) \sin x$$

In Problems 31–40 we list first the complementary function y_c, the trial solution y_{tr} for the method of undetermined coefficients, and the corresponding general solution $y_g = y_c + y_p$ where y_p results from determining the coefficients in y_{tr} so as to satisfy the given nonhomogeneous differential equation. Then we list the linear equations obtained by imposing the given initial conditions, and finally the resulting particular solution $y(x)$.

31. $$y_c = c_1 \cos 2x + c_2 \sin 2x; \qquad y_{tr} = A + Bx$$
$$y_g = c_1 \cos 2x + c_2 \sin 2x + x/2$$

$$c_1 = 1, \quad 2c_2 + 1/2 = 2$$
$$y(x) = \cos 2x + (3/4)\sin 2x + x/2$$

32. $y_c = c_1e^{-x} + c_2e^{-2x}; \qquad y_{tr} = Ae^x$

$$y_g = c_1e^{-x} + c_2e^{-2x} + e^x/6$$

$$c_1 + c_2 + 1/6 = 0, \quad -c_1 - 2c_2 + 1/6 = 3$$
$$y(x) = \left(15e^{-x} - 16e^{-2x} + e^x\right)/6$$

33. $y_c = c_1\cos 3x + c_2\sin 3x; \qquad y_{tr} = A\cos 2x + B\sin 2x$

$$y_g = c_1\cos 3x + c_2\sin 3x + (1/5)\sin 2x$$

$$c_1 = 1, \quad 3c_2 + 2/5 = 0$$
$$y(x) = \left(15\cos 3x - 2\sin 3x + 3\sin 2x\right)/15$$

34. $y_c = c_1\cos x + c_2\sin x; \qquad y_{tr} = x\cdot\left(A\cos x + B\sin x\right)$

$$y_g = c_1\cos x + c_2\sin x + \tfrac{1}{2}x\sin x$$

$$c_1 = 1, \quad c_2 = -1; \qquad y(x) = \cos x - \sin x + \tfrac{1}{2}x\sin x$$

35. $y_c = e^x\left(c_1\cos x + c_2\sin x\right); \qquad y_{tr} = A + Bx$

$$y_g = e^x\left(c_1\cos x + c_2\sin x\right) + 1 + x/2$$

$$c_1 + 1 = 3, \quad c_1 + c_2 + 1/2 = 0$$
$$y(x) = e^x\left(4\cos x - 5\sin x\right)/2 + 1 + x/2$$

36. $y_c = c_1 + c_2x + c_3e^{-2x} + c_4e^{2x}; \qquad y_{tr} = x^2\cdot\left(A + Bx + Cx^2\right)$

$$y_g = c_1 + c_2x + c_3e^{-2x} + c_4e^{2x} - x^2/16 - x^4/48$$

$$c_1 + c_3 + c_4 = 1, \quad c_2 - 2c_3 + 2c_4 = 1, \quad 4c_3 + 4c_4 - 1/8 = -1, \quad -8c_3 + 8c_4 = -1$$

$$y(x) = \left(234 + 240x - 9e^{-2x} - 33e^{2x} - 12x^2 - 4x^4\right)/192$$

37. $y_c = c_1 + c_2e^x + c_3xe^x; \qquad y_{tr} = x\cdot\left(A\right) + x^2\cdot\left(B + Cx\right)e^x$

$$y_g = c_1 + c_2e^x + c_3xe^x + x - x^2e^x/2 + x^3e^x/6$$

$$c_1 + c_2 = 0, \quad c_2 + c_3 + 1 = 0, \quad c_2 + 2c_3 - 1 = 1$$

$$y(x) = 4 + x + e^x\left(-24 + 18x - 3x^2 + x^3\right)/6$$

38. $$y_c = e^{-x}(c_1 \cos x + c_2 \sin x); \qquad y_{tr} = A\cos 3x + B\sin 3x$$

$$y_g = e^{-x}(c_1 \cos x + c_2 \sin x) - (6\cos 3x + 7\sin 3x)/85$$

$$c_1 - 6/185 = 2, \quad -c_1 + c_2 - 21/85 = 0$$

$$y(x) = \left[e^{-x}(176\cos x + 197\sin x) - (6\cos 3x + 7\sin 3x)\right]/85$$

39. $$y_c = c_1 + c_2 x + c_3 e^{-x}; \qquad y_{tr} = x^2 \cdot (A + Bx) + x \cdot \left(Ce^{-x}\right)$$

$$y_g = c_1 + c_2 x + c_3 e^{-x} - x^2/2 + x^3/6 + x\,e^{-x}$$

$$c_1 + c_3 = 1, \quad c_2 - c_3 + 1 = 0, \quad c_3 - 3 = 1$$

$$y(x) = \left(-18 + 18x - 3x^2 + x^3\right)/6 + (4 + x)e^{-x}$$

40. $$y_c = c_1 e^{-x} + c_2 e^{x} + c_3 \cos x + c_4 \sin x; \qquad y_{tr} = A$$

$$y_g = c_1 e^{-x} + c_2 e^{x} + c_3 \cos x + c_4 \sin x - 5$$

$$c_1 + c_2 + c_3 - 5 = 0, \quad -c_1 + c_2 + c_4 = 0, \quad c_1 + c_2 - c_3 = 0, \quad -c_1 + c_2 - c_4 = 0$$

$$y(x) = \left(5e^{-x} + 5e^{x} + 10\cos x - 20\right)/4$$

41. The trial solution $y_{tr} = A + Bx + Cx^2 + Dx^3 + Ex^4 + Fx^5$ leads to the equations

$$\begin{aligned}
&2A - B - 2C - 6D + 24E = 0 \\
&-2B - 2C - 6D - 24E + 120F = 0 \\
&-2C - 3D - 12E - 60F = 0 \\
&-2D - 4E - 20F = 0 \\
&-2E - 5F = 0 \\
&-2F = 8
\end{aligned}$$

that are readily solve by back-substitution. The resulting particular solution is

$$y(x) = -255 - 450x + 30x^2 + 20x^3 + 10x^4 - 4x^5.$$

42. The characteristic equation $r^4 - r^3 - r^2 - r - 2 = 0$ has roots $r = -1, 2, \pm i$ so the complementary function is $y_c = c_1 e^{-x} + c_2 e^{2x} + c_3 \cos x + c_4 \sin x$. We find that the coefficients satisfy the equations

$$\begin{aligned} c_1 + c_2 + c_3 - 255 &= 0 \\ -c_1 + 2c_2 + c_4 - 450 &= 0 \\ c_1 + 4c_2 - c_3 + 60 &= 0 \\ -c_1 + 8c_2 - c_4 + 120 &= 0 \end{aligned}$$

Solution of this system gives finally the particular solution $y = y_c + y_p$ where y_p is the particular solution of Problem 41 and

$$y_c = 10e^{-x} + 35e^{2x} + 210\cos x + 390\sin x.$$

43. **(a)**

$$\begin{aligned} \cos 3x + i\sin 3x &= (\cos x + i\sin x)^3 \\ &= \cos^3 x + 3i\cos^2 x\sin x - 3\cos x\sin^2 x - i\sin^3 x \end{aligned}$$

When we equate real parts we get the equation

$$\cos^3 x - 3(\cos x)\left(1 - \cos^2 x\right) = 4\cos^3 x - 3\cos x$$

and readily solve for $\cos^3 x = \tfrac{3}{4}\cos x + \tfrac{1}{4}\cos 3x$. The formula for $\sin^3 x$ is derived similarly by equating imaginary parts in the first equation above.

(b) Upon substituting the trial solution $y_p = A\cos x + B\sin x + C\cos 3x + D\sin 3x$ in the differential equation $y'' + 4y = \tfrac{3}{4}\cos x + \tfrac{1}{4}\cos 3x$, we find that $A = 1/4$, $B = 0$, $C = -1/20$, $D = 0$. The resulting general solution is

$$y(x) = c_1 \cos 2x + c_2 \sin 2x + (1/4)\cos x - (1/20)\cos 3x.$$

44. We use the identity $\sin x \sin 3x = \tfrac{1}{2}\cos 2x - \tfrac{1}{2}\cos 4x$, and hence substitute the trial solution $y_p = A\cos 2x + B\sin 2x + C\cos 4x + D\sin 4x$ in the differential equation $y'' + y' + y = \tfrac{1}{2}\cos 2x - \tfrac{1}{2}\cos 4x$. We find that $A = -3/26$, $B = 1/13$, $C = -14/482$, $D = 2/141$. The resulting general solution is

$$\begin{aligned} y(x) &= e^{-x/2}(c_1 \cos x\sqrt{3}/2 + c_2 \sin x\sqrt{3}/2) \\ &\quad + (-3\cos 2x + 2\sin 2x)/26 + (-15\cos 4x + 4\sin 4x)/482. \end{aligned}$$

45. We substitute

$$\sin^4 x = (1 - \cos 2x)^2/4$$

$$= (1 - 2\cos 2x + \cos^2 2x)/4 = (3 - 4\cos 2x + \cos 4x)/8$$

on the right-hand side of the differential equation, and then substitute the trial solution $y_p = A\cos 2x + B\sin 2x + C\cos 4x + D\sin 4x + E$. We find that $A = -1/10$, $B = 0$, $C = -1/56$, $D = 0$, $E = 1/24$. The resulting general solution is

$$y = c_1\cos 3x + c_2\sin 3x + 1/24 - (1/10)\cos 2x - (1/56)\cos 4x.$$

46. By the formula for $\cos^3 x$ in Problem 43, the differential equation can be written as

$$y'' + y = \tfrac{3}{4}x\cos x + \tfrac{1}{4}x\cos 3x.$$

The complementary solution is $y_c = c_1\cos x + c_2\sin x$, so we substitute the trial solution

$$y_p = x\cdot\left[(A+Bx)\cos x + (C+Dx)\sin x\right] + \left[(E+Fx)\cos 3x + (G+Hx)\sin 3x\right].$$

We find that $A = 3/16, B = C = 0, D = 3/16, E = 0, F = -1/32, G = 3/128, H = 0$. Hence the general solution is given by $y = y_c + y_1 + y_2$ where

$$y_1 = (3x\cos x + 3x^2\sin x)/16 \quad \text{and} \quad y_2 = (3\sin 3x - 4x\cos 3x)/128.$$

In Problems 47–49 we list the independent solutions y_1 and y_2 of the associated homogeneous equation, their Wronskian $W = W(y_1, y_2)$, the coefficient functions

$$u_1(x) = -\int \frac{y_2(x)f(x)}{W(x)}\,dx \quad \text{and} \quad u_2(x) = \int \frac{y_1(x)f(x)}{W(x)}\,dx$$

in the particular solution $y_p = u_1y_1 + u_2y_2$ of Eq. (32) in the text, and finally y_p itself.

47. $y_1 = e^{-2x}$, $\quad y_2 = e^{-x}$, $\quad W = e^{-3x}$

$u_1 = -(4/3)e^{3x}$, $\quad u_2 = 2e^{2x}$,

$y_p = (2/3)e^x$

48. $y_1 = e^{-2x}$, $\quad y_2 = e^{4x}$, $\quad W = 6e^{2x}$

$u_1 = -x/2$, $\quad u_2 = -e^{-6x}/12$,

$y_p = -(6x+1)e^{-2x}/12$

49. $y_1 = e^{2x}$, $\quad y_2 = xe^{2x}$, $\quad W = e^{4x}$

$u_1 = -x^2$, $\quad u_2 = 2x$,

$y_p = x^2e^{2x}$

50. The complementary function is $y_1 = c_1 \cosh 2x + c_2 \sinh 2x$, so the Wronskian is

$$W = 2\cosh^2 2x - 2\sinh^2 2x = 2,$$

so when we solve Equations (31) simultaneously for u_1' and u_2', integrate each and substitute in $y_p = y_1 u_1 + y_2 u_2$, the result is

$$y_p = -(\cosh 2x)\int \tfrac{1}{2}(\sinh 2x)(\sinh 2x)\,dx + (\sinh 2x)\int \tfrac{1}{2}(\cosh 2x)(\sinh 2x)\,dx\,.$$

Using the identities $2\sinh^2 x = \cosh 2x - 1$ and $2\sinh x\cosh x = \sinh 2x$, we evaluate the integrals and find that

$$y_p = (4x\cosh 2x - \sinh 4x\cosh 2x + \cosh 4x\sinh 2x)/16,$$

$$y_p = (4x\cosh 2x - \sinh 2x)/16.$$

51. $y_1 = \cos 2x, \qquad y_2 = \sin 2x, \qquad W = 2$

Liberal use of trigonometric sum and product identities yields

$u_1 = (\cos 5x - 5\cos x)/20, \qquad u_1 = (\sin 5x - 5\sin x)/20$

$y_p = -(1/4)(\cos 2x\cos x - \sin 2x\sin x) + (1/20)(\cos 5x\cos 2x + \sin 5x\sin 2x)$

$\quad = -(1/5)\cos 3x$ (!)

52. $y_1 = \cos 3x, \qquad y_2 = \sin 3x, \qquad W = 3$

$u_1 = -(6x - \sin 6x)/36, \qquad u_1 = -(1 + \cos 6x)/36$

$y_p = -(x\cos 3x)/6$

53. $y_1 = \cos 3x, \qquad y_2 = \sin 3x, \qquad W = 3$

$u_1' = -(2/3)\tan 3x, \qquad u_2' = 2/3$

$y_p = (2/9)[3x\sin 3x + (\cos 3x)\ln|\cos 3x|]$

54. $y_1 = \cos x, \qquad y_2 = \sin x, \qquad W = 1$

$u_1' = -\csc x, \qquad u_2' = \cos x\csc^2 x$

$y_p = -1 - (\cos x)\ln|\csc x - \cot x|$

55. $y_1 = \cos 2x, \qquad y_2 = \sin 2x, \qquad W = 2$

$u_1' = -(1/2)\sin^2 x\sin 2x = -(1/4)(1 - \cos 2x)\sin 2x$

$u_2' = (1/2)\sin^2 x \cos 2x = (1/4)(1 - \cos 2x)\cos 2x$

$y_p = (1 - x \sin 2x)/8$

56. $y_1 = e^{-2x}$, $y_2 = e^{2x}$, $W = 4$

$u_1 = -(3x - 1)e^{3x}/36$, $u_2 = -(x + 1)e^{-x}/4$

$y_p = -e^x(3x + 2)/9$

57. With $y_1 = x$, $y_2 = x^{-1}$, and $f(x) = 72x^3$, Equations (31) in the text take the form

$$\begin{aligned} x u_1' + x^{-1} u_2' &= 0, \\ u_1' - x^{-2} u_2' &= 72x^3. \end{aligned}$$

Upon multiplying the second equation by x and then adding, we readily solve first for

$$u_1' = 36x^3, \quad \text{so} \quad u_1 = 9x^4$$

and then

$$u_2' = -x^2 u_1' = -36x^5, \quad \text{so} \quad u_2 = -6x^6.$$

Then it follows that

$$y_p = y_1 u_1 + y_2 u_2 = (x)(9x^4) + (x^{-1})(-6x^6) = 3x^5.$$

58. Here it is important to remember that — for variation of parameters — the differential equation must be written in standard form with leading coefficient 1. We therefore rewrite the given equation with complementary function $y_c = c_1 x^2 + c_2 x^3$ as

$$y'' - (4/x)y' + (6/x^2)y = x.$$

Thus $f(x) = x$, and $W = x^4$, so simultaneous solution of Equations (31) as in Problem 50 (followed by integration of u_1' and u_2') yields

$$\begin{aligned} y_p &= -x^2 \int x^3 \cdot x \cdot x^{-4}\, dx + x^3 \int x^2 \cdot x \cdot x^{-4}\, dx \\ &= -x^2 \int dx + x^3 \int (1/x)\, dx = x^3(\ln x - 1). \end{aligned}$$

59. $y_1 = x^2$, $y_2 = x^2 \ln x$,

$W = x^3$, $f(x) = x^2$

$u_1' = -x \ln x$, $u_2' = x$

$y_p = x^4/4$

60. $y_1 = x^{1/2}, \qquad y_2 = x^{3/2}$

$f(x) = 2x^{-2/3}; \qquad W = x$

$u_1 = -12x^{5/6}/5, \qquad u_2 = -12x^{-1/6}$

$y_p = -72x^{4/3}/5$

61. $y_1 = \cos(\ln x), \qquad y_2 = \sin(\ln x), \qquad W = 1/x,$

$f(x) = (\ln x)/x^2$

$u_1 = (\ln x)\cos(\ln x) - \sin(\ln x)$

$u_2 = (\ln x)\sin(\ln x) + \cos(\ln x)$

$y_p = \ln x \quad (!)$

62. $y_1 = x, \qquad y_2 = 1 + x^2,$

$W = x^2 - 1, \qquad f(x) = 1$

$u_1' = (1 + x^2)/(1 - x^2), \qquad u_2' = x/(x^2 - 1)$

$y_p = -x^2 + x \ln|(1 + x)/(1 - x)| + (1/2)(1 + x^2)\ln|1 - x^2|$

63. This is simply a matter of solving the equations in (31) for the derivatives

$$u_1' = -\frac{y_2(x)f(x)}{W(x)} \quad \text{and} \quad u_2' = \frac{y_1(x)f(x)}{W(x)},$$

integrating each, and then substituting the results in (32).

64. Here we have $y_1(x) = \cos x,\ y_2(x) = \sin x,\ W(x) = 1,\ f(x) = 2\sin x$, so (33) gives

$$\begin{aligned}
y_p(x) &= -(\cos x)\int \sin x \cdot 2\sin x\,dx + (\sin x)\int \cos x \cdot 2\sin x\,dx \\
&= -(\cos x)\int (1-\cos 2x)\,dx + (\sin x)\int 2(\sin x)\cdot \cos x\,dx \\
&= -(\cos x)(x - \sin x\cos x) + (\sin x)(\sin^2 x) \\
&= -x\cos x + (\sin x)(\cos^2 x + \sin^2 x) \\
y_p(x) &= -x\cos x + \sin x
\end{aligned}$$

But we can drop the term $\sin x$ because it satisfies the associated homogeneous equation $y'' + y = 0$.

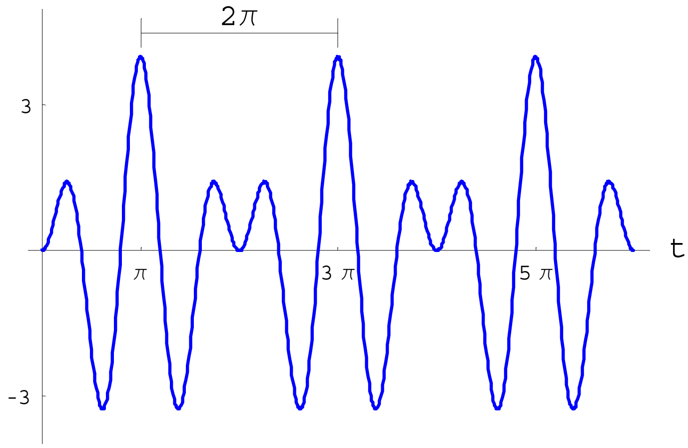
SECTION 5.6

FORCED OSCILLATIONS AND RESONANCE

1. Trial of $x = A\cos 2t$ yields the particular solution $x_p = 2\cos 2t$. (Can you see that — because the differential equation contains no first-derivative term — there is no need to include a $\sin 2t$ term in the trial solution?) Hence the general solution is

$$x(t) = c_1\cos 3t + c_2\sin 3t + 2\cos 2t.$$

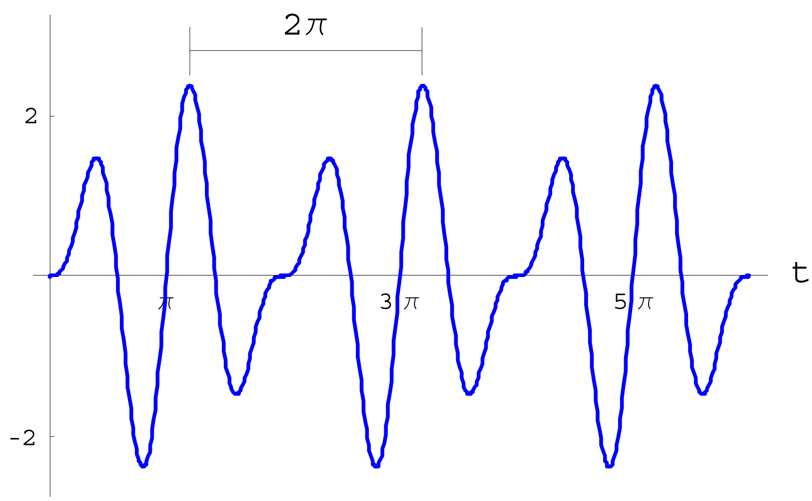
The initial conditions imply that $c_1 = -2$ and $c_2 = 0$, so $x(t) = 2\cos 2t - 2\cos 3t$. The following figure shows the graph of $x(t)$.



2. Trial of $x = A\sin 3t$ yields the particular solution $x_p = -\sin 3t$. Then we impose the initial conditions $x(0) = x'(0) = 0$ on the general solution

$$x(t) = c_1\cos 2t + c_2\sin 2t - \sin 3t,$$

and find that $x(t) = \frac{3}{2}\sin 2t - \sin 3t$. The following figure shows the graph of $x(t)$.



3. First we apply the method of undetermined coefficients — with trial solution $x = A\cos 5t + B\sin 5t$ — to find the particular solution

$$x_p = 3\cos 5t + 4\sin 5t$$

$$= 5\left[\frac{3}{5}\cos 5t + \frac{4}{5}\sin 5t\right] = 5\cos(5t - \beta)$$

where $\beta = \tan^{-1}(4/3) \approx 0.9273$. Hence the general solution is

$$x(t) = c_1\cos 10t + c_2\sin 10t + 3\cos 5t + 4\sin 5t.$$

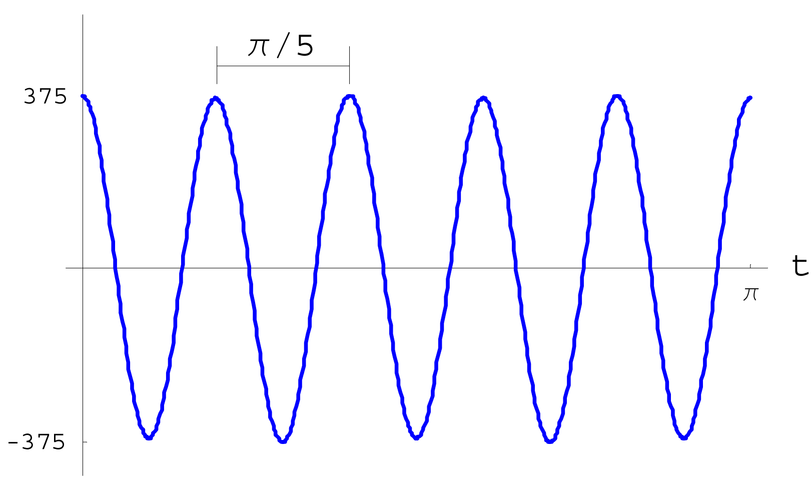
The initial conditions $x(0) = 375$, $x'(0) = 0$ now yield $c_1 = 372$ and $c_2 = -2$, so the part of the solution with frequency $\omega = 10$ is

$$x_c = 372\cos 10t - 2\sin 10t$$

$$= \sqrt{138388}\left[\frac{372}{\sqrt{138388}}\cos 10t - \frac{2}{\sqrt{138388}}\sin 10t\right]$$

$$= \sqrt{138388}\cos(10t - \alpha)$$

where $\alpha = 2\pi - \tan^{-1}(1/186) \approx 6.2778$ is a fourth-quadrant angle. The following figure shows the graph of $x(t)$.

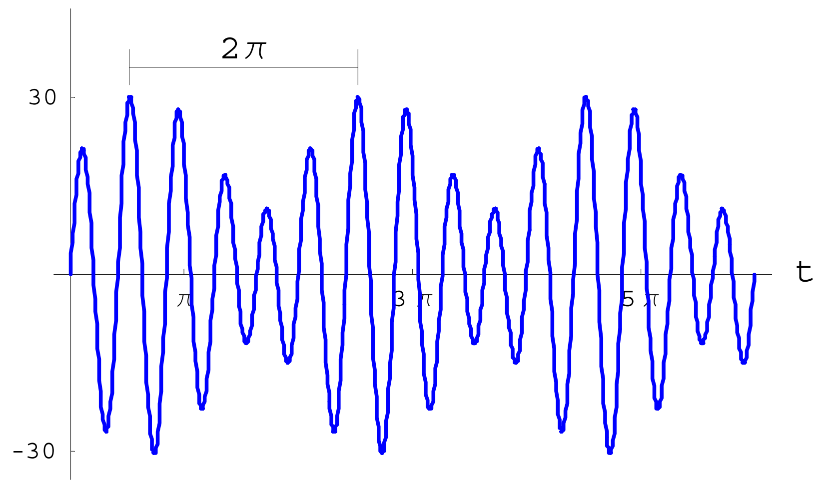


4. Noting that there is no first-derivative term, we try $x = A\cos 4t$ and find the particular solution $x_p = 10\cos 4t$. Then imposition of the initial conditions on the general solution $x(t) = c_1\cos 5t + c_2\sin 5t + 10\cos 4t$ yields

$$x(t) = (-10\cos 5t + 18\sin 5t) + 10\cos 4t$$

$$
\begin{aligned}
&= 2(-5\cos 5t + 9\sin 5t) + 10\cos 4t \\
&= 2\sqrt{106}\left(-\frac{5}{\sqrt{106}}\cos 5t + \frac{9}{\sqrt{106}}\sin 5t\right) + 10\cos 4t \\
&= 2\sqrt{106}\cos(5t - \alpha)
\end{aligned}
$$

where $\alpha = \pi - \tan^{-1}(9/5) \approx 2.0779$ is a second-quadrant angle. The following figure shows the graph of $x(t)$.



5. Substitution of the trial solution $x = C\cos\omega t$ gives $C = F_0/(k - m\omega^2)$. Then imposition of the initial conditions $x(0) = x_0$, $x'(0) = 0$ on the general solution

$$x(t) = c_1\cos\omega_0 t + c_2\sin\omega_0 t + C\cos\omega t \qquad (\text{where } \omega_0 = \sqrt{k/m})$$

gives the particular solution $x(t) = (x_0 - C)\cos\omega_0 t + C\cos\omega t$.

6. First, let's write the differential equation in the form $x'' + \omega_0^2 x = (F_0/m)\cos\omega_0 t$, which is the same as Eq. (13) in the text, and therefore has the particular solution $x_p = (F_0/2m\omega_0)t\sin\omega_0 t$ given in Eq. (14). When we impose the initial conditions $x(0) = 0$, $x'(0) = v_0$ on the general solution

$$x(t) = c_1\cos\omega_0 t + c_2\sin\omega_0 t + (F_0/2m\omega_0)\, t\sin\omega_0 t$$

we find that $c_1 = 0$, $c_2 = v_0/\omega_0$. The resulting resonance solution of our initial value problem is

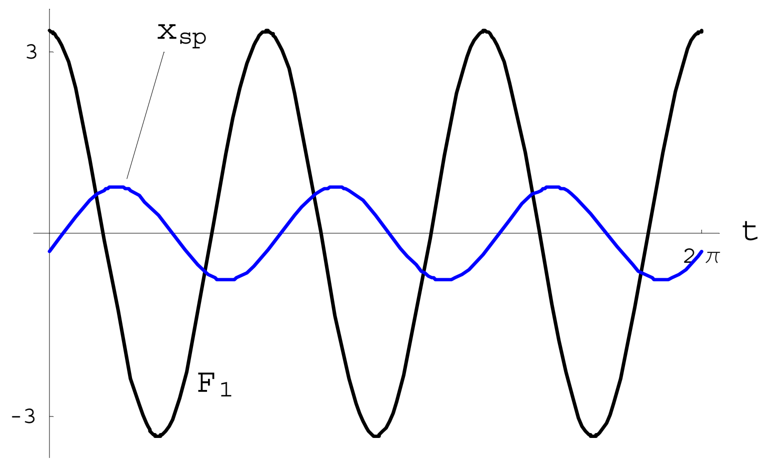
$$x(t) = \frac{2mv_0 + F_0 t}{2m\omega_0}\sin\omega_0 t.$$

In Problems 7–10 we give first the trial solution x_p involving undetermined coefficients A and B, then the equations that determine these coefficients, and finally the resulting steady periodic solution x_{sp}. In each case the figure shows the graphs of $x_{sp}(t)$ and the adjusted forcing function $F_1(t) = F(t)/m\omega$.

7. $x_p = A\cos 3t + B\sin 3t; \qquad -5A + 12B = 10, \quad 12A + 5B = 0$

$$x_{sp}(t) = -\frac{50}{169}\cos 3t + \frac{120}{169}\sin 3t = \frac{10}{13}\left(-\frac{5}{13}\cos 3t + \frac{12}{13}\sin 3t\right) = \frac{10}{13}\cos(3t - \alpha)$$

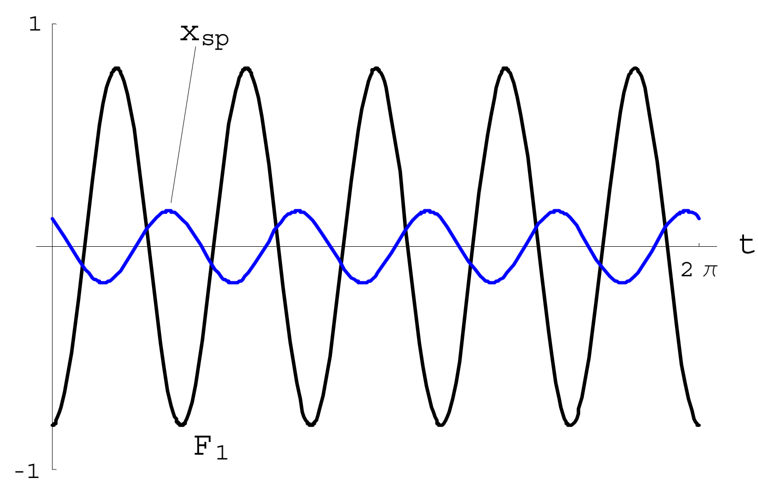
$\alpha = \pi - \tan^{-1}(12/5) \approx 1.9656 \qquad$ (2nd quadrant angle)



8. $x_p = A\cos 5t + B\sin 5t; \qquad -20A + 15B = -4, \quad 15A + 20B = 0$

$$x_{sp}(t) = \frac{16}{125}\cos 5t - \frac{12}{125}\sin 5t = \frac{4}{25}\left(\frac{4}{5}\cos 5t - \frac{3}{5}\sin 5t\right) = \frac{4}{25}\cos(5t - \alpha)$$

$\alpha = 2\pi - \tan^{-1}(3/4) \approx 5.6397 \qquad$ (4th quadrant angle)

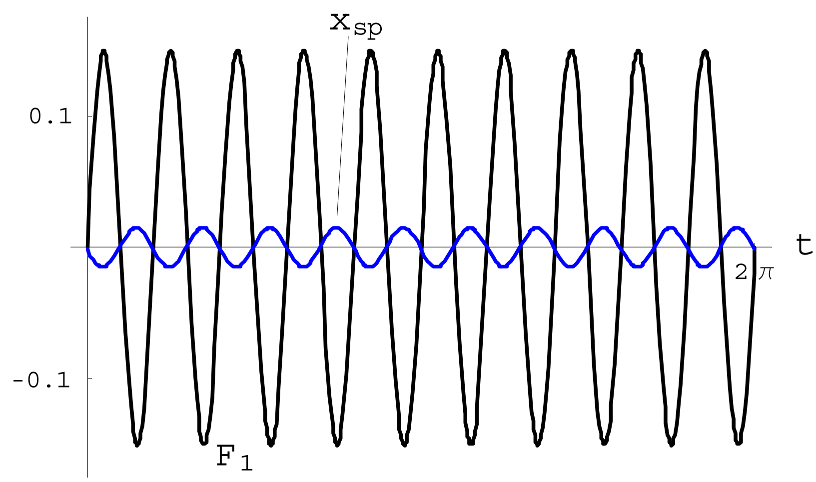


9. $x_p = A\cos 10t + 10\sin 5t; \qquad -199A + 20B = 0, \quad 20A + 199B = -3$

$$x_{sp}(t) = -\frac{60}{40001}\cos 10t - \frac{597}{40001}\sin 10t$$

$$= \frac{3}{\sqrt{40001}}\left(-\frac{20}{\sqrt{40001}}\cos 10t - \frac{199}{\sqrt{40001}}\sin 10t\right) = \frac{3}{\sqrt{40001}}\cos(10t-\alpha)$$

$\alpha = \pi + \tan^{-1}(199/20) \approx 4.6122$ (3rd quadrant angle)

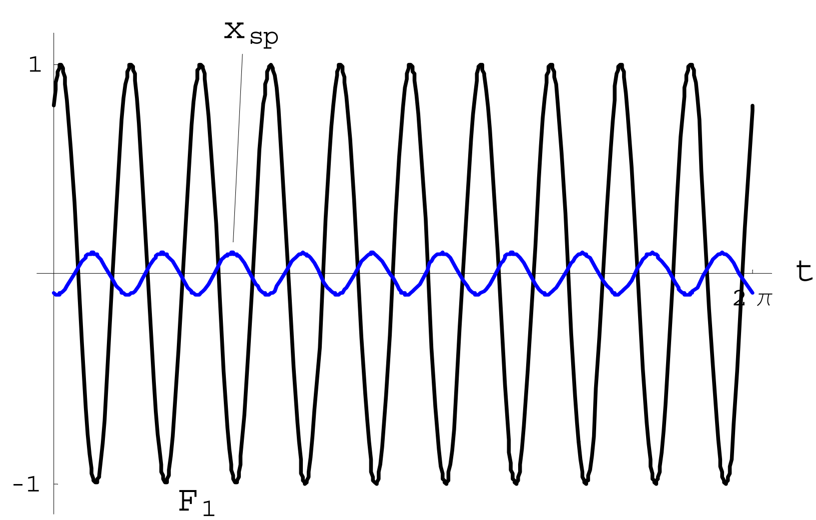


10. $x_p = A\cos 10t + 10\sin 5t; \qquad -97A + 30B = 8, \quad 30A + 97B = -6$

$$x_{sp}(t) = -\frac{956}{10309}\cos 10t - \frac{342}{10309}\sin 10t$$

$$= \frac{2\sqrt{257725}}{10309}\left(-\frac{478}{\sqrt{257725}}\cos 10t - \frac{171}{\sqrt{257725}}\sin 10t\right) = \frac{10}{793}\sqrt{61}\cos(10t-\alpha)$$

$\alpha = \pi + \tan^{-1}(171/478) \approx 3.4851$ (3rd quadrant angle)



Each solution in Problems 11–14 has two parts. For the first part, we give first the trial solution x_p involving undetermined coefficients A and B, then the equations that determine these coefficients, and finally the resulting steady periodic solution x_{sp}. For the second part, we give first the general solution $x(t)$ involving the coefficients c_1 and c_2 in the transient solution, then the equations that determine these coefficients, and finally the resulting transient solution x_{tr} so that $x(t) = x_{tr}(t) + x_{sp}(t)$ satisfies the given initial conditions. For each problem, the graph shows the graphs of both $x(t)$ and $x_{sp}(t)$.

11. $x_p = A\cos 3t + B\sin 3t; \qquad -4A + 12B = 10, \quad 12A + 4B = 0$

$$x_{sp}(t) = -\frac{1}{4}\cos 3t + \frac{3}{4}\sin 3t = \frac{\sqrt{10}}{4}\left(-\frac{1}{\sqrt{10}}\cos 3t + \frac{3}{\sqrt{10}}\sin 3t\right) = \frac{\sqrt{10}}{4}\cos(3t - \alpha)$$

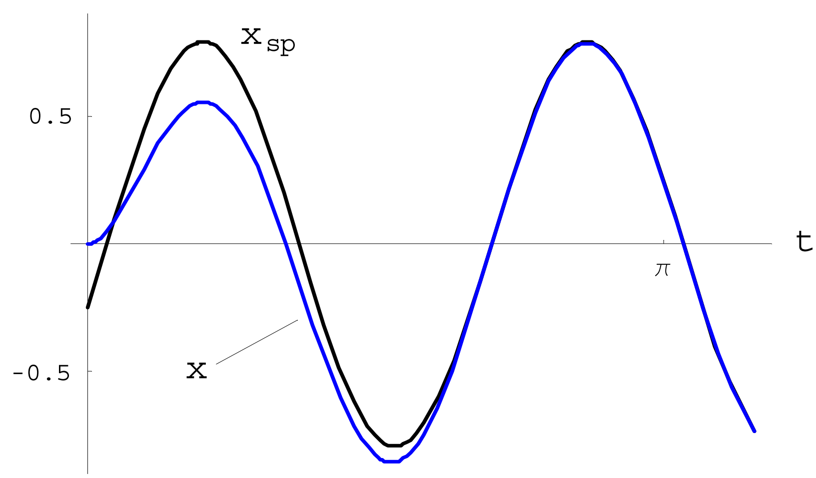
$\alpha = \pi - \tan^{-1}(3) \approx 1.8925$ (2nd quadrant angle)

$x(t) = e^{-2t}(c_1\cos t + c_2\sin t) + x_{sp}(t); \qquad c_1 - \tfrac{1}{4} = 0, \quad -2c_1 + c_2 + \tfrac{9}{4} = 0$

$$x_{tr}(t) = e^{-2t}\left(\frac{1}{4}\cos t - \frac{7}{4}\sin t\right) = \frac{\sqrt{50}}{4}e^{-2t}\left(\frac{1}{\sqrt{50}}\cos t - \frac{7}{\sqrt{50}}\sin t\right)$$

$$= \frac{5}{4}\sqrt{2}\,e^{-2t}\cos(t - \beta)$$

$\beta = 2\pi - \tan^{-1}(7) \approx 4.8543$ (4th quadrant angle)



12. $x_p = A\cos 5t + B\sin 5t; \qquad 12A - 30B = 0, \quad 30A + 12B = -10$

$$x_{sp}(t) = -\frac{25}{87}\cos 5t - \frac{10}{87}\sin 5t$$

$$= \frac{5\sqrt{29}}{87}\left(-\frac{5}{\sqrt{29}}\cos 3t - \frac{2}{\sqrt{29}}\sin 3t\right) = \frac{5}{3\sqrt{29}}\cos(3t-\alpha)$$

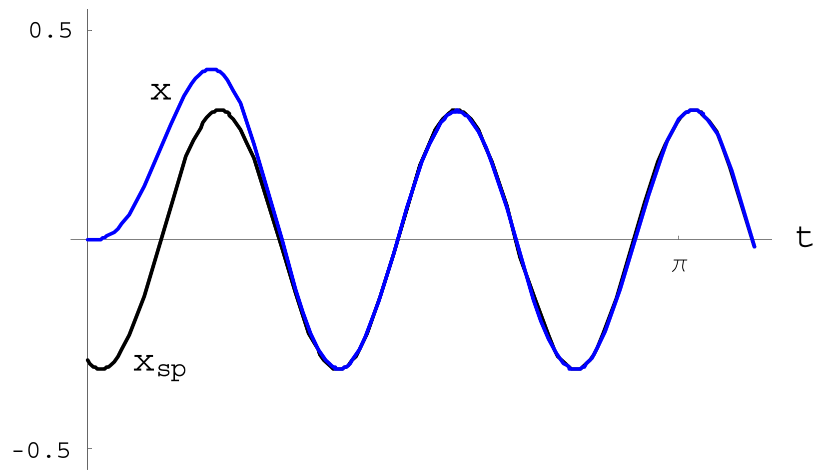
$\alpha = \pi + \tan^{-1}(2/5) \approx 3.5221 \qquad$ (3rd quadrant angle)

$x(t) = e^{-3t}\left(c_1\cos 2t + c_2\sin 2t\right) + x_{\text{sp}}(t); \qquad c_1 - 25/87 = 0, \quad -3c_1 + 2c_2 - 50/87 = 0$

$$x_{\text{tr}}(t) = e^{-3t}\left(\frac{50}{174}\cos 2t + \frac{125}{174}\sin 2t\right)$$

$$= \frac{25\sqrt{29}}{174}e^{-3t}\left(\frac{2}{\sqrt{29}}\cos 2t + \frac{5}{\sqrt{29}}\sin 2t\right) = \frac{25}{6\sqrt{29}}e^{-3t}\cos(2t-\beta)$$

$\beta = \tan^{-1}(5/2) \approx 1.1903 \qquad$ (1st quadrant angle)



13. $\quad x_{\text{p}} = A\cos 10t + B\sin 10t; \qquad -74A + 20B = 600, \quad 20A + 74B = 0$

$$x_{\text{sp}}(t) = -\frac{11100}{1469}\cos 10t + \frac{3000}{1469}\sin 10t$$

$$= \frac{300}{\sqrt{1469}}\left(-\frac{37}{\sqrt{1469}}\cos 10t + \frac{10}{\sqrt{1469}}\sin 10t\right) = \frac{300}{\sqrt{1469}}\cos(10t-\alpha)$$

$\alpha = \pi - \tan^{-1}(10/37) \approx 2.9320 \qquad$ (2nd quadrant angle)

$x(t) = e^{-t}\left(c_1\cos 5t + c_2\sin 5t\right) + x_{\text{sp}}(t);$

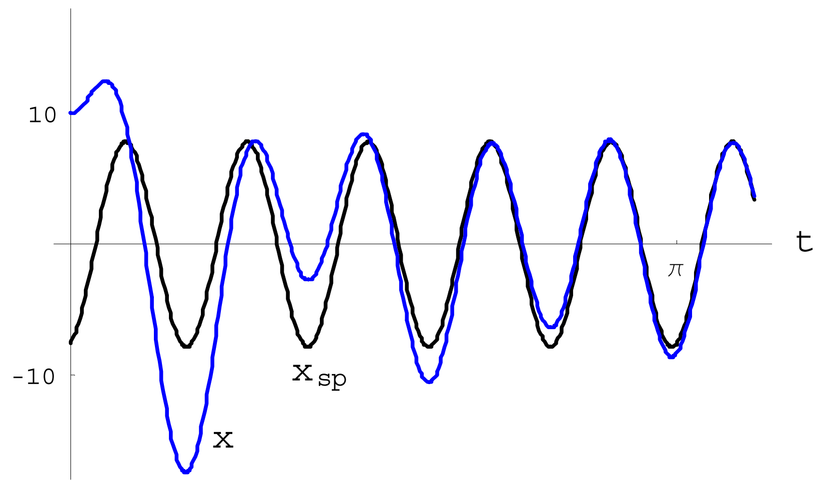
$c_1 - 11100/1469 = 10, \qquad -c_1 + 5c_2 = -30000/1469$

$$x_{\text{tr}}(t) = \frac{e^{-t}}{1469}\left(25790\cos 5t - 842\sin 5t\right)$$

$$= \frac{2\sqrt{166458266}}{1469} e^{-t} \left(\frac{12895}{\sqrt{166458266}} \cos 5t - \frac{421}{\sqrt{166458266}} \sin 5t \right)$$

$$= 2\sqrt{\frac{113314}{1469}}\, e^{-t} \cos(5t - \beta)$$

$\beta \;=\; 2\pi - \tan^{-1}(421/12895) \;\approx\; 6.2505$ (4th quadrant angle)



14. $x_p \;=\; A\cos t + B\sin t; \qquad 24A + 8B \;=\; 200, \quad -8A + 24B \;=\; 520$

$$x_{sp}(t) \;=\; \cos t + 22\sin t \;=\; \sqrt{485}\left(\frac{1}{\sqrt{485}} \cos t + \frac{22}{\sqrt{485}} \sin t \right) \;=\; \sqrt{485}\cos(t - \alpha)$$

$\alpha \;=\; \tan^{-1}(22) \;\approx\; 1.5254$ (1st quadrant angle)

$x(t) \;=\; e^{-4t}\left(c_1 \cos 3t + c_2 \sin 3t\right) + x_{sp}(t);$

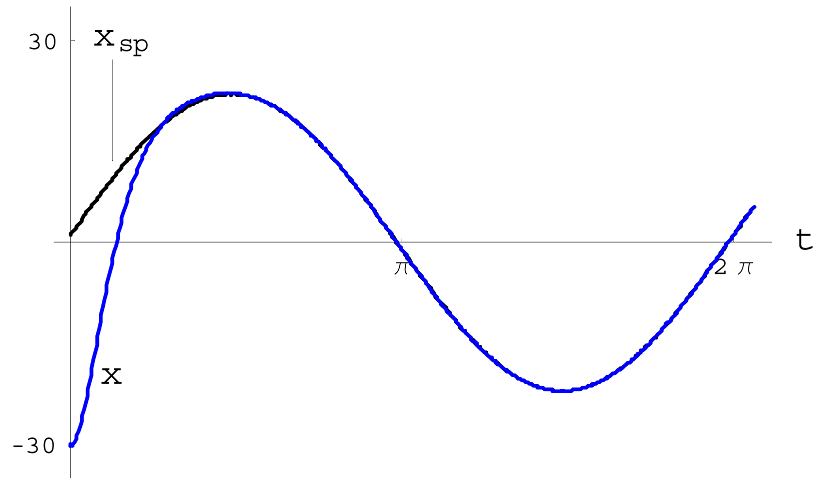
$c_1 + 1 \;=\; -30, \qquad -4c_1 + 3c_2 + 22 \;=\; -10$

$x_{tr}(t) \;=\; e^{-4t}\left(-31\cos 3t - 52\sin 3t\right)$

$$= \sqrt{3665}\, e^{-4t} \left(-\frac{31}{\sqrt{3665}} \cos 3t - \frac{52}{\sqrt{3665}} \sin 3t \right) \;=\; \sqrt{3665}\, e^{-4t} \cos(3t - \beta)$$

$\beta \;=\; \pi + \tan^{-1}(52/31) \;\approx\; 4.1748$ (3rd quadrant angle)

The figure at the top of the next page shows the graphs of $x(t)$ and $x_{sp}(t)$.

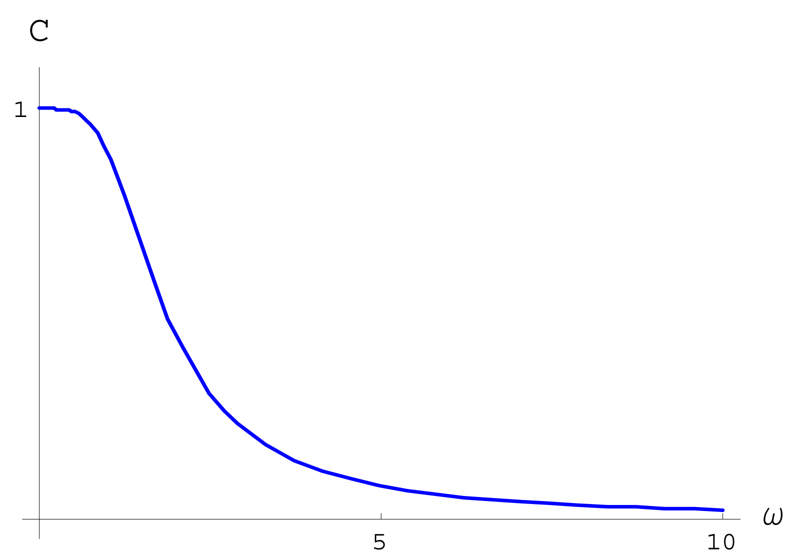


In Problems 15–18 we substitute $x(t) = A(\omega)\cos\omega t + B(\omega)\sin\omega t$ into the differential equation $mx'' + cx' + kx = F_0\cos\omega t$ with the given numerical values of m, c, k, and F_0. We give first the equations in A and B that result upon collection of coefficients of $\cos\omega t$ and $\sin\omega t$, and then the values of $A(\omega)$ and $B(\omega)$ that we get by solving these equations. Finally, $C = \sqrt{A^2 + B^2}$ gives the amplitude of the resulting forced oscillations as a function of the forcing frequency ω, and we show the graph of the function $C(\omega)$.

15. $(2-\omega^2)A + 2\omega B = 2, \quad -2\omega A + (2-\omega^2)B = 0$

$$A = \frac{2\left(2-\omega^2\right)}{4+\omega^4}, \qquad B = \frac{4\omega}{4+\omega^4}$$

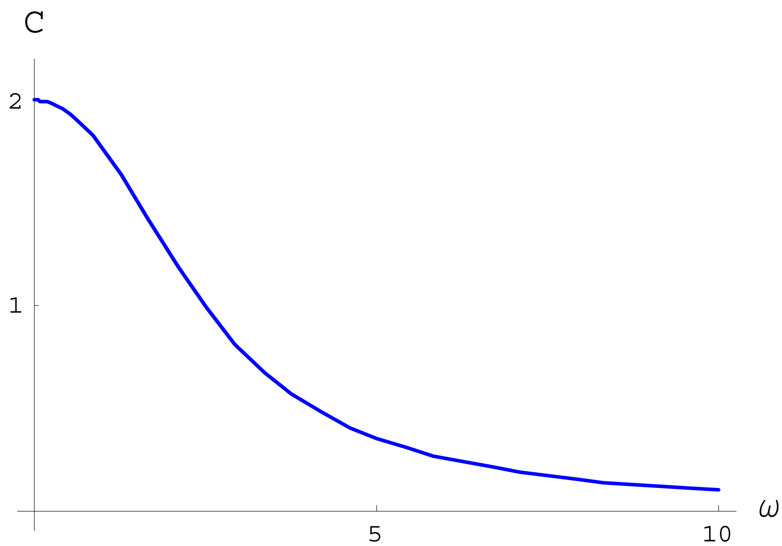
$C(\omega) = 2/\sqrt{4+\omega^4}$ begins with $C(0) = 1$ and steadily decreases as ω increases. Hence there is no practical resonance frequency.



16. $(5-\omega^2)A+4\omega B = 10, \quad -4\omega A+(5-\omega^2)B = 0$

$$A = \frac{10\left(5-\omega^2\right)}{25+6\omega^2+\omega^4}, \quad B = \frac{40\omega}{25+6\omega^2+\omega^4}$$

$C(\omega) = 10/\sqrt{25+6\omega^2+\omega^4}$ begins with $C(0)=2$ and steadily decreases as ω increases. Hence there is no practical resonance frequency.



17. $(45-\omega^2)A+6\omega B = 50, \quad -6\omega A+(45-\omega^2)B = 0$

$$A = \frac{50\left(45-\omega^2\right)}{2025-54\omega^2+\omega^4}, \quad B = \frac{300\omega}{2025-54\omega^2+\omega^4}$$

$C(\omega) = 50/\sqrt{2025-54\omega^2+\omega^4}$ so, to find its maximum value, we calculate the derivative

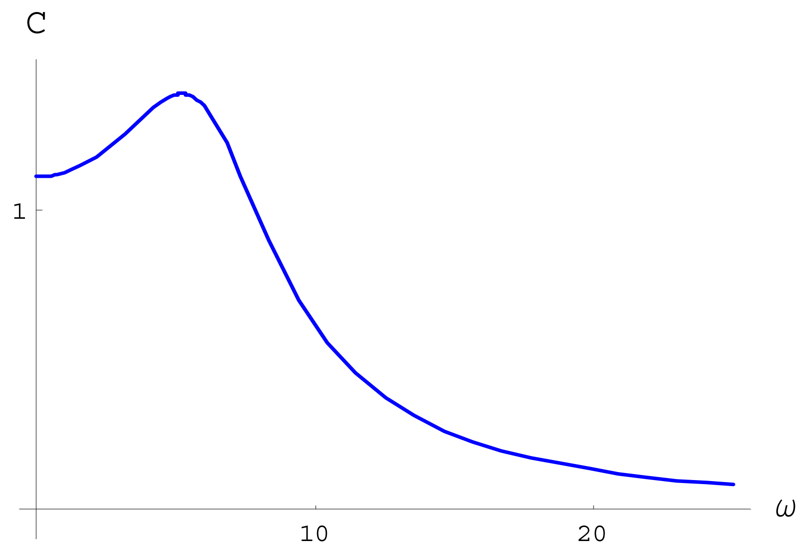
$$C'(\omega) = \frac{-100\,\omega(-27+\omega^2)}{(2025-54\omega^2+\omega^4)^{3/2}}.$$

Hence the practical resonance frequency (where the derivative vanishes) is $\omega = \sqrt{27} = 3\sqrt{3}$. The graph of $C(\omega)$ is shown at the top of the next page.

18. $(650-\omega^2)A+10\omega B = 100, \quad -10\omega A+(650-\omega^2)B = 0$

$$A = \frac{100\left(650-\omega^2\right)}{422500-1200\omega^2+\omega^4}, \quad B = \frac{1000\omega}{422500-1200\omega^2+\omega^4}$$

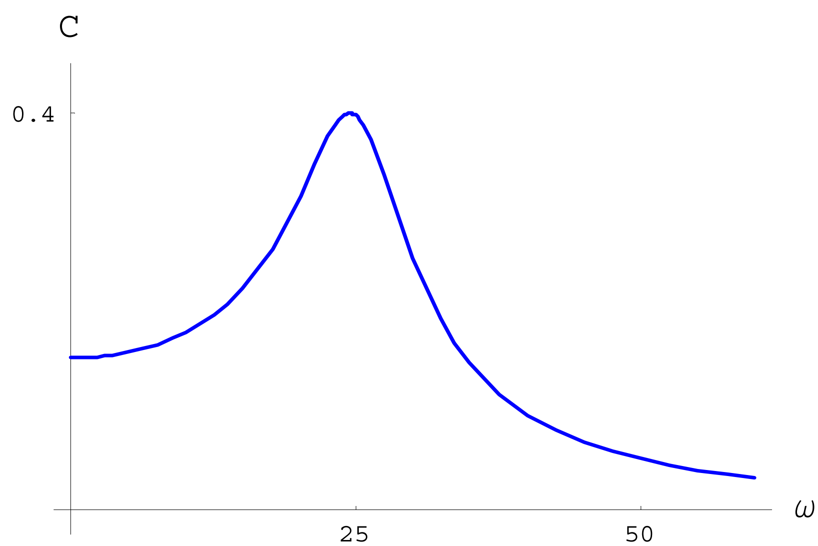
$C(\omega) = 100/\sqrt{422500-1200\omega^2+\omega^4}$ so, to find its maximum value,



we calculate the derivative

$$C'(\omega) \;=\; \frac{-200\,\omega(-600+\omega^2)}{(422500-1200\omega^2+\omega^4)^{3/2}}.$$

Hence the practical resonance frequency (where the derivative vanishes) is $\omega = \sqrt{600} = 10\sqrt{6}.$



19. m = 100/32 slugs and k = 1200 lb/ft, so the critical frequency is $\omega_0 = \sqrt{k/m}$ $= \sqrt{384}$ rad/sec $= \sqrt{384}/2\pi \approx 3.12$ Hz.

20. Let the machine have mass m. Then the force $F = mg = 9.8m$ (the machine's weight) causes a displacement of x = 0.5 cm = 1/200 meters, so Hooke's law

$$F = kx, \quad \text{that is,} \quad mg = k(1/200)$$

gives the spring constant is $k = 200mg$ (N/m). Hence the resonance frequency is

$$\omega = \sqrt{k/m} = \sqrt{200g} \approx \sqrt{200 \times 9.8} \approx 44.27 \ \text{rad/sec} \approx 7.05 \ \text{Hz},$$

which is about 423 rpm (revolutions per minute).

21. If θ is the angular displacement from the vertical, then the (essentially horizontal) displacement of the mass is $x = L\theta$, so twice its total energy (KE + PE) is

$$m(x')^2 + kx^2 + 2mgh = mL^2(\theta')^2 + kL^2\theta^2 + 2mgL(1 - \cos\theta) = C.$$

Differentiation, substitution of $\theta \approx \sin\theta$, and simplification yields

$$\theta'' + (k/m + g/L)\theta = 0$$

so

$$\omega_0 = \sqrt{k/m + g/L}.$$

22. Let x denote the displacement of the mass from its equilibrium position, $v = x'$ its velocity, and $\omega = v/a$ the angular velocity of the pulley. Then conservation of energy yields

$$mv^2/2 + I\omega^2/2 + kx^2/2 - mgx = C.$$

When we differentiate both sides with respect to t and simplify the result, we get the differential equation

$$(m + I/a^2)x'' + kx = mg.$$

Hence $\omega = \sqrt{k/\left(m + I/a^2\right)}$.

23. **(a)** In ft-lb-sec units we have $m = 1000$ and $k = 10000$, so $\omega_0 = \sqrt{10}$ rad/sec $\approx$ 0.50 Hz.

(b) We are given that $\omega = 2\pi/2.25 \approx 2.79$ rad/sec, and the equation $mx'' + kx = F(t)$ simplifies to

$$x'' + 10x = (1/4)\omega^2 \sin \omega t.$$

When we substitute $x(t) = A \sin \omega t$ we find that the amplitude is

$$A = \omega^2/4\left(10 - \omega^2\right) \approx 0.8854 \ \text{ft} \approx 10.63 \ \text{in}.$$

24. By the identity of Problem 43 in Section 5.5, the differential equation is

$$mx'' + kx = F_0(3\cos\omega t + \cos 3\omega t)/4.$$

Hence resonance occurs when either ω or 3ω equals $\omega_0 = \sqrt{k/m}$, that is, when either $\omega = \omega_0$ or $\omega = \omega_0/3$.

25. Substitution of the trial solution $x = A\cos\omega t + B\sin\omega t$ in the differential equation, and then collection of coefficients as usual yields the equations

$$(k - m\omega^2)A + (c\omega)B = 0, \quad -(c\omega)A + (k - m\omega^2)B = F_0$$

with coefficient determinant $\Delta = (k - m\omega^2)^2 + (c\omega)^2$ and solution $A = -(c\omega)F_0/\Delta$, $B = (k - m\omega^2)F_0/\Delta$. Hence

$$x(t) = \frac{F_0}{\sqrt{\Delta}}\left[\frac{k - m\omega^2}{\sqrt{\Delta}}\sin\omega t - \frac{c\omega}{\sqrt{\Delta}}\cos\omega t\right] = C\sin(\omega t - \alpha),$$

where $C = F_0/\sqrt{\Delta}$ and $\sin\alpha = c\omega/\sqrt{\Delta}$, $\cos\alpha = (k - m\omega^2)/\sqrt{\Delta}$.

26. Let $G_0 = \sqrt{E_0^2 + F_0^2}$ and $\rho = 1/\sqrt{(k - m\omega^2) + (c\omega)^2}$. Then

$$\begin{aligned}
x_{sp}(t) &= \rho E_0\cos(\omega t - \alpha) + \rho F_0\sin(\omega t - \alpha) \\
&= \rho G_0\left[\frac{E_0}{G_0}\cos(\omega t - \alpha) + \frac{F_0}{G_0}\sin(\omega t - \alpha)\right] \\
&= \rho G_0\left[\cos\beta\,\cos(\omega t - \alpha) + \sin\beta\sin(\omega t - \alpha)\right] \\
x_{sp}(t) &= \rho G_0\cos(\omega t - \alpha - \beta)
\end{aligned}$$

where $\tan\beta = F_0/E_0$. The desired formula now results when we substitute the value of ρ defined above.

27. The derivative of $C(\omega) = F_0/\sqrt{(k - m\omega^2)^2 + (c\omega)^2}$ is given by

$$C'(\omega) = -\frac{\omega F_0}{2}\frac{(c^2 - 2km) + 2(m\omega)^2}{\left[(k - m\omega^2)^2 + (c\omega)^2\right]^{3/2}}.$$

(a) Therefore, if $c^2 \geq 2km$, it is clear from the numerator that $C'(\omega) < 0$ for all ω, so $C(\omega)$ steadily decreases as ω increases.

(b) But if $c^2 < 2km$, then the numerator (and hence $C'(\omega)$) vanishes when $\omega = \omega_m = \sqrt{k/m - c^2/2m^2} < \sqrt{k/m} = \omega_0$. Calculation then shows that

$$C''(\omega_m) = \frac{16F_0 m^3(c^2 - 2km)}{c^3\left(4km - c^2\right)^{3/2}} < 0,$$

so it follows from the second-derivative test that $C(\omega_m)$ is a local maximum value.

28. **(a)** The given differential equation corresponds to Equation (17) with $F_0 = mA\omega^2$. It therefore follows from Equation (21) that the amplitude of the steady periodic vibrations at frequency ω is

$$C(\omega) = \frac{F_0}{\sqrt{(k - m\omega^2)^2 + (c\omega)^2}} = \frac{mA\omega^2}{\sqrt{(k - m\omega^2)^2 + (c\omega)^2}}.$$

(b) Now we calculate

$$C'(\omega) = \frac{mA\omega\left[2k^2 - (2mk - c^2)\omega^2\right]}{\left[\left(k - m\omega^2\right)^2 + \left(c\omega\right)^2\right]^{3/2}},$$

and we see that the numerator vanishes when

$$\omega = \sqrt{\frac{2k^2}{2mk - c^2}} = \sqrt{\frac{k}{m}\left(\frac{2mk}{2mk - c^2}\right)} > \sqrt{\frac{k}{m}} = \omega_0.$$

29. We need only substitute $E_0 = ac\omega$ and $F_0 = ak$ in the result of Problem 26.

30. When we substitute the values $\omega = 2\pi v/L$, $m = 800$, $k = 7\times10^4$, $c = 3000$ and $L = 10$, $a = 0.05$ in the formula of Problem 29, simplify, and square, we get the function

$$Csq(v) = \frac{25\left(9\pi^2 v^2 + 122500\right)}{16\left(16\pi^4 v^4 - 64375\pi^2 v^2 + 76562500\right)^2}$$

giving the *square* of the amplitude C (in meters) as a function of the velocity v (in meters per second). Differentiation gives

$$Csq'(v) = -\frac{50\pi^2 v(9\pi^4 v^4 + 245000\pi^2 v^2 - 535937500)}{(16\pi^4 v^4 - 64375\pi^2 v^2 - 76562500)^2}.$$

Because the principal factor in the numerator is a quadratic in v^2, it is easy to solve the equation $Csq'(v) = 0$ to find where the maximum amplitude occurs; we find that the only positive solution is $v \approx 14.36$ m/sec ≈ 32.12 mi/hr. The corresponding amplitude of the car's vibrations is $\sqrt{Csq(14.36)} \approx 0.1364 \text{ m} = 13.64 \text{ cm}.$

CHAPTER 6

EIGENVALUES AND EIGENVECTORS

SECTION 6.1

INTRODUCTION TO EIGENVALUES

In each of Problems 1–32 we first list the characteristic polynomial $p(\lambda) = |\mathbf{A} - \lambda\mathbf{I}|$ of the given matrix $\mathbf{A}$, and then the roots of $p(\lambda)$ — which are the eigenvalues of $\mathbf{A}$. All of the eigenvalues that appear in Problems 1–26 are integers, so each characteristic polynomial factors readily. For each eigenvalue λ_j of the matrix $\mathbf{A}$, we determine the associated eigenvector(s) by finding a basis for the solution space of the linear system $\left(\mathbf{A} - \lambda_j\mathbf{I}\right)\mathbf{v} = \mathbf{0}$. We write this linear system in scalar form in terms of the components of $\mathbf{v} = \begin{bmatrix} a & b & \cdots \end{bmatrix}^T$. In most cases an associated eigenvector is then apparent. If $\mathbf{A}$ is a 2×2 matrix, for instance, then our two scalar equations will be multiples one of the other, so we can substitute a convenient numerical value for the first component a of $\mathbf{v}$ and then solve either equation for the second component b (or vice versa).

1. Characteristic polynomial: $p(\lambda) = \lambda^2 - 5\lambda + 6 = (\lambda-2)(\lambda-3)$

Eigenvalues: $\lambda_1 = 2, \quad \lambda_2 = 3$

With $\lambda_1 = 2$: $\left.\begin{aligned} 2a-2b &= 0 \\ a-b &= 0 \end{aligned}\right\}$ $\qquad \mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

With $\lambda_2 = 3$: $\left.\begin{aligned} a-2b &= 0 \\ a-2b &= 0 \end{aligned}\right\}$ $\qquad \mathbf{v}_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

2. Characteristic polynomial: $p(\lambda) = \lambda^2 - \lambda - 2 = (\lambda+1)(\lambda-2)$

Eigenvalues: $\lambda_1 = -1, \quad \lambda_2 = 2$

With $\lambda_1 = -1$: $\left.\begin{aligned} 6a-6b &= 0 \\ 3a-3b &= 0 \end{aligned}\right\}$ $\qquad \mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

With $\lambda_2 = 2$: $\left.\begin{aligned} 3a-6b &= 0 \\ 3a-6b &= 0 \end{aligned}\right\}$ $\qquad \mathbf{v}_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

3. Characteristic polynomial: $p(\lambda) = \lambda^2 - 7\lambda + 10 = (\lambda - 2)(\lambda - 5)$

Eigenvalues: $\lambda_1 = 2, \quad \lambda_2 = 5$

With $\lambda_1 = 2$: $\left.\begin{aligned} 6a - 6b &= 0 \\ 3a - 3b &= 0 \end{aligned}\right\}$ $\quad \mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

With $\lambda_2 = 5$: $\left.\begin{aligned} 3a - 6b &= 0 \\ 3a - 6b &= 0 \end{aligned}\right\}$ $\quad \mathbf{v}_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

4. Characteristic polynomial: $p(\lambda) = \lambda^2 - 3\lambda + 2 = (\lambda - 1)(\lambda - 2)$

Eigenvalues: $\lambda_1 = 2, \quad \lambda_2 = 5$

With $\lambda_1 = 1$: $\left.\begin{aligned} 3a - 3b &= 0 \\ 2a - 2b &= 0 \end{aligned}\right\}$ $\quad \mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

With $\lambda_2 = 2$: $\left.\begin{aligned} 2a - 3b &= 0 \\ 2a - 3b &= 0 \end{aligned}\right\}$ $\quad \mathbf{v}_2 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$

5. Characteristic polynomial: $p(\lambda) = \lambda^2 - 5\lambda + 4 = (\lambda - 1)(\lambda - 4)$

Eigenvalues: $\lambda_1 = 1, \quad \lambda_2 = 4$

With $\lambda_1 = 1$: $\left.\begin{aligned} 9a - 9b &= 0 \\ 6a - 6b &= 0 \end{aligned}\right\}$ $\quad \mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

With $\lambda_2 = 4$: $\left.\begin{aligned} 6a - 9b &= 0 \\ 6a - 9b &= 0 \end{aligned}\right\}$ $\quad \mathbf{v}_2 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$

6. Characteristic polynomial: $p(\lambda) = \lambda^2 - 5\lambda + 6 = (\lambda - 2)(\lambda - 3)$

Eigenvalues: $\lambda_1 = 1, \quad \lambda_2 = 4$

With $\lambda_1 = 2$: $\left.\begin{aligned} 4a - 4b &= 0 \\ 3a - 3b &= 0 \end{aligned}\right\}$ $\quad \mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

With $\lambda_2 = 3$: $\left.\begin{aligned} 3a - 4b &= 0 \\ 3a - 4b &= 0 \end{aligned}\right\}$ $\quad \mathbf{v}_2 = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$

7. Characteristic polynomial: $p(\lambda) = \lambda^2 - 6\lambda + 8 = (\lambda - 2)(\lambda - 4)$

Eigenvalues: $\lambda_1 = 2, \quad \lambda_2 = 4$

With $\lambda_1 = 2$: $\left.\begin{aligned} 8a-8b &= 0 \\ 6a-6b &= 0 \end{aligned}\right\}$ $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

With $\lambda_2 = 4$: $\left.\begin{aligned} 6a-8b &= 0 \\ 6a-8b &= 0 \end{aligned}\right\}$ $\mathbf{v}_2 = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$

8. Characteristic polynomial: $p(\lambda) = \lambda^2+3\lambda+2 = (\lambda+2)(\lambda+1)$

Eigenvalues: $\lambda_1 = -2, \quad \lambda_2 = -1$

With $\lambda_1 = -2$: $\left.\begin{aligned} 9a-6b &= 0 \\ 12a-8b &= 0 \end{aligned}\right\}$ $\mathbf{v}_1 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

With $\lambda_2 = -1$: $\left.\begin{aligned} 8a-6b &= 0 \\ 12a-9b &= 0 \end{aligned}\right\}$ $\mathbf{v}_2 = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$

9. Characteristic polynomial: $p(\lambda) = \lambda^2-7\lambda+12 = (\lambda-3)(\lambda-4)$

Eigenvalues: $\lambda_1 = 3, \quad \lambda_2 = 4$

With $\lambda_1 = 3$: $\left.\begin{aligned} 5a-10b &= 0 \\ 2a-4b &= 0 \end{aligned}\right\}$ $\mathbf{v}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

With $\lambda_2 = 4$: $\left.\begin{aligned} 4a-10b &= 0 \\ 2a-5b &= 0 \end{aligned}\right\}$ $\mathbf{v}_2 = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$

10. Characteristic polynomial: $p(\lambda) = \lambda^2-9\lambda+20 = (\lambda-4)(\lambda-5)$

Eigenvalues: $\lambda_1 = 4, \quad \lambda_2 = 5$

With $\lambda_1 = 4$: $\left.\begin{aligned} 5a-10b &= 0 \\ 2a-4b &= 0 \end{aligned}\right\}$ $\mathbf{v}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

With $\lambda_2 = 5$: $\left.\begin{aligned} 4a-10b &= 0 \\ 2a-5b &= 0 \end{aligned}\right\}$ $\mathbf{v}_2 = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$

11. Characteristic polynomial: $p(\lambda) = \lambda^2-9\lambda+20 = (\lambda-4)(\lambda-5)$

Eigenvalues: $\lambda_1 = 4, \quad \lambda_2 = 5$

With $\lambda_1 = 4$: $\left.\begin{aligned} 15a-10b &= 0 \\ 21a-14b &= 0 \end{aligned}\right\}$ $\mathbf{v}_1 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

With $\lambda_2 = 5$: $\left.\begin{aligned} 14a-10b &= 0 \\ 21a-15b &= 0 \end{aligned}\right\}$ $\mathbf{v}_2 = \begin{bmatrix} 5 \\ 7 \end{bmatrix}$

12. Characteristic polynomial: $p(\lambda) = \lambda^2 - 7\lambda + 212 = (\lambda-3)(\lambda-4)$

Eigenvalues: $\lambda_1 = 3, \quad \lambda_2 = 4$

With $\lambda_1 = 3$: $\left.\begin{aligned} 10a - 15b &= 0 \\ 6a - 9b &= 0 \end{aligned}\right\}$ $\quad \mathbf{v}_1 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$

With $\lambda_2 = 4$: $\left.\begin{aligned} 9a - 15b &= 0 \\ 6a - 10b &= 0 \end{aligned}\right\}$ $\quad \mathbf{v}_2 = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$

13. Characteristic polynomial: $p(\lambda) = -\lambda^3 + 3\lambda^2 - 2\lambda = -\lambda(\lambda-1)(\lambda-2)$

Eigenvalues: $\lambda_1 = 0, \quad \lambda_2 = 1, \quad \lambda_3 = 2$

With $\lambda_1 = 0$: $\left.\begin{aligned} 2a &= 0 \\ 2a - 2b - c &= 0 \\ -2a + 6b + 3c &= 0 \end{aligned}\right\}$ $\quad \mathbf{v}_1 = \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix}$

With $\lambda_2 = 1$: $\left.\begin{aligned} a &= 0 \\ 2a - 3b - c &= 0 \\ -2a + 6b + 2c &= 0 \end{aligned}\right\}$ $\quad \mathbf{v}_2 = \begin{bmatrix} 0 \\ -1 \\ 3 \end{bmatrix}$

With $\lambda_3 = 2$: $\left.\begin{aligned} 0 &= 0 \\ 2a - 4b - c &= 0 \\ -2a + 6b + c &= 0 \end{aligned}\right\}$ $\quad \mathbf{v}_3 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$

14. Characteristic polynomial: $p(\lambda) = -\lambda^3 + 7\lambda^2 - 10\lambda = -\lambda(\lambda-2)(\lambda-5)$

Eigenvalues: $\lambda_1 = 0, \quad \lambda_2 = 2, \quad \lambda_3 = 5$

With $\lambda_1 = 0$: $\left.\begin{aligned} 5a &= 0 \\ 4a - 4b - 2c &= 0 \\ -2a + 12b + 6c &= 0 \end{aligned}\right\}$ $\quad \mathbf{v}_1 = \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix}$

With $\lambda_2 = 2$: $\left.\begin{aligned} 3a &= 0 \\ 4a - 6b - 2c &= 0 \\ -2a + 12b + 4c &= 0 \end{aligned}\right\}$ $\quad \mathbf{v}_2 = \begin{bmatrix} 0 \\ -1 \\ 3 \end{bmatrix}$

With $\lambda_3 = 5$: $\left.\begin{aligned} 0 &= 0 \\ 4a - 9b - 2c &= 0 \\ -2a + 12b + c &= 0 \end{aligned}\right\}$ $\quad \mathbf{v}_3 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$

15. Characteristic polynomial: $p(\lambda) = -\lambda^3 + 3\lambda^2 - 2\lambda = -\lambda(\lambda-1)(\lambda-2)$

Eigenvalues: $\lambda_1 = 0, \quad \lambda_2 = 1, \quad \lambda_3 = 2$

With $\lambda_1 = 0$: $\left.\begin{aligned} 2a-2b &= 0 \\ 2a-2b-c &= 0 \\ -2a+2b+3c &= 0 \end{aligned}\right\}$ $\quad \mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

With $\lambda_2 = 1$: $\left.\begin{aligned} a-2b &= 0 \\ 2a-3b-c &= 0 \\ -2a+2b+2c &= 0 \end{aligned}\right\}$ $\quad \mathbf{v}_2 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$

With $\lambda_3 = 2$: $\left.\begin{aligned} -2b &= 0 \\ 2a-4b-c &= 0 \\ -2a+2b+c &= 0 \end{aligned}\right\}$ $\quad \mathbf{v}_3 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$

16. Characteristic polynomial: $p(\lambda) = -\lambda^3 + 4\lambda^2 - 3\lambda = -\lambda(\lambda-1)(\lambda-3)$

Eigenvalues: $\lambda_1 = 0, \quad \lambda_2 = 1, \quad \lambda_3 = 3$

With $\lambda_1 = 0$: $\left.\begin{aligned} a-c &= 0 \\ -2a+3b-c &= 0 \\ -6a+6b &= 0 \end{aligned}\right\}$ $\quad \mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

With $\lambda_2 = 1$: $\left.\begin{aligned} -c &= 0 \\ -2a+2b-c &= 0 \\ -6a+6b-c &= 0 \end{aligned}\right\}$ $\quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

With $\lambda_3 = 3$: $\left.\begin{aligned} -2a-c &= 0 \\ -2a-c &= 0 \\ -6a+6b-3c &= 0 \end{aligned}\right\}$ $\quad \mathbf{v}_3 = \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}$

17. Characteristic polynomial: $p(\lambda) = -\lambda^3 + 6\lambda^2 - 11\lambda + 6 = -(\lambda-1)(\lambda-2)(\lambda-3)$

Eigenvalues: $\lambda_1 = 1, \quad \lambda_2 = 2, \quad \lambda_3 = 3$

With $\lambda_1 = 1$: $\left.\begin{aligned} 2a+5b-2c &= 0 \\ b &= 0 \\ 2b &= 0 \end{aligned}\right\}$ $\quad \mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

With $\lambda_2 = 2:$ $\quad \left.\begin{array}{r} a+5b-2c = 0 \\ 0 = 0 \\ 2b-c = 0 \end{array}\right\}$ $\quad \mathbf{v}_2 = \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$

With $\lambda_3 = 3:$ $\quad \left.\begin{array}{r} 5b-2c = 0 \\ -b = 0 \\ 2b-2c = 0 \end{array}\right\}$ $\quad \mathbf{v}_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

18. Characteristic polynomial: $\quad p(\lambda) = -\lambda^3 + 6\lambda^2 - 11\lambda + 6 = -(\lambda-1)(\lambda-2)(\lambda-3)$

Eigenvalues: $\lambda_1 = 1, \quad \lambda_2 = 2, \quad \lambda_3 = 3$

With $\lambda_1 = 1:$ $\quad \left.\begin{array}{r} 0 = 0 \\ -6a+7b+2c = 0 \\ 12a-15b-4c = 0 \end{array}\right\}$ $\quad \mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}$

With $\lambda_2 = 2:$ $\quad \left.\begin{array}{r} -a = 0 \\ -6a+6b+2c = 0 \\ 12a-15b-5c = 0 \end{array}\right\}$ $\quad \mathbf{v}_2 = \begin{bmatrix} 0 \\ -1 \\ 3 \end{bmatrix}$

With $\lambda_3 = 3:$ $\quad \left.\begin{array}{r} -2a = 0 \\ -6a+5b+2c = 0 \\ 12a-15b-6c = 0 \end{array}\right\}$ $\quad \mathbf{v}_3 = \begin{bmatrix} 0 \\ -2 \\ 5 \end{bmatrix}$

19. Characteristic polynomial: $\quad p(\lambda) = -\lambda^3 + 5\lambda^2 - 7\lambda + 3 = -(\lambda-1)^2(\lambda-3)$

Eigenvalues: $\lambda_1 = \lambda_2 = 1, \quad \lambda_3 = 3$

With $\lambda_1 = 1:$ $\quad \left.\begin{array}{r} 2a+6b-2c = 0 \\ 0 = 0 \\ 0 = 0 \end{array}\right\}$ $\quad \mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix}$

The eigenspace of $\lambda_1 = 1$ is 2-dimensional. We get the eigenvector $\mathbf{v}_1$ with $b = 0,\ c = 1,$ and the eigenvector $\mathbf{v}_2$ with $b = 1,\ c = 0.$

With $\lambda_3 = 3:$ $\quad \left.\begin{array}{r} 6b-2c = 0 \\ -2b = 0 \\ -2c = 0 \end{array}\right\}$ $\quad \mathbf{v}_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

20. Characteristic polynomial: $\quad p(\lambda) = -\lambda^3 + 4\lambda^2 - 5\lambda + 2 = -(\lambda-1)^2(\lambda-2)$

Eigenvalues: $\lambda_1 = \lambda_2 = 1, \quad \lambda_3 = 2$

With $\lambda_1 = 1$: $\left.\begin{aligned} 0 &= 0 \\ -4a+6b+2c &= 0 \\ 10a-15b-5c &= 0 \end{aligned}\right\}$ $\quad \mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix}$

The eigenspace of $\lambda_1 = 1$ is 2-dimensional. We get the eigenvector $\mathbf{v}_1$ with $b = 0,\ c = 2,$ and the eigenvector $\mathbf{v}_2$ with $b = 2,\ c = 0.$

With $\lambda_3 = 2$: $\left.\begin{aligned} -a &= 0 \\ -4a+5b+2c &= 0 \\ 10a-15b-5c &= 0 \end{aligned}\right\}$ $\quad \mathbf{v}_3 = \begin{bmatrix} 0 \\ -2 \\ 5 \end{bmatrix}$

21. Characteristic polynomial: $\quad p(\lambda) = -\lambda^3 + 5\lambda^2 - 8\lambda + 4 = -(\lambda-1)(\lambda-2)^2$

Eigenvalues: $\lambda_1 = 1, \quad \lambda_2 = \lambda_3 = 2$

With $\lambda_1 = 1$: $\left.\begin{aligned} 3a-3b+c &= 0 \\ 2a-2b+c &= 0 \\ c &= 0 \end{aligned}\right\}$ $\quad \mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

With $\lambda_2 = 2$: $\left.\begin{aligned} 2a-3b+c &= 0 \\ 2a-3b+c &= 0 \\ 0 &= 0 \end{aligned}\right\}$ $\quad \mathbf{v}_2 = \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix} \quad \mathbf{v}_3 = \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}$

The eigenspace of $\lambda_2 = 2$ is 2-dimensional. We get the eigenvector $\mathbf{v}_2$ with $b = 2,\ c = 0,$ and the eigenvector $\mathbf{v}_3$ with $b = 0,\ c = 2.$

22. Characteristic polynomial: $\quad p(\lambda) = -\lambda^3 + 3\lambda + 2 = -(\lambda+1)^2(\lambda-2)$

Eigenvalues: $\lambda_1 = \lambda_2 = -1, \quad \lambda_3 = 2$

With $\lambda_1 = -1$: $\left.\begin{aligned} 6a-6b+3c &= 0 \\ 6a-6b+3c &= 0 \\ 6a-6b+3c &= 0 \end{aligned}\right\}$ $\quad \mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}$

The eigenspace of $\lambda_1 = 1$ is 2-dimensional. We get the eigenvector $\mathbf{v}_1$ with $b = 1,\ c = 0,$ and the eigenvector $\mathbf{v}_2$ with $b = 0,\ c = 2.$

With $\lambda_3 = 2$: $\left.\begin{aligned} 3a-6b+3c &= 0 \\ 6a-9b+3c &= 0 \\ 6a-6b &= 0 \end{aligned}\right\}$ $\quad \mathbf{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

23. Characteristic polynomial: $p(\lambda) = (\lambda-1)(\lambda-2)(\lambda-3)(\lambda-4)$

Eigenvalues: $\lambda_1 = 1, \quad \lambda_2 = 2, \quad \lambda_3 = 3, \quad \lambda_4 = 4$

With $\lambda_1 = 1$: $\left.\begin{array}{r} 2b+2c+2d = 0 \\ b+2c+2d = 0 \\ 2c+2d = 0 \\ 3d = 0 \end{array}\right\} \qquad \mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

With $\lambda_2 = 2$: $\left.\begin{array}{r} -a+2b+2c+2d = 0 \\ 2c+2d = 0 \\ c+2d = 0 \\ 2d = 0 \end{array}\right\} \qquad \mathbf{v}_2 = \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}$

With $\lambda_3 = 3$: $\left.\begin{array}{r} -2a+2b+2c+2d = 0 \\ b+2c+2d = 0 \\ 2d = 0 \\ d = 0 \end{array}\right\} \qquad \mathbf{v}_3 = \begin{bmatrix} 3 \\ 2 \\ 1 \\ 0 \end{bmatrix}$

With $\lambda_4 = 4$: $\left.\begin{array}{r} -3a+2b+2c+2d = 0 \\ -2b+2c+2d = 0 \\ -c+2d = 0 \\ 0 = 0 \end{array}\right\} \qquad \mathbf{v}_4 = \begin{bmatrix} 4 \\ 3 \\ 2 \\ 1 \end{bmatrix}$

24. Characteristic polynomial: $p(\lambda) = (\lambda-1)^2(\lambda-3)^2$

Eigenvalues: $\lambda_1 = \lambda_2 = 1, \quad \lambda_3 = \lambda_4 = 3$

With $\lambda_1 = 1$: $\left.\begin{array}{r} 4c = 0 \\ 4c = 0 \\ 2c = 0 \\ 2d = 0 \end{array}\right\} \qquad \mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$

The eigenspace of $\lambda_1 = 1$ is 2-dimensional. We note that $c = d = 0$, but a and b are arbitrary.

With $\lambda_3 = 3$: $\left.\begin{array}{r} -2a+4c = 0 \\ -2b+4c = 0 \\ 0 = 0 \\ 0 = 0 \end{array}\right\} \qquad \mathbf{v}_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{v}_4 = \begin{bmatrix} 2 \\ 2 \\ 1 \\ 0 \end{bmatrix}$

The eigenspace of $\lambda_3 = 3$ is 2-dimensional. We get the eigenvector $\mathbf{v}_3$ with $b = 0, \; c = 1,$ and the eigenvector $\mathbf{v}_2$ with $b = 1, \; c = 0.$

25. Characteristic polynomial: $p(\lambda) = (\lambda-1)^2(\lambda-2)^2$

Eigenvalues: $\lambda_1 = \lambda_2 = 1, \quad \lambda_3 = \lambda_4 = 2$

With $\lambda_1 = 1$: $\left.\begin{aligned} c &= 0 \\ c &= 0 \\ c &= 0 \\ d &= 0 \end{aligned}\right\}$ $\quad \mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$

The eigenspace of $\lambda_1 = 1$ is 2-dimensional. We note that $c = d = 0$, but a and b are arbitrary.

With $\lambda_3 = 2$: $\left.\begin{aligned} -a+c &= 0 \\ -b+c &= 0 \\ 0 &= 0 \\ 0 &= 0 \end{aligned}\right\}$ $\quad \mathbf{v}_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{v}_4 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}$

The eigenspace of $\lambda_3 = 2$ is 2-dimensional. We get the eigenvector $\mathbf{v}_3$ with $b = 0,\ c = 1,$ and the eigenvector $\mathbf{v}_4$ with $b = 1,\ c = 0.$

26. Characteristic polynomial:

$$p(\lambda) = \lambda^4 - 5\lambda^2 + 4 = (\lambda^2-1)(\lambda^2-4) = (\lambda+1)(\lambda-1)(\lambda+2)(\lambda-2)$$

Eigenvalues: $\lambda_1 = -2, \quad \lambda_2 = -1, \quad \lambda_3 = 1, \quad \lambda_4 = 2$

With $\lambda_1 = -2$: $\left.\begin{aligned} 6a-3d &= 0 \\ 4b &= 0 \\ c &= 0 \\ 6a-3d &= 0 \end{aligned}\right\}$ $\quad \mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 2 \end{bmatrix}$

With $\lambda_2 = -1$: $\left.\begin{aligned} 5a-3d &= 0 \\ 3b &= 0 \\ 0 &= 0 \\ 6a-4d &= 0 \end{aligned}\right\}$ $\quad \mathbf{v}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$

With $\lambda_3 = 1$: $\left.\begin{aligned} 3a-3d &= 0 \\ b &= 0 \\ -2c &= 0 \\ 6a-6d &= 0 \end{aligned}\right\}$ $\quad \mathbf{v}_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

With $\lambda_4 = 2:$ $\left.\begin{aligned} 2a-3d &= 0 \\ 0 &= 0 \\ -3c &= 0 \\ 6a-7d &= 0 \end{aligned}\right\}$ $\mathbf{v}_4 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$

27. Characteristic polynomial: $p(\lambda) = \lambda^2 + 1$

Eigenvalues: $\lambda_1 = -i, \quad \lambda_2 = +i$

With $\lambda_1 = -i:$ $\left.\begin{aligned} ia+b &= 0 \\ -a+ib &= 0 \end{aligned}\right\}$ $\mathbf{v}_1 = \begin{bmatrix} i \\ 1 \end{bmatrix}$

With $\lambda_2 = +i:$ $\left.\begin{aligned} -ia+b &= 0 \\ -a-ib &= 0 \end{aligned}\right\}$ $\mathbf{v}_2 = \begin{bmatrix} -i \\ 1 \end{bmatrix}$

28. Characteristic polynomial: $p(\lambda) = \lambda^2 + 36$

Eigenvalues: $\lambda_1 = -6i, \quad \lambda_2 = +6i$

With $\lambda_1 = -6i:$ $\left.\begin{aligned} 6ia-6b &= 0 \\ 6a+6ib &= 0 \end{aligned}\right\}$ $\mathbf{v}_1 = \begin{bmatrix} -i \\ 1 \end{bmatrix}$

With $\lambda_2 = +6i:$ $\left.\begin{aligned} -6ia-6b &= 0 \\ 6a-6ib &= 0 \end{aligned}\right\}$ $\mathbf{v}_2 = \begin{bmatrix} i \\ 1 \end{bmatrix}$

29. Characteristic polynomial: $p(\lambda) = \lambda^2 + 36$

Eigenvalues: $\lambda_1 = -6i, \quad \lambda_2 = +6i$

With $\lambda_1 = -6i:$ $\left.\begin{aligned} 6ia-3b &= 0 \\ 12a+6ib &= 0 \end{aligned}\right\}$ $\mathbf{v}_1 = \begin{bmatrix} -i \\ 2 \end{bmatrix}$

With $\lambda_2 = +6i:$ $\left.\begin{aligned} -6ia-3b &= 0 \\ 12a-6ib &= 0 \end{aligned}\right\}$ $\mathbf{v}_2 = \begin{bmatrix} i \\ 2 \end{bmatrix}$

30. Characteristic polynomial: $p(\lambda) = \lambda^2 + 144$

Eigenvalues: $\lambda_1 = -12i, \quad \lambda_2 = +12i$

With $\lambda_1 = -12i:$ $\left.\begin{aligned} 12ia-12b &= 0 \\ 12a+12ib &= 0 \end{aligned}\right\}$ $\mathbf{v}_1 = \begin{bmatrix} -i \\ 1 \end{bmatrix}$

With $\lambda_2 = +12i:$ $\left.\begin{aligned} -12ia-12b &= 0 \\ 12a-12ib &= 0 \end{aligned}\right\}$ $\mathbf{v}_2 = \begin{bmatrix} i \\ 1 \end{bmatrix}$

31. Characteristic polynomial: $p(\lambda) = \lambda^2 + 144$

Eigenvalues: $\lambda_1 = -12i, \quad \lambda_2 = +12i$

With $\lambda_1 = -12i$: $\left.\begin{aligned} 12i\,a + 24b &= 0 \\ -6a + 12i\,b &= 0 \end{aligned}\right\}$ $\quad \mathbf{v}_1 = \begin{bmatrix} 2i \\ 1 \end{bmatrix}$

With $\lambda_2 = +12i$: $\left.\begin{aligned} -12i\,a + 24b &= 0 \\ -6a - 12i\,b &= 0 \end{aligned}\right\}$ $\quad \mathbf{v}_2 = \begin{bmatrix} -2i \\ 1 \end{bmatrix}$

32. Characteristic polynomial: $p(\lambda) = \lambda^2 + 144$

Eigenvalues: $\lambda_1 = -12i, \quad \lambda_2 = +12i$

With $\lambda_1 = -12i$: $\left.\begin{aligned} -12i\,a - 4b &= 0 \\ 36a + 12i\,b &= 0 \end{aligned}\right\}$ $\quad \mathbf{v}_1 = \begin{bmatrix} -i \\ 3 \end{bmatrix}$

With $\lambda_2 = +12i$: $\left.\begin{aligned} -12i\,a - 4b &= 0 \\ 36a - 12i\,b &= 0 \end{aligned}\right\}$ $\quad \mathbf{v}_2 = \begin{bmatrix} i \\ 3 \end{bmatrix}$

33. If $\mathbf{Av} = \lambda\mathbf{v}$ and we assume that $\mathbf{A}^{n-1}\mathbf{v} = \lambda^{n-1}\mathbf{v}$ — meaning that λ^{n-1} is an eigenvalue of $\mathbf{A}^{n-1}$ with associated eigenvector $\mathbf{v}$, then multiplication by $\mathbf{A}$ yields

$$\mathbf{A}^n\mathbf{v} = \mathbf{A}\cdot\mathbf{A}^{n-1}\mathbf{v} = \mathbf{A}\cdot\lambda^{n-1}\mathbf{v} = \lambda^{n-1}\cdot\mathbf{Av} = \lambda^{n-1}\cdot\lambda\mathbf{v} = \lambda^n\mathbf{v}.$$

Thus λ^n is an eigenvalue of the matrix $\mathbf{A}^n$ with associated eigenvector $\mathbf{v}$.

34. By the remark following Example 6, any eigenvalue of an invertible matrix is nonzero. If $\lambda \neq 0$ is an eigenvalue of the invertible matrix $\mathbf{A}$ with associated eigenvalue $\mathbf{v}$, then

$$\begin{aligned} \mathbf{Av} &= \lambda\mathbf{v}, \\ \mathbf{v} &= \mathbf{A}^{-1}\cdot\lambda\mathbf{v} = \lambda\cdot\mathbf{A}^{-1}\mathbf{v}, \\ \mathbf{A}^{-1}\mathbf{v} &= \lambda^{-1}\mathbf{v}. \end{aligned}$$

Thus λ^{-1} is an eigenvalue of $\mathbf{A}^{-1}$ with associated eigenvector $\mathbf{v}$.

35. **(a)** Note first that $(\mathbf{A} - \lambda\mathbf{I})^T = (\mathbf{A}^T - \lambda\mathbf{I})$ because $\mathbf{I}^T = \mathbf{I}$. Since the determinant of a square matrix equals the determinant of its transpose, it follows that

$$|\mathbf{A} - \lambda\mathbf{I}| = |\mathbf{A}^T - \lambda\mathbf{I}|.$$

Thus the matrices $\mathbf{A}$ and $\mathbf{A}^T$ have the same characteristic polynomial, and therefore have the same eigenvalues.

(b) Consider the matrix $\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ with characteristic equation $(\lambda - 1)^2 = 0$ and the single eigenvalue $\lambda = 1$. Then $\mathbf{A} - \mathbf{I} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ and it follows that the only associated eigenvector is a multiple of $\begin{bmatrix} 0 & 1 \end{bmatrix}^T$. The transpose $\mathbf{A}^T = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ has the same characteristic equation and eigenvalue, but we see similarly that its only eigenvector is a multiple of $\begin{bmatrix} 1 & 0 \end{bmatrix}^T$. Thus $\mathbf{A}$ and $\mathbf{A}^T$ have the same eigenvalue but different eigenvectors.

36. If the $n \times n$ matrix $\mathbf{A} = \left[a_{ij} \right]$ is either upper or lower triangular, then obviously its characteristic equation is

$$(a_{11} - \lambda)(a_{22} - \lambda) \cdots\cdots (a_{nn} - \lambda) = 0.$$

This observation makes it clear that the eigenvalues of the matrix $\mathbf{A}$ are its diagonal elements $a_{11}, a_{22}, \cdots, a_{nn}$.

37. If $|\mathbf{A} - \lambda \mathbf{I}| = (-1)^n \lambda^n + c_{n-1} \lambda^{n-1} + \cdots + c_1 \lambda + c_0$, then substitution of $\lambda = 0$ yields $c_0 = |\mathbf{A} - 0\mathbf{I}| = |\mathbf{A}|$ for the constant term in the characteristic polynomial.

38. The characteristic polynomial of the 2×2 matrix $\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is

$(a - \lambda)(d - \lambda) - bc = 0$, that is, $\lambda^2 - (a + d)\lambda + (ad - bc) = 0$. Thus the coefficient of λ in the characteristic equation is $-(a + d) = -\operatorname{trace} \mathbf{A}$.

39. If the characteristic equation of the $n \times n$ matrix $\mathbf{A}$ with eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_n$ (not necessarily distinct) is written in the factored form

$$(\lambda - \lambda_1)(\lambda - \lambda_2) \cdots\cdots (\lambda - \lambda_n) = 0,$$

then it should be clear that upon multiplying out the factors the coefficient of λ^{n-1} will be $-(\lambda_1 + \lambda_2 + \cdots + \lambda_n)$. But according to Problem 38, this coefficient also equals $-(\operatorname{trace} \mathbf{A})$. Therefore $\lambda_1 + \lambda_2 + \cdots + \lambda_n = \operatorname{trace} \mathbf{A} = a_{11} + a_{22} + \cdots a_{nn}$.

40. We find that $\operatorname{trace} \mathbf{A} = 12$ and $\det \mathbf{A} = 60$, so the characteristic polynomial of the given matrix $\mathbf{A}$ is

$$p(\lambda) = -\lambda^3 + 12\lambda^2 + c_1 \lambda + 60.$$

Substitution of $\lambda = 1$ and

$$p(1) = |\mathbf{A}-\mathbf{I}| = \begin{vmatrix} 31 & -67 & 47 \\ 7 & -15 & 13 \\ -7 & 15 & -7 \end{vmatrix} = 24$$

yields $c_1 = -47$, so the characteristic equation of $\mathbf{A}$ is (after multiplication by -1)

$$\lambda^3 - 12\lambda^2 + 47\lambda - 60 = 0.$$

Trying $\lambda = \pm 1, \pm 2, \pm 3$ (all divisors of 60) in turn, we discover the eigenvalue $\lambda_1 = 3$. Then division of the cubic by $(\lambda - 3)$ yields

$$\lambda^2 - 9\lambda + 20 = (\lambda - 4)(\lambda - 5),$$

so the other two eigenvalues are $\lambda_2 = 4$ and $\lambda_3 = 5$. We proceed to find the eigenvectors associated with these three eigenvalues.

With $\lambda_1 = 3$:

$$\left.\begin{aligned} 29a - 67b + 47c &= 0 \\ 7a - 17b + 13c &= 0 \\ -7a + 15b - 9c &= 0 \end{aligned}\right\} \rightarrow \left.\begin{aligned} a - 3c &= 0 \\ b - 2c &= 0 \\ 0 &= 0 \end{aligned}\right\} \qquad \mathbf{v}_1 = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$$

With $\lambda_2 = 4$:

$$\left.\begin{aligned} 28a - 67b + 47c &= 0 \\ 7a - 18b + 13c &= 0 \\ -7a + 15b - 10c &= 0 \end{aligned}\right\} \rightarrow \left.\begin{aligned} a - (5/7)c &= 0 \\ b - c &= 0 \\ 0 &= 0 \end{aligned}\right\} \qquad \mathbf{v}_2 = \begin{bmatrix} 5 \\ 7 \\ 7 \end{bmatrix}$$

With $\lambda_3 = 5$:

$$\left.\begin{aligned} 27a - 67b + 47c &= 0 \\ 7a - 19b + 13c &= 0 \\ -7a + 15b - 11c &= 0 \end{aligned}\right\} \rightarrow \left.\begin{aligned} a + (1/2)c &= 0 \\ b - (1/2)c &= 0 \\ 0 &= 0 \end{aligned}\right\} \qquad \mathbf{v}_3 = \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$$

41. We find that trace $\mathbf{A} = 8$ and $\det \mathbf{A} = -60$, so the characteristic polynomial of the given matrix $\mathbf{A}$ is

$$p(\lambda) = \lambda^4 - 8\lambda^3 + c_2\lambda^2 + c_1\lambda - 60.$$

Substitution of $\lambda = 1$, $p(1) = \det(\mathbf{A}-\mathbf{I}) = -24$ and $\lambda = -1$, $p(-1) = \det(\mathbf{A}+\mathbf{I}) = -72$ yields the equations

$$c_2 + c_1 = 43, \quad c_2 - c_1 = -21$$

that we solve readily for $c_1 = 32$, $c_2 = 11$. Hence the characteristic equation of $\mathbf{A}$ is

$$\lambda^4 - 8\lambda^3 + 11\lambda^2 + 32\lambda - 60 = 0.$$

Trying $\lambda = \pm 1, \pm 2$ in turn, we discover the eigenvalues $\lambda_1 = -2$ and $\lambda_2 = 2$. Then division of the quartic by $(\lambda^2 - 4)$ yields

$$\lambda^2 - 8\lambda + 15 = (\lambda - 3)(\lambda - 5),$$

so the other two eigenvalues are $\lambda_3 = 3$ and $\lambda_4 = 5$. We proceed to find the eigenvectors associated with these four eigenvalues.

With $\lambda_1 = -2$:

$$\left.\begin{aligned} 24a - 9b - 8c - 8d &= 0 \\ 10a - 5b - 14c + 2d &= 0 \\ 10a + 10c - 10d &= 0 \\ 29a - 9b - 3c - 13d &= 0 \end{aligned}\right\} \rightarrow \left.\begin{aligned} a - (1/2)d &= 0 \\ b &= 0 \\ c - (1/2)d &= 0 \\ 0 &= 0 \end{aligned}\right\} \qquad \mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 2 \end{bmatrix}$$

With $\lambda_2 = 2$:

$$\left.\begin{aligned} 20a - 9b - 8c - 8d &= 0 \\ 10a - 9b - 14c + 2d &= 0 \\ 10a + 6c - 10d &= 0 \\ 29a - 9b - 3c - 17d &= 0 \end{aligned}\right\} \rightarrow \left.\begin{aligned} a - d &= 0 \\ b - (4/3)d &= 0 \\ c &= 0 \\ 0 &= 0 \end{aligned}\right\} \qquad \mathbf{v}_2 = \begin{bmatrix} 3 \\ 4 \\ 0 \\ 3 \end{bmatrix}$$

With $\lambda_3 = 3$:

$$\left.\begin{aligned} 19a - 9b - 8c - 8d &= 0 \\ 10a - 10b - 14c + 2d &= 0 \\ 10a + 5c - 10d &= 0 \\ 29a - 9b - 3c - 18d &= 0 \end{aligned}\right\} \rightarrow \left.\begin{aligned} a - (3/4)d &= 0 \\ b - (1/4)d &= 0 \\ c - (1/2)d &= 0 \\ 0 &= 0 \end{aligned}\right\} \qquad \mathbf{v}_3 = \begin{bmatrix} 3 \\ 1 \\ 2 \\ 4 \end{bmatrix}$$

With $\lambda_4 = 5$:

$$\left.\begin{aligned} 17a - 9b - 8c - 8d &= 0 \\ 10a - 12b - 14c + 2d &= 0 \\ 10a + 3c - 10d &= 0 \\ 29a - 9b - 3c - 20d &= 0 \end{aligned}\right\} \rightarrow \left.\begin{aligned} a - d &= 0 \\ b - d &= 0 \\ c &= 0 \\ 0 &= 0 \end{aligned}\right\} \qquad \mathbf{v}_4 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

SECTION 6.2

DIAGONALIZATION OF MATRICES

In Problems 1–28 we first find the eigenvalues and associated eigenvectors of the given $n \times n$ matrix $\mathbf{A}$. If $\mathbf{A}$ has n linearly independent eigenvectors, then we can proceed to set up the desired diagonalizing matrix $\mathbf{P} = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_n \end{bmatrix}$ and diagonal matrix $\mathbf{D}$ such that $\mathbf{P}^{-1}\mathbf{AP} = \mathbf{D}$. If you write the eigenvalues in a different order on the diagonal of $\mathbf{D}$, then naturally the eigenvector columns of $\mathbf{P}$ must be rearranged in the same order.

1. Characteristic polynomial: $p(\lambda) = \lambda^2 - 4\lambda + 3 = (\lambda - 1)(\lambda - 3)$

Eigenvalues: $\lambda_1 = 1, \quad \lambda_2 = 3$

With $\lambda_1 = 1$: $\left.\begin{aligned} 4a - 4b &= 0 \\ 2a - 2b &= 0 \end{aligned}\right\}$ $\qquad \mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

With $\lambda_2 = 3$: $\left.\begin{aligned} 2a - 4b &= 0 \\ 2a - 4b &= 0 \end{aligned}\right\}$ $\qquad \mathbf{v}_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$$\mathbf{P} = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}, \qquad \mathbf{D} = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$$

2. Characteristic polynomial: $p(\lambda) = \lambda^2 - 2\lambda = \lambda(\lambda - 2)$

Eigenvalues: $\lambda_1 = 0, \quad \lambda_2 = 2$

With $\lambda_1 = 0$: $\left.\begin{aligned} 6a - 6b &= 0 \\ 4a - 4b &= 0 \end{aligned}\right\}$ $\qquad \mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

With $\lambda_2 = 2$: $\left.\begin{aligned} 4a - 6b &= 0 \\ 4a - 6b &= 0 \end{aligned}\right\}$ $\qquad \mathbf{v}_2 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$

$$\mathbf{P} = \begin{bmatrix} 1 & 3 \\ 1 & 2 \end{bmatrix}, \qquad \mathbf{D} = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}$$

3. Characteristic polynomial: $p(\lambda) = \lambda^2 - 5\lambda + 6 = (\lambda - 2)(\lambda - 3)$

Eigenvalues: $\lambda_1 = 2, \quad \lambda_2 = 3$

With $\lambda_1 = 2$: $\left.\begin{aligned} 3a - 3b &= 0 \\ 2a - 2b &= 0 \end{aligned}\right\}$ $\qquad \mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

With $\lambda_2 = 3$: $\left.\begin{aligned} 2a-3b &= 0 \\ 2a-3b &= 0 \end{aligned}\right\}$ $\mathbf{v}_2 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$

$\mathbf{P} = \begin{bmatrix} 1 & 3 \\ 1 & 2 \end{bmatrix}$, $\mathbf{D} = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$

4. Characteristic polynomial: $p(\lambda) = \lambda^2 - 3\lambda + 2 = (\lambda-1)(\lambda-2)$

Eigenvalues: $\lambda_1 = 1$, $\lambda_2 = 2$

With $\lambda_1 = 1$: $\left.\begin{aligned} 4a-4b &= 0 \\ 3a-3b &= 0 \end{aligned}\right\}$ $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

With $\lambda_2 = 2$: $\left.\begin{aligned} 3a-4b &= 0 \\ 3a-4b &= 0 \end{aligned}\right\}$ $\mathbf{v}_2 = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$

$\mathbf{P} = \begin{bmatrix} 1 & 4 \\ 1 & 3 \end{bmatrix}$, $\mathbf{D} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$

5. Characteristic polynomial: $p(\lambda) = \lambda^2 - 4\lambda + 3 = (\lambda-1)(\lambda-3)$

Eigenvalues: $\lambda_1 = 1$, $\lambda_2 = 3$

With $\lambda_1 = 1$: $\left.\begin{aligned} 8a-8b &= 0 \\ 6a-6b &= 0 \end{aligned}\right\}$ $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

With $\lambda_2 = 3$: $\left.\begin{aligned} 6a-8b &= 0 \\ 6a-8b &= 0 \end{aligned}\right\}$ $\mathbf{v}_2 = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$

$\mathbf{P} = \begin{bmatrix} 1 & 4 \\ 1 & 3 \end{bmatrix}$, $\mathbf{D} = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$

6. Characteristic polynomial: $p(\lambda) = \lambda^2 - 3\lambda + 2 = (\lambda-1)(\lambda-2)$

Eigenvalues: $\lambda_1 = 1$, $\lambda_2 = 2$

With $\lambda_1 = 1$: $\left.\begin{aligned} 9a-6b &= 0 \\ 12a-8b &= 0 \end{aligned}\right\}$ $\mathbf{v}_1 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

With $\lambda_2 = 2$: $\left.\begin{aligned} 8a-6b &= 0 \\ 12a-9b &= 0 \end{aligned}\right\}$ $\mathbf{v}_2 = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$

$\mathbf{P} = \begin{bmatrix} 2 & 3 \\ 3 & 4 \end{bmatrix}$, $\mathbf{D} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$

7. Characteristic polynomial: $p(\lambda) = \lambda^2 - 3\lambda + 2 = (\lambda - 1)(\lambda - 2)$

Eigenvalues: $\lambda_1 = 1, \quad \lambda_2 = 2$

With $\lambda_1 = 1$: $\left.\begin{aligned} 5a - 10b &= 0 \\ 2a - 4b &= 0 \end{aligned}\right\}$ $\mathbf{v}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

With $\lambda_2 = 2$: $\left.\begin{aligned} 4a - 10b &= 0 \\ 2a - 5b &= 0 \end{aligned}\right\}$ $\mathbf{v}_2 = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$

$\mathbf{P} = \begin{bmatrix} 2 & 5 \\ 1 & 2 \end{bmatrix}, \qquad \mathbf{D} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$

8. Characteristic polynomial: $p(\lambda) = \lambda^2 - 3\lambda + 2 = (\lambda - 1)(\lambda - 2)$

Eigenvalues: $\lambda_1 = 1, \quad \lambda_2 = 2$

With $\lambda_1 = 1$: $\left.\begin{aligned} 10a - 15b &= 0 \\ 6a - 9b &= 0 \end{aligned}\right\}$ $\mathbf{v}_1 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$

With $\lambda_2 = 2$: $\left.\begin{aligned} 9a - 15b &= 0 \\ 6a - 10b &= 0 \end{aligned}\right\}$ $\mathbf{v}_2 = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$

$\mathbf{P} = \begin{bmatrix} 3 & 5 \\ 2 & 3 \end{bmatrix}, \qquad \mathbf{D} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$

9. Characteristic polynomial: $p(\lambda) = \lambda^2 - 2\lambda + 1 = (\lambda - 1)^2$

Eigenvalues: $\lambda_1 = 1, \quad \lambda_2 = 1$

With $\lambda_1 = 1$: $\left.\begin{aligned} 4a - 2b &= 0 \\ 2a - b &= 0 \end{aligned}\right\}$ $\mathbf{v}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

Because the given matrix $\mathbf{A}$ has only the single eigenvector $\mathbf{v}_1$, it is not diagonalizable.

10. Characteristic polynomial: $p(\lambda) = \lambda^2 - 4\lambda + 4 = (\lambda - 2)^2$

Eigenvalues: $\lambda_1 = 2, \quad \lambda_2 = 2$

With $\lambda_1 = 2$: $\left.\begin{aligned} a - b &= 0 \\ a - b &= 0 \end{aligned}\right\}$ $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Because the given matrix $\mathbf{A}$ has only the single eigenvector $\mathbf{v}_1$, it is not diagonalizable.

11. Characteristic polynomial: $p(\lambda) = \lambda^2 - 4\lambda + 4 = (\lambda - 2)^2$

Eigenvalues: $\lambda_1 = 2, \quad \lambda_2 = 2$

With $\lambda_1 = 2$: $\left.\begin{aligned} 3a + b &= 0 \\ -9a - 3b &= 0 \end{aligned}\right\}$ $\quad \mathbf{v}_1 = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$

Because the given matrix $\mathbf{A}$ has only the single eigenvector $\mathbf{v}_1$, it is not diagonalizable.

12. Characteristic polynomial: $p(\lambda) = \lambda^2 + 2\lambda + 1 = (\lambda + 1)^2$

Eigenvalues: $\lambda_1 = -1, \quad \lambda_2 = -1$

With $\lambda_1 = -1$: $\left.\begin{aligned} 12a + 9b &= 0 \\ -16a - 12b &= 0 \end{aligned}\right\}$ $\quad \mathbf{v}_1 = \begin{bmatrix} -3 \\ 4 \end{bmatrix}$

Because the given matrix $\mathbf{A}$ has only the single eigenvector $\mathbf{v}_1$, it is not diagonalizable.

13. Characteristic polynomial: $p(\lambda) = -\lambda^3 + 5\lambda^2 - 8\lambda + 4 = -(\lambda - 1)(\lambda - 2)^2$

Eigenvalues: $\lambda_1 = 1, \quad \lambda_2 = 2, \quad \lambda_3 = 2$

With $\lambda_1 = 1$: $\left.\begin{aligned} 3b &= 0 \\ b &= 0 \\ c &= 0 \end{aligned}\right\}$ $\quad \mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

With $\lambda_2 = 2$: $\left.\begin{aligned} 3b - a &= 0 \\ 0 &= 0 \\ 0 &= 0 \end{aligned}\right\}$ $\quad \mathbf{v}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}$

The eigenspace of $\lambda_2 = 2$ is 2-dimensional. We get the eigenvector $\mathbf{v}_2$ with $b = 0,\ c = 1$, and the eigenvector $\mathbf{v}_3$ with $b = 1,\ c = 0$.

$$\mathbf{P} = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \qquad \mathbf{D} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

14. Characteristic polynomial: $p(\lambda) = -\lambda^3 + \lambda^2 = -\lambda^2(\lambda - 1)$

Eigenvalues: $\lambda_1 = 0, \quad \lambda_2 = 0, \quad \lambda_3 = 1$

With $\lambda_1 = 0$: $\left.\begin{aligned} 2a - 2b + c &= 0 \\ 2a - 2b + c &= 0 \\ 2a - 2b + c &= 0 \end{aligned}\right\}$ $\quad \mathbf{v}_1 = \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

The eigenspace of $\lambda_1 = 0$ is 2-dimensional. We get the eigenvector $\mathbf{v}$ with $b=0,\ c=2,$ and the eigenvector $\mathbf{v}_2$ with $b=1,\ c=0.$

With $\lambda_3 = 1$: $\quad \left.\begin{aligned} a-2b+c &= 0 \\ 2a-3b+c &= 0 \\ 2a-2b &= 0 \end{aligned}\right\} \qquad \mathbf{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

$$\mathbf{P} = \begin{bmatrix} -1 & 1 & 1 \\ 0 & 1 & 1 \\ 2 & 0 & 1 \end{bmatrix}, \qquad \mathbf{D} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

15. Characteristic polynomial: $\quad p(\lambda) = -\lambda^3 + 2\lambda^2 - \lambda = -\lambda(\lambda-1)^2$

Eigenvalues: $\lambda_1 = 0, \quad \lambda_2 = 1, \quad \lambda_3 = 1$

With $\lambda_1 = 0$: $\quad \left.\begin{aligned} 3a-3b+c &= 0 \\ 2a-2b+c &= 0 \\ c &= 0 \end{aligned}\right\} \qquad \mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

With $\lambda_2 = 1$: $\quad \left.\begin{aligned} 2a-3b+c &= 0 \\ 2a-3b+c &= 0 \\ 0 &= 0 \end{aligned}\right\} \qquad \mathbf{v}_2 = \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix}$

The eigenspace of $\lambda_2 = 2$ is 2-dimensional. We get the eigenvector $\mathbf{v}_2$ with $b=0,\ c=2,$ and the eigenvector $\mathbf{v}_3$ with $b=2,\ c=0.$

$$\mathbf{P} = \begin{bmatrix} 1 & -1 & 3 \\ 1 & 0 & 2 \\ 0 & 2 & 0 \end{bmatrix}, \qquad \mathbf{D} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

16. Characteristic polynomial: $\quad p(\lambda) = -\lambda^3 + 5\lambda^2 - 7\lambda + 3 = -(\lambda-1)^2(\lambda-3)$

Eigenvalues: $\lambda_1 = 1, \quad \lambda_2 = 1, \quad \lambda_3 = 3$

With $\lambda_1 = 1$: $\quad \left.\begin{aligned} 2a-2b &= 0 \\ 0 &= 0 \\ 4a-4b &= 0 \end{aligned}\right\} \qquad \mathbf{v}_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

The eigenspace of $\lambda_1 = 1$ is 2-dimensional. We get the eigenvector $\mathbf{v}_2$ with $b=0,\ c=1,$ and the eigenvector $\mathbf{v}_3$ with $b=1,\ c=0.$

With $\lambda_3 = 3$: $\left.\begin{aligned} -2b &= 0 \\ -2b &= 0 \\ -4a+4b-2c &= 0 \end{aligned}\right\}$ $\mathbf{v}_3 = \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}$

$$\mathbf{P} = \begin{bmatrix} 0 & 1 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 2 \end{bmatrix}, \qquad \mathbf{D} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

17. Characteristic polynomial: $p(\lambda) = -\lambda^3 + 2\lambda^2 + \lambda - 2 = -(\lambda+1)(\lambda-1)(\lambda-2)$

Eigenvalues: $\lambda_1 = -1, \quad \lambda_2 = 1, \quad \lambda_3 = 2$

With $\lambda_1 = -1$: $\left.\begin{aligned} 8a-8b+3c &= 0 \\ 6a-6b+3c &= 0 \\ 2a-2b+3c &= 0 \end{aligned}\right\}$ $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

With $\lambda_2 = 1$: $\left.\begin{aligned} 6a-8b+3c &= 0 \\ 6a-8b+3c &= 0 \\ 2a-2b+c &= 0 \end{aligned}\right\}$ $\mathbf{v}_2 = \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}$

With $\lambda_3 = 2$: $\left.\begin{aligned} 5a-8b+3c &= 0 \\ 6a-9b+3c &= 0 \\ 2a-2b &= 0 \end{aligned}\right\}$ $\mathbf{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

$$\mathbf{P} = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 2 & 1 \end{bmatrix}, \qquad \mathbf{D} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

18. Characteristic polynomial: $p(\lambda) = -\lambda^3 + 6\lambda^2 - 11\lambda + 6 = -(\lambda-1)(\lambda-2)(\lambda-3)$

Eigenvalues: $\lambda_1 = 1, \quad \lambda_2 = 2, \quad \lambda_3 = 3$

With $\lambda_1 = 1$: $\left.\begin{aligned} 5a-5b+2c &= 0 \\ 4a-4b+2c &= 0 \\ 2a-2b+2c &= 0 \end{aligned}\right\}$ $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

With $\lambda_2 = 2$: $\left.\begin{aligned} 4a-5b+2c &= 0 \\ 4a-5b+2c &= 0 \\ 2a-2b+c &= 0 \end{aligned}\right\}$ $\mathbf{v}_2 = \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}$

With $\lambda_3 = 3$: $\left.\begin{aligned} 3a-5b+2c &= 0 \\ 4a-6b+2c &= 0 \\ 2a-2b &= 0 \end{aligned}\right\}$ $\mathbf{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

$$\mathbf{P} = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 2 & 1 \end{bmatrix}, \qquad \mathbf{D} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

19. Characteristic polynomial: $p(\lambda) = -\lambda^3 + 6\lambda^2 - 11\lambda + 6 = -(\lambda-1)(\lambda-2)(\lambda-3)$

Eigenvalues: $\lambda_1 = 1, \quad \lambda_2 = 2, \quad \lambda_3 = 3$

With $\lambda_1 = 1$: $\left.\begin{aligned} b-c &= 0 \\ -2a+3b-c &= 0 \\ -4a+4b &= 0 \end{aligned}\right\}$ $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

With $\lambda_2 = 2$: $\left.\begin{aligned} -a+b-c &= 0 \\ -2a+2b-c &= 0 \\ -4a+4b-c &= 0 \end{aligned}\right\}$ $\mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

With $\lambda_3 = 3$: $\left.\begin{aligned} -2a+b-c &= 0 \\ -2a+b-c &= 0 \\ -4a+4b-2c &= 0 \end{aligned}\right\}$ $\mathbf{v}_3 = \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}$

$$\mathbf{P} = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 1 & 0 \\ 1 & 0 & 2 \end{bmatrix}, \qquad \mathbf{D} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

20. Characteristic polynomial: $p(\lambda) = -\lambda^3 + 13\lambda^2 - 52\lambda + 60 = -(\lambda-2)(\lambda-5)(\lambda-6)$

Eigenvalues: $\lambda_1 = 2, \quad \lambda_2 = 5, \quad \lambda_3 = 6$

With $\lambda_1 = 2$: $\left.\begin{aligned} 0 &= 0 \\ -6a+9b+2c &= 0 \\ 6a-15b-2c &= 0 \end{aligned}\right\}$ $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}$

With $\lambda_2 = 5$: $\left.\begin{aligned} -3a &= 0 \\ -6a+6b+2c &= 0 \\ 6a-15b-5c &= 0 \end{aligned}\right\}$ $\mathbf{v}_2 = \begin{bmatrix} 0 \\ -1 \\ 3 \end{bmatrix}$

With $\lambda_3 = 6$: $\left.\begin{aligned} -4a &= 0 \\ -6a+5b+2c &= 0 \\ 6a-15b-6c &= 0 \end{aligned}\right\}$ $\mathbf{v}_3 = \begin{bmatrix} 0 \\ -2 \\ 5 \end{bmatrix}$

$$\mathbf{P} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & -2 \\ 3 & 3 & 5 \end{bmatrix}, \qquad \mathbf{D} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 6 \end{bmatrix}$$

21. Characteristic polynomial: $p(\lambda) = -\lambda^3 + 3\lambda^2 - 3\lambda + 1 = -(\lambda-1)^3$

Eigenvalues: $\lambda_1 = 1, \quad \lambda_2 = 1, \quad \lambda_3 = 1$

With $\lambda_1 = 1$: $\left.\begin{aligned} b-a &= 0 \\ b-a &= 0 \\ b-a &= 0 \end{aligned}\right\}$ $\mathbf{v}_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

The eigenspace of $\lambda_1 = 1$ is 2-dimensional. We get the eigenvector $\mathbf{v}_1$ with $b = 0,\ c = 1,$ and the eigenvector $\mathbf{v}_2$ with $b = 1,\ c = 0.$ Because the given matrix $\mathbf{A}$ has only two linearly independent eigenvectors, it is not diagonalizable.

22. Characteristic polynomial: $p(\lambda) = -\lambda^3 + 3\lambda^2 - 3\lambda + 1 = -(\lambda-1)^3$

Eigenvalues: $\lambda_1 = 1, \quad \lambda_2 = 1, \quad \lambda_3 = 1$

With $\lambda_1 = 1$: $\left.\begin{aligned} a-2b+c &= 0 \\ -a+b &= 0 \\ -5a+7b-2c &= 0 \end{aligned}\right\}$ $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

The eigenspace of $\lambda_1 = 1$ is 1-dimensional. Because the given matrix $\mathbf{A}$ has only one eigenvector, it is not diagonalizable.

23. Characteristic polynomial: $p(\lambda) = -\lambda^3 + 4\lambda^2 - 5\lambda + 2 = -(\lambda-1)^2(\lambda-2)$

Eigenvalues: $\lambda_1 = 1, \quad \lambda_2 = 1, \quad \lambda_3 = 2$

With $\lambda_1 = 1$: $\left.\begin{aligned} -3a+4b-c &= 0 \\ -3a+4b-c &= 0 \\ -a+b &= 0 \end{aligned}\right\}$ $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

$$\text{With } \lambda_3 = 2: \qquad \left.\begin{aligned} -4a+4b-c &= 0 \\ -3a+3b-c &= 0 \\ -a+b-c &= 0 \end{aligned}\right\} \qquad \mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

The given matrix $\mathbf{A}$ has only the two linearly independent eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$, and therefore is not diagonalizable.

24. Characteristic polynomial: $p(\lambda) = -\lambda^3 + 5\lambda^2 - 8\lambda + 4 = -(\lambda-1)(\lambda-2)^2$

Eigenvalues: $\lambda_1 = 1, \quad \lambda_2 = 2, \quad \lambda_3 = 2$

$$\text{With } \lambda_1 = 1: \qquad \left.\begin{aligned} 2a-2b+c &= 0 \\ a-b+c &= 0 \\ -a+b+c &= 0 \end{aligned}\right\} \qquad \mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$$\text{With } \lambda_2 = 2: \qquad \left.\begin{aligned} a-2b+c &= 0 \\ a-2b+c &= 0 \\ -a+b &= 0 \end{aligned}\right\} \qquad \mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

The given matrix $\mathbf{A}$ has only the two linearly independent eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$, and therefore is not diagonalizable.

25. Characteristic polynomial: $p(\lambda) = (\lambda+1)^2(\lambda-1)^2$

Eigenvalues: $\lambda_1 = -1, \quad \lambda_2 = -1, \quad \lambda_3 = 1, \quad \lambda_4 = 1$

$$\text{With } \lambda_1 = -1: \qquad \left.\begin{aligned} 2a-2c &= 0 \\ 2b-2c &= 0 \\ 0 &= 0 \\ 0 &= 0 \end{aligned}\right\} \qquad \mathbf{v}_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}$$

The eigenspace of $\lambda_1 = -1$ is 2-dimensional. We get the eigenvector $\mathbf{v}_1$ with $c = 0,\ d = 1,$ and the eigenvector $\mathbf{v}_2$ with $c = 1,\ d = 0.$

$$\text{With } \lambda_3 = 1: \qquad \left.\begin{aligned} -2c &= 0 \\ -2c &= 0 \\ -2c &= 0 \\ -2d &= 0 \end{aligned}\right\} \qquad \mathbf{v}_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{v}_4 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

The eigenspace of $\lambda_3 = 1$ is also 2-dimensional. We get the eigenvector $\mathbf{v}_3$ with $a = 0,\ b = 1,$ and the eigenvector $\mathbf{v}_4$ with $a = 1,\ b = 0.$

$$\mathbf{P} = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \qquad \mathbf{D} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

26. Characteristic polynomial: $p(\lambda) = (\lambda - 1)^3(\lambda - 2)$

Eigenvalues: $\lambda_1 = 1, \quad \lambda_2 = 1, \quad \lambda_3 = 1, \quad \lambda_4 = 2$

With $\lambda_1 = 1$: $\left.\begin{aligned} d &= 0 \\ d &= 0 \\ d &= 0 \\ d &= 0 \end{aligned}\right\}$ $\quad \mathbf{v}_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

The eigenspace of $\lambda_1 = 1$ is 3-dimensional, and we have taken advantage of the fact that we can select a, b, and c independently.

With $\lambda_4 = 2$: $\left.\begin{aligned} d - a &= 0 \\ d - b &= 0 \\ d - c &= 0 \\ 0 &= 0 \end{aligned}\right\}$ $\quad \mathbf{v}_4 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$

$$\mathbf{P} = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \qquad \mathbf{D} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

27. Characteristic polynomial: $p(\lambda) = (\lambda - 1)^3(\lambda - 2)$

Eigenvalues: $\lambda_1 = 1, \quad \lambda_2 = 1, \quad \lambda_3 = 1, \quad \lambda_4 = 2$

With $\lambda_1 = 1$: $\left.\begin{aligned} b &= 0 \\ c &= 0 \\ d &= 0 \\ d &= 0 \end{aligned}\right\}$ $\quad \mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

The eigenspace of $\lambda_1 = 1$ is 1-dimensional, with only a single associated eigenvector.

$$\text{With } \lambda_4 = 2: \qquad \left.\begin{aligned} b-a &= 0 \\ c-b &= 0 \\ d-c &= 0 \\ 0 &= 0 \end{aligned}\right\} \qquad \mathbf{v}_4 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

The given matrix $\mathbf{A}$ has only the two linearly independent eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_4$, and therefore is not diagonalizable.

28. Characteristic polynomial: $p(\lambda) = (\lambda-1)^2(\lambda-2)^2$

Eigenvalues: $\lambda_1 = 1, \quad \lambda_2 = 1, \quad \lambda_3 = 2, \quad \lambda_4 = 2$

$$\text{With } \lambda_1 = 1: \qquad \left.\begin{aligned} b+d &= 0 \\ c+d &= 0 \\ c+d &= 0 \\ d &= 0 \end{aligned}\right\} \qquad \mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

The eigenspace of $\lambda_1 = 1$ is 1-dimensional, with only a single associated eigenvector.

$$\text{With } \lambda_3 = 2: \qquad \left.\begin{aligned} b-a+d &= 0 \\ c-b+d &= 0 \\ d &= 0 \\ 0 &= 0 \end{aligned}\right\} \qquad \mathbf{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}$$

The eigenspace of $\lambda_3 = 2$ is also 1-dimensional, with only a single associated eigenvector. Thus the given matrix $\mathbf{A}$ has only the two linearly independent eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_4$, and therefore is not diagonalizable.

29. If $\mathbf{A}$ is similar to $\mathbf{B}$ and $\mathbf{B}$ is similar to $\mathbf{C}$, so $\mathbf{A} = \mathbf{P}^{-1}\mathbf{BP}$ and $\mathbf{B} = \mathbf{Q}^{-1}\mathbf{CQ}$, then

$$\mathbf{A} = \mathbf{P}^{-1}(\mathbf{Q}^{-1}\mathbf{CQ})\mathbf{P} = (\mathbf{P}^{-1}\mathbf{Q}^{-1})\mathbf{C}(\mathbf{QP}) = (\mathbf{QP})^{-1}\mathbf{C}(\mathbf{QP}) = \mathbf{R}^{-1}\mathbf{CR}$$

with $\mathbf{R} = \mathbf{QP}$, so $\mathbf{A}$ is similar to $\mathbf{C}$.

30. If $\mathbf{A}$ is similar to $\mathbf{B}$ so $\mathbf{A} = \mathbf{P}^{-1}\mathbf{BP}$ then

$$\begin{aligned} \mathbf{A}^n &= (\mathbf{P}^{-1}\mathbf{BP})(\mathbf{P}^{-1}\mathbf{BP})(\mathbf{P}^{-1}\mathbf{BP})\cdots\cdots(\mathbf{P}^{-1}\mathbf{BP})(\mathbf{P}^{-1}\mathbf{BP}) \\ &= \mathbf{P}^{-1}\mathbf{B}(\mathbf{PP}^{-1})\mathbf{B}(\mathbf{PP}^{-1})\mathbf{B}\cdots\cdots(\mathbf{PP}^{-1})\mathbf{B}(\mathbf{PP}^{-1})\mathbf{BP} \\ &= \mathbf{P}^{-1}\mathbf{B}(\mathbf{I})\mathbf{B}(\mathbf{I})\mathbf{B}\cdots\cdots(\mathbf{I})\mathbf{B}(\mathbf{I})\mathbf{BP} \\ &= \mathbf{P}^{-1}\mathbf{B}^n\mathbf{P} \end{aligned}$$

so we see that $\mathbf{A}^n$ is similar to $\mathbf{B}^n$.

31. If $\mathbf{A}$ is similar to $\mathbf{B}$ so $\mathbf{A} = \mathbf{P}^{-1}\mathbf{BP}$ then $\mathbf{A}^{-1} = (\mathbf{P}^{-1}\mathbf{BP})^{-1} = \mathbf{P}^{-1}\mathbf{B}^{-1}\mathbf{P}$, so $\mathbf{A}^{-1}$ is similar to $\mathbf{B}^{-1}$.

32. If $\mathbf{A}$ is similar to $\mathbf{B}$ so $\mathbf{A} = \mathbf{P}^{-1}\mathbf{BP}$, then $|\mathbf{P}^{-1}||\mathbf{P}| = |\mathbf{P}^{-1}\mathbf{P}| = |\mathbf{I}| = 1$ so

$$\begin{aligned} |\mathbf{A} - \lambda\mathbf{I}| &= |\mathbf{P}^{-1}||\mathbf{A} - \lambda\mathbf{I}||\mathbf{P}| = |\mathbf{P}^{-1}(\mathbf{A} - \lambda\mathbf{I})\mathbf{P}| \\ &= |\mathbf{P}^{-1}\mathbf{AP} - \lambda\mathbf{P}^{-1}\mathbf{IP}| = |\mathbf{B} - \lambda\mathbf{I}|. \end{aligned}$$

Thus $\mathbf{A}$ and $\mathbf{B}$ have the same characteristic polynomial.

33. If $\mathbf{A}$ and $\mathbf{B}$ are similar with $\mathbf{A} = \mathbf{P}^{-1}\mathbf{BP}$, then

$$|\mathbf{A}| = |\mathbf{P}^{-1}\mathbf{BP}| = |\mathbf{P}^{-1}||\mathbf{B}||\mathbf{P}| = |\mathbf{P}|^{-1}|\mathbf{B}||\mathbf{P}| = |\mathbf{B}|.$$

Moreover, by Problem 32 the two matrices have the same eigenvalues, and by Problem 39 in Section 6.1, the trace of a square matrix with real eigenvalues is equal to the sum of those eigenvalues. Therefore trace **A** = (eigenvalue sum) = trace **B**.

34. The characteristic equation of the 2×2 matrix $\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is

$\lambda^2 - (a+d)\lambda + (ad - bc) = 0$, and the discriminant of this quadratic equation is

$$\Delta = (a+d)^2 - 4(ad - bc) = (a-d)^2 + 4bc.$$

(a) If $\Delta > 0$, then $\mathbf{A}$ has two distinct eigenvalues and hence has two linearly independent eigenvectors, and is therefore diagonalizable.

(b) If $\Delta < 0$, then $\mathbf{A}$ has no (real) eigenvalues and hence no real eigenvectors, and therefore is not diagonalizable.

(c) Finally, note that $\Delta = 0$ for both

$$\mathbf{I} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{A} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix},$$

but $\mathbf{A}$ has only the single eigenvalue $\lambda = 1$ and the single eigenvector $\mathbf{v} = \begin{bmatrix} 1 & 0 \end{bmatrix}^T$, and is therefore not diagonalizable.

35. Three eigenvectors associated with three distinct eigenvalues can be arranged in six different orders as the column vectors of the diagonalizing matrix $\mathbf{P} = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 \end{bmatrix}^T$.

36. The fact that the matrices $\mathbf{A}$ and $\mathbf{B}$ have the same eigenvalues (with the same multiplicities) implies that they are both similar to the same diagonal matrix $\mathbf{D}$ having these eigenvalues as its diagonal elements. But two matrices that are similar to a third matrix are (by Problem 29) similar to one another.

37. If $\mathbf{A} = \mathbf{PDP}^{-1}$ with $\mathbf{P}$ the eigenvector matrix of $\mathbf{A}$ and $\mathbf{D}$ its diagonal matrix of eigenvalues, then $\mathbf{A}^2 = (\mathbf{PDP}^{-1})(\mathbf{PDP}^{-1}) = \mathbf{PD}(\mathbf{P}^{-1}\mathbf{P})\mathbf{DP}^{-1} = \mathbf{PD}^2\mathbf{P}^{-1}$. Thus the same (eigenvector) matrix $\mathbf{P}$ diagonalizes $\mathbf{A}^2$, but the resulting diagonal (eigenvalue) matrix $\mathbf{D}^2$ is the square of the one for $\mathbf{A}$. The diagonal elements of $\mathbf{D}^2$ are the eigenvalues of $\mathbf{A}^2$ and the diagonal elements of $\mathbf{D}$ are the eigenvalues of $\mathbf{A}$, so the former are the squares of the latter.

38. If the $n\times n$ matrix $\mathbf{A}$ has n linearly independent eigenvectors associated with the single eigenvalue λ, then $\mathbf{A} = \mathbf{PDP}^{-1}$ with $\mathbf{D} = \lambda\mathbf{I}$, so $\mathbf{A} = \mathbf{P}(\lambda\mathbf{I})\mathbf{P}^{-1} = \lambda\mathbf{PP}^{-1} = \lambda\mathbf{I} = \mathbf{D}$.

39. Let the $n\times n$ matrix $\mathbf{A}$ have $k \le n$ distinct eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_k$. Then the definition of algebraic multiplicity and the fact that all solutions of the nth degree polynomial equation $|\mathbf{A} - \lambda\mathbf{I}| = 0$ are real imply that the sum of the multiplicities of the eigenvalues equals n,

$$p_1 + p_2 + \cdots + p_k = n.$$

Now Theorem 4 in this section implies that A is diagonalizable if and only if

$$q_1 + q_2 + \cdots + q_k = n$$

where q_i denotes the geometric multiplicity of λ_i $(i = 1, 2, \cdots, k)$. But, because $p_i \ge q_i$ for each $i = 1, 2, \cdots, k$, the two equations displayed above can both be satisfied if and only if $p_i = q_i$ for each i.

SECTION 6.3

APPLICATIONS INVOLVING POWERS OF MATRICES

In Problems 1–10 we first find the eigensystem of the given matrix $\mathbf{A}$ so as to determine its eigenvector matrix $\mathbf{P}$ and its diagonal eigenvalue matrix $\mathbf{D}$. Then we calculate the matrix power $\mathbf{A}^5 = \mathbf{PD}^5\mathbf{P}^{-1}$.

1. Characteristic polynomial: $p(\lambda) = \lambda^2 - 3\lambda + 2 = (\lambda - 1)(\lambda - 2)$

Eigenvalues: $\lambda_1 = 1, \quad \lambda_2 = 2$

With $\lambda_1 = 1$: $\left.\begin{aligned} 2a-2b &= 0 \\ a-b &= 0 \end{aligned}\right\}$ $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

With $\lambda_2 = 2$: $\left.\begin{aligned} a-2b &= 0 \\ a-2b &= 0 \end{aligned}\right\}$ $\mathbf{v}_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$$\mathbf{P} = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}, \quad \mathbf{D} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}, \quad \mathbf{P}^{-1} = \begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix}$$

$$\mathbf{A}^5 = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 32 \end{bmatrix}\begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 63 & -62 \\ 31 & -30 \end{bmatrix}$$

2. Characteristic polynomial: $p(\lambda) = \lambda^2 - \lambda - 2 = (\lambda+1)(\lambda-2)$

Eigenvalues: $\lambda_1 = -1, \quad \lambda_2 = 2$

With $\lambda_1 = -1$: $\left.\begin{aligned} 6a-6b &= 0 \\ 3a-3b &= 0 \end{aligned}\right\}$ $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

With $\lambda_2 = 2$: $\left.\begin{aligned} 3a-6b &= 0 \\ 3a-6b &= 0 \end{aligned}\right\}$ $\mathbf{v}_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$$\mathbf{P} = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}, \quad \mathbf{D} = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}, \quad \mathbf{P}^{-1} = \begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix}$$

$$\mathbf{A}^5 = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 0 & 32 \end{bmatrix}\begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 65 & -66 \\ 33 & -34 \end{bmatrix}$$

3. Characteristic polynomial: $p(\lambda) = \lambda^2 - 2\lambda = \lambda(\lambda-2)$

Eigenvalues: $\lambda_1 = 0, \quad \lambda_2 = 2$

With $\lambda_1 = 0$: $\left.\begin{aligned} 6a-6b &= 0 \\ 4a-4b &= 0 \end{aligned}\right\}$ $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

With $\lambda_2 = 2$: $\left.\begin{aligned} 4a-6b &= 0 \\ 4a-6b &= 0 \end{aligned}\right\}$ $\mathbf{v}_2 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$

$$\mathbf{P} = \begin{bmatrix} 1 & 3 \\ 1 & 2 \end{bmatrix}, \quad \mathbf{D} = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}, \quad \mathbf{P}^{-1} = \begin{bmatrix} -2 & 3 \\ 1 & -1 \end{bmatrix}$$

$$\mathbf{A}^5 = \begin{bmatrix} 1 & 3 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 32 \end{bmatrix}\begin{bmatrix} -2 & 3 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 96 & -96 \\ 64 & -64 \end{bmatrix}$$

4. Characteristic polynomial: $p(\lambda) = \lambda^2 - 3\lambda + 2 = (\lambda - 1)(\lambda - 2)$

Eigenvalues: $\lambda_1 = 1, \quad \lambda_2 = 2$

With $\lambda_1 = 1$: $\left.\begin{aligned} 3a - 3b &= 0 \\ 2a - 2b &= 0 \end{aligned}\right\}$ $\quad \mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

With $\lambda_2 = 2$: $\left.\begin{aligned} 2a - 3b &= 0 \\ 2a - 3b &= 0 \end{aligned}\right\}$ $\quad \mathbf{v}_2 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$

$$\mathbf{P} = \begin{bmatrix} 1 & 3 \\ 1 & 2 \end{bmatrix}, \quad \mathbf{D} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}, \quad \mathbf{P}^{-1} = \begin{bmatrix} -2 & 3 \\ 1 & -1 \end{bmatrix}$$

$$\mathbf{A}^5 = \begin{bmatrix} 1 & 3 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 32 \end{bmatrix}\begin{bmatrix} -2 & 3 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 94 & -93 \\ 62 & -61 \end{bmatrix}$$

5. Characteristic polynomial: $p(\lambda) = \lambda^2 - 3\lambda + 2 = (\lambda - 1)(\lambda - 2)$

Eigenvalues: $\lambda_1 = 1, \quad \lambda_2 = 2$

With $\lambda_1 = 1$: $\left.\begin{aligned} 4a - 4b &= 0 \\ 3a - 3b &= 0 \end{aligned}\right\}$ $\quad \mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

With $\lambda_2 = 2$: $\left.\begin{aligned} 3a - 4b &= 0 \\ 3a - 4b &= 0 \end{aligned}\right\}$ $\quad \mathbf{v}_2 = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$

$$\mathbf{P} = \begin{bmatrix} 1 & 4 \\ 1 & 3 \end{bmatrix}, \quad \mathbf{D} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}, \quad \mathbf{P}^{-1} = \begin{bmatrix} -3 & 4 \\ 1 & -1 \end{bmatrix}$$

$$\mathbf{A}^5 = \begin{bmatrix} 1 & 4 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 32 \end{bmatrix}\begin{bmatrix} -3 & 4 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 125 & -124 \\ 93 & -92 \end{bmatrix}$$

6. Characteristic polynomial: $p(\lambda) = \lambda^2 - 3\lambda + 2 = (\lambda - 1)(\lambda - 2)$

Eigenvalues: $\lambda_1 = 1, \quad \lambda_2 = 2$

With $\lambda_1 = 1$: $\left.\begin{aligned} 5a - 10b &= 0 \\ 2a - 4b &= 0 \end{aligned}\right\}$ $\quad \mathbf{v}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

With $\lambda_2 = 2$: $\left.\begin{aligned} 4a - 10b &= 0 \\ 2a - 5b &= 0 \end{aligned}\right\}$ $\quad \mathbf{v}_2 = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$

$$\mathbf{P} = \begin{bmatrix} 2 & 5 \\ 1 & 2 \end{bmatrix}, \quad \mathbf{D} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}, \quad \mathbf{P}^{-1} = \begin{bmatrix} -2 & 5 \\ 1 & -2 \end{bmatrix}$$

$$\mathbf{A}^5 = \begin{bmatrix} 2 & 5 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 32 \end{bmatrix}\begin{bmatrix} -2 & 5 \\ 1 & -2 \end{bmatrix} = \begin{bmatrix} 156 & -310 \\ 62 & -123 \end{bmatrix}$$

7. Characteristic polynomial: $p(\lambda) = -(\lambda-1)(\lambda-2)^2$

Eigenvalues: $\lambda_1 = 1, \quad \lambda_2 = 2, \quad \lambda_3 = 2$

With $\lambda_1 = 1$: $\left.\begin{aligned} 3b &= 0 \\ b &= 0 \\ c &= 0 \end{aligned}\right\}$ $\qquad \mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

With $\lambda_2 = 2$: $\left.\begin{aligned} 3b - a &= 0 \\ 0 &= 0 \\ 0 &= 0 \end{aligned}\right\}$ $\qquad \mathbf{v}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}$

$$\mathbf{P} = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad \mathbf{D} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}, \quad \mathbf{P}^{-1} = \begin{bmatrix} 1 & -3 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$\mathbf{A}^5 = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 32 & 0 \\ 0 & 0 & 32 \end{bmatrix}\begin{bmatrix} 1 & -3 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 93 & 0 \\ 0 & 32 & 0 \\ 0 & 0 & 32 \end{bmatrix}$$

8. Characteristic polynomial: $p(\lambda) = -\lambda^3 + 4\lambda^2 - 5\lambda + 2 = -(\lambda-1)^2(\lambda-2)$

Eigenvalues: $\lambda_1 = 1, \quad \lambda_2 = 1, \quad \lambda_3 = 2$

With $\lambda_1 = 1$: $\left.\begin{aligned} c - 2b &= 0 \\ 0 &= 0 \\ c - 2b &= 0 \end{aligned}\right\}$ $\qquad \mathbf{v}_1 = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

With $\lambda_3 = 2$: $\left.\begin{aligned} -a - 2b + c &= 0 \\ -b &= 0 \\ -2b &= 0 \end{aligned}\right\}$ $\qquad \mathbf{v}_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

$$\mathbf{P} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 2 & 0 & 1 \end{bmatrix}, \quad \mathbf{D} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}, \quad \mathbf{P}^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 2 & -1 \\ 0 & -2 & 1 \end{bmatrix}$$

$$\mathbf{A}^5 = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 32 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 2 & -1 \\ 0 & -2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -62 & 31 \\ 0 & 1 & 0 \\ 0 & -62 & 32 \end{bmatrix}$$

9. Characteristic polynomial: $p(\lambda) = -(\lambda-1)(\lambda-2)^2$

Eigenvalues: $\lambda_1 = 1, \quad \lambda_2 = 2, \quad \lambda_3 = 2$

With $\lambda_1 = 1$: $\left.\begin{aligned} c-3b &= 0 \\ b &= 0 \\ c &= 0 \end{aligned}\right\}$ $\quad \mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

With $\lambda_2 = 2$: $\left.\begin{aligned} -a-3b+c &= 0 \\ 0 &= 0 \\ 0 &= 0 \end{aligned}\right\}$ $\quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix}$

$$\mathbf{P} = \begin{bmatrix} 1 & 1 & -3 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad \mathbf{D} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}, \quad \mathbf{P}^{-1} = \begin{bmatrix} 1 & 3 & -1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$\mathbf{A}^5 = \begin{bmatrix} 1 & 1 & -3 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 32 & 0 \\ 0 & 0 & 32 \end{bmatrix} \begin{bmatrix} 1 & 3 & -1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & -93 & 31 \\ 0 & 32 & 0 \\ 0 & 0 & 32 \end{bmatrix}$$

10. Characteristic polynomial: $p(\lambda) = -(\lambda-1)(\lambda-2)^2$

Eigenvalues: $\lambda_1 = 1, \quad \lambda_2 = 2, \quad \lambda_3 = 2$

With $\lambda_1 = 1$: $\left.\begin{aligned} 3a-3b+c &= 0 \\ 2a-2b+c &= 0 \\ c &= 0 \end{aligned}\right\}$ $\quad \mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

With $\lambda_2 = 2$: $\left.\begin{aligned} 2a-3b+c &= 0 \\ 2a-3b+c &= 0 \\ 0 &= 0 \end{aligned}\right\}$ $\quad \mathbf{v}_2 = \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix}$

$$\mathbf{P} = \begin{bmatrix} 1 & -1 & 3 \\ 1 & 0 & 2 \\ 0 & 2 & 0 \end{bmatrix}, \quad \mathbf{D} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}, \quad \mathbf{P}^{-1} = \frac{1}{2}\begin{bmatrix} -4 & 6 & -2 \\ 0 & 0 & 1 \\ 2 & -2 & 1 \end{bmatrix}$$

$$\mathbf{A}^5 = \begin{bmatrix} 1 & -1 & 3 \\ 1 & 0 & 2 \\ 0 & 2 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 32 & 0 \\ 0 & 0 & 32 \end{bmatrix}\frac{1}{2}\begin{bmatrix} -4 & 6 & -2 \\ 0 & 0 & 1 \\ 2 & -2 & 1 \end{bmatrix} = \begin{bmatrix} 94 & -93 & 31 \\ 62 & -61 & 31 \\ 0 & 0 & 32 \end{bmatrix}$$

11. Characteristic polynomial: $p(\lambda) = -\lambda^3 + \lambda = -\lambda(\lambda+1)(\lambda-1)$

Eigenvalues: $\lambda_1 = -1, \quad \lambda_2 = 0, \quad \lambda_3 = 1$

With $\lambda_1 = -1$: $\left.\begin{aligned} 2a &= 0 \\ 6a+6b+2c &= 0 \\ 21a-15b-5c &= 0 \end{aligned}\right\}$ $\mathbf{v}_1 = \begin{bmatrix} 0 \\ -1 \\ 3 \end{bmatrix}$

With $\lambda_2 = 0$: $\left.\begin{aligned} a &= 0 \\ 6a+5b+2c &= 0 \\ 21a-15b-6c &= 0 \end{aligned}\right\}$ $\mathbf{v}_2 = \begin{bmatrix} 0 \\ -2 \\ 5 \end{bmatrix}$

With $\lambda_3 = 1$: $\left.\begin{aligned} 0 &= 0 \\ 6a+4b+2c &= 0 \\ 21a-15b-7c &= 0 \end{aligned}\right\}$ $\mathbf{v}_3 = \begin{bmatrix} -1 \\ -42 \\ 87 \end{bmatrix}$

$$\mathbf{P} = \begin{bmatrix} 0 & 0 & -1 \\ -1 & -2 & -42 \\ 3 & 5 & 87 \end{bmatrix}, \quad \mathbf{D} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{P}^{-1} = \begin{bmatrix} -36 & 5 & 2 \\ 39 & -3 & -1 \\ -1 & 0 & 0 \end{bmatrix}$$

$$\mathbf{A}^{10} = \begin{bmatrix} 0 & 0 & -1 \\ -1 & -2 & -42 \\ 3 & 5 & 87 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} -36 & 5 & 2 \\ 39 & -3 & -1 \\ -1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 78 & -5 & -2 \\ -195 & 15 & 6 \end{bmatrix}$$

12. Characteristic polynomial: $p(\lambda) = (1-\lambda)(\lambda^2-1) = -(\lambda+1)(\lambda-1)^2$

Eigenvalues: $\lambda_1 = -1, \quad \lambda_2 = 1, \quad \lambda_3 = 1$

With $\lambda_1 = -1$: $\left.\begin{aligned} 12a-6b-2c &= 0 \\ 20a-10b-4c &= 0 \\ 2c &= 0 \end{aligned}\right\}$ $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$

With $\lambda_2 = 1$: $\left.\begin{aligned} 10a-6b-2c &= 0 \\ 20a-12b-4c &= 0 \\ 0 &= 0 \end{aligned}\right\}$ $\mathbf{v}_2 = \begin{bmatrix} 1 \\ 0 \\ 5 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} 3 \\ 5 \\ 0 \end{bmatrix}$

$$\mathbf{P} = \begin{bmatrix} 1 & 1 & 3 \\ 2 & 0 & 5 \\ 0 & 5 & 0 \end{bmatrix}, \quad \mathbf{D} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{P}^{-1} = \frac{1}{5}\begin{bmatrix} -25 & 15 & 5 \\ 0 & 0 & 1 \\ 10 & -5 & -2 \end{bmatrix}$$

$$\mathbf{A}^{10} = \begin{bmatrix} 1 & 1 & 3 \\ 2 & 0 & 5 \\ 0 & 5 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\frac{1}{5}\begin{bmatrix} -25 & 15 & 5 \\ 0 & 0 & 1 \\ 10 & -5 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

13. Characteristic polynomial: $p(\lambda) = -\lambda^3 + \lambda = -\lambda(\lambda+1)(\lambda-1)$

Eigenvalues: $\lambda_1 = -1, \quad \lambda_2 = 0, \quad \lambda_3 = 1$

With $\lambda_1 = -1$: $\left.\begin{aligned} 2a-b+c &= 0 \\ 2a-b+c &= 0 \\ 4a-4b+2c &= 0 \end{aligned}\right\}$ $\mathbf{v}_1 = \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}$

With $\lambda_2 = 0$: $\left.\begin{aligned} a-b+c &= 0 \\ 2a-2b+c &= 0 \\ 4a-4b+c &= 0 \end{aligned}\right\}$ $\mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

With $\lambda_3 = 1$: $\left.\begin{aligned} -b+c &= 0 \\ 2a-3b+c &= 0 \\ 4a-4b &= 0 \end{aligned}\right\}$ $\mathbf{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

$$\mathbf{P} = \begin{bmatrix} -1 & 1 & 1 \\ 0 & 1 & 1 \\ 2 & 0 & 1 \end{bmatrix}, \quad \mathbf{D} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{P}^{-1} = \begin{bmatrix} -1 & 1 & 0 \\ -2 & 3 & -1 \\ 2 & -2 & 1 \end{bmatrix}$$

$$\mathbf{A}^{10} = \begin{bmatrix} -1 & 1 & 1 \\ 0 & 1 & 1 \\ 2 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} -1 & 1 & 0 \\ -2 & 3 & -1 \\ 2 & -2 & 1 \end{bmatrix} = \begin{bmatrix} 3 & -3 & 1 \\ 2 & -2 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

14. Characteristic polynomial: $p(\lambda) = -\lambda^3 + \lambda = -\lambda(\lambda+1)(\lambda-1)$

Eigenvalues: $\lambda_1 = -1, \quad \lambda_2 = 0, \quad \lambda_3 = 1$

With $\lambda_1 = -1$: $\left.\begin{aligned} 6a-5b-3c &= 0 \\ 2a-b-c &= 0 \\ 4a-4b-2c &= 0 \end{aligned}\right\}$ $\quad \mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$

With $\lambda_2 = 0$: $\left.\begin{aligned} 5a-5b-3c &= 0 \\ 2a-2b-c &= 0 \\ 4a-4b-3c &= 0 \end{aligned}\right\}$ $\quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

With $\lambda_3 = 1$: $\left.\begin{aligned} 4a-5b-3c &= 0 \\ 2a-3b-c &= 0 \\ 4a-4b-4c &= 0 \end{aligned}\right\}$ $\quad \mathbf{v}_3 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$

$$\mathbf{P} = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 2 & 0 & 1 \end{bmatrix}, \quad \mathbf{D} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{P}^{-1} = \begin{bmatrix} -1 & 1 & 1 \\ -2 & 3 & 1 \\ 2 & -2 & -1 \end{bmatrix}$$

$$\mathbf{A}^{10} = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 2 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} -1 & 1 & 1 \\ -2 & 3 & 1 \\ 2 & -2 & -1 \end{bmatrix} = \begin{bmatrix} 3 & -3 & -1 \\ 2 & -2 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

15. $p(\lambda) = \lambda^2 - 3\lambda + 2$ so $\mathbf{A}^2 - 3\mathbf{A} + 2\mathbf{I} = 0$

$$\mathbf{A}^2 = 3\mathbf{A} - 2\mathbf{I} = 3\begin{bmatrix} 5 & -4 \\ 3 & -2 \end{bmatrix} - 2\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 13 & -12 \\ 9 & -8 \end{bmatrix}$$

$$\mathbf{A}^3 = 3\mathbf{A}^2 - 2\mathbf{A} = 3\begin{bmatrix} 13 & -12 \\ 9 & -8 \end{bmatrix} - 2\begin{bmatrix} 5 & -4 \\ 3 & -2 \end{bmatrix} = \begin{bmatrix} 29 & -28 \\ 21 & -20 \end{bmatrix}$$

$$\mathbf{A}^4 = 3\mathbf{A}^3 - 2\mathbf{A}^2 = 3\begin{bmatrix} 29 & -28 \\ 21 & -20 \end{bmatrix} - 2\begin{bmatrix} 13 & -12 \\ 9 & -8 \end{bmatrix} = \begin{bmatrix} 61 & -60 \\ 45 & -44 \end{bmatrix}$$

$$\mathbf{A}^{-1} = \frac{1}{2}\left(-\mathbf{A} + 3\mathbf{I}\right) = \frac{1}{2}\left(-\begin{bmatrix} 5 & -4 \\ 3 & -2 \end{bmatrix} + 3\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = \frac{1}{2}\begin{bmatrix} -2 & 4 \\ -3 & 5 \end{bmatrix}$$

16. $p(\lambda) = \lambda^2 - 3\lambda + 2$ so $\mathbf{A}^2 - 3\mathbf{A} + 2\mathbf{I} = 0$

$$\mathbf{A}^2 = 3\mathbf{A} - 2\mathbf{I} = 3\begin{bmatrix} 6 & -10 \\ 2 & -3 \end{bmatrix} - 2\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 16 & -30 \\ 6 & -11 \end{bmatrix}$$

$$\mathbf{A}^3 = 3\mathbf{A}^2 - 2\mathbf{A} = 3\begin{bmatrix} 16 & -30 \\ 6 & -11 \end{bmatrix} - 2\begin{bmatrix} 6 & -10 \\ 2 & -3 \end{bmatrix} = \begin{bmatrix} 36 & -70 \\ 14 & -27 \end{bmatrix}$$

$$\mathbf{A}^4 = 3\mathbf{A}^3 - 2\mathbf{A}^2 = 3\begin{bmatrix} 36 & -70 \\ 14 & -27 \end{bmatrix} - 2\begin{bmatrix} 16 & -30 \\ 6 & -11 \end{bmatrix} = \begin{bmatrix} 76 & -150 \\ 30 & -59 \end{bmatrix}$$

$$\mathbf{A}^{-1} = \frac{1}{2}(-\mathbf{A} + 3\mathbf{I}) = \frac{1}{2}\left(-\begin{bmatrix} 6 & -10 \\ 2 & -3 \end{bmatrix} + 3\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = \frac{1}{2}\begin{bmatrix} -3 & 10 \\ -2 & 6 \end{bmatrix}$$

17. $p(\lambda) = -\lambda^3 + 5\lambda^2 - 8\lambda + 4$ so $-\mathbf{A}^3 + 5\mathbf{A}^2 - 8\mathbf{A} + 4\mathbf{I} = \mathbf{0}$

$$\mathbf{A}^2 = \begin{bmatrix} 1 & 9 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}, \qquad \mathbf{A}^3 = 5\mathbf{A}^2 - 8\mathbf{A} + 4\mathbf{I} = \begin{bmatrix} 1 & 21 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 8 \end{bmatrix}$$

$$\mathbf{A}^4 = 5\mathbf{A}^3 - 8\mathbf{A}^2 + 4\mathbf{A} = \begin{bmatrix} 1 & 45 & 0 \\ 0 & 16 & 0 \\ 0 & 0 & 16 \end{bmatrix}$$

$$\mathbf{A}^{-1} = \frac{1}{4}(\mathbf{A}^2 - 5\mathbf{A} + 8\mathbf{I}) = \frac{1}{2}\begin{bmatrix} 2 & -3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

18. $p(\lambda) = -\lambda^3 + 4\lambda^2 - 5\lambda + 2$ so $-\mathbf{A}^3 + 4\mathbf{A}^2 - 5\mathbf{A} + 2\mathbf{I} = \mathbf{0}$

$$\mathbf{A}^2 = \begin{bmatrix} 1 & -6 & 3 \\ 0 & 1 & 0 \\ 0 & -6 & 4 \end{bmatrix}, \qquad \mathbf{A}^3 = 4\mathbf{A}^2 - 5\mathbf{A} + 2\mathbf{I} = \begin{bmatrix} 1 & -14 & 7 \\ 0 & 1 & 0 \\ 0 & -14 & 8 \end{bmatrix}$$

$$\mathbf{A}^4 = 4\mathbf{A}^3 - 5\mathbf{A}^2 + 2\mathbf{A} = \begin{bmatrix} 1 & -30 & 15 \\ 0 & 1 & 0 \\ 0 & -30 & 16 \end{bmatrix}$$

$$\mathbf{A}^{-1} = \frac{1}{2}(\mathbf{A}^2 - 4\mathbf{A} + 5\mathbf{I}) = \frac{1}{2}\begin{bmatrix} 2 & 2 & -1 \\ 0 & 2 & 0 \\ 0 & 2 & 1 \end{bmatrix}$$

19. $p(\lambda) = -\lambda^3 + 5\lambda^2 - 8\lambda + 4$ so $-\mathbf{A}^3 + 5\mathbf{A}^2 - 8\mathbf{A} + 4\mathbf{I} = \mathbf{0}$

$$\mathbf{A}^2 = \begin{bmatrix} 1 & -9 & 3 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}, \qquad \mathbf{A}^3 = 5\mathbf{A}^2 - 8\mathbf{A} + 4\mathbf{I} = \begin{bmatrix} 1 & -21 & 7 \\ 0 & 8 & 0 \\ 0 & 0 & 8 \end{bmatrix}$$

$$\mathbf{A}^4 = 5\mathbf{A}^3 - 8\mathbf{A}^2 + 4\mathbf{A} = \begin{bmatrix} 1 & -45 & 15 \\ 0 & 16 & 0 \\ 0 & 0 & 16 \end{bmatrix}$$

$$\mathbf{A}^{-1} = \frac{1}{4}\left(\mathbf{A}^2 - 5\mathbf{A} + 8\mathbf{I}\right) = \frac{1}{2}\begin{bmatrix} 2 & 3 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

20. $p(\lambda) = -\lambda^3 + 5\lambda^2 - 8\lambda + 4$ so $-\mathbf{A}^3 + 5\mathbf{A}^2 - 8\mathbf{A} + 4\mathbf{I} = \mathbf{0}$

$$\mathbf{A}^2 = \begin{bmatrix} 10 & -9 & 3 \\ 6 & -5 & 3 \\ 0 & 0 & 4 \end{bmatrix}, \qquad \mathbf{A}^3 = 5\mathbf{A}^2 - 8\mathbf{A} + 4\mathbf{I} = \begin{bmatrix} 22 & -21 & 7 \\ 14 & -13 & 7 \\ 0 & 0 & 8 \end{bmatrix}$$

$$\mathbf{A}^4 = 5\mathbf{A}^3 - 8\mathbf{A}^2 + 4\mathbf{A} = \begin{bmatrix} 46 & -45 & 15 \\ 30 & -29 & 15 \\ 0 & 0 & 16 \end{bmatrix}$$

$$\mathbf{A}^{-1} = \frac{1}{4}\left(\mathbf{A}^2 - 5\mathbf{A} + 8\mathbf{I}\right) = \frac{1}{2}\begin{bmatrix} -1 & 3 & -1 \\ -2 & 4 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

21. $p(\lambda) = -\lambda^3 + \lambda$ so $-\mathbf{A}^3 + \mathbf{A} = \mathbf{0}$

$$\mathbf{A}^2 = \begin{bmatrix} 1 & 0 & 0 \\ 78 & -5 & -2 \\ -195 & 15 & 6 \end{bmatrix}, \qquad \mathbf{A}^3 = \mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \\ 6 & 5 & 2 \\ 21 & -15 & -6 \end{bmatrix}$$

$$\mathbf{A}^4 = \mathbf{A}^2 = \begin{bmatrix} 1 & 0 & 0 \\ 78 & -5 & -2 \\ -195 & 15 & 6 \end{bmatrix}$$

Because $\lambda = 0$ is an eigenvalue, $\mathbf{A}$ is singular and $\mathbf{A}^{-1}$ does not exist.

22. $p(\lambda) = -\lambda^3 + \lambda^2 + \lambda - 1$ so $-\mathbf{A}^3 + \mathbf{A}^2 + \mathbf{A} - \mathbf{I} = \mathbf{0}$

$$\mathbf{A}^2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \mathbf{I}, \qquad \mathbf{A}^3 = \mathbf{A}^2 + \mathbf{A} - \mathbf{I} = \begin{bmatrix} 11 & -6 & -2 \\ 20 & -11 & -4 \\ 0 & 0 & 1 \end{bmatrix} = \mathbf{A}$$

$$\mathbf{A}^4 = \mathbf{A}^3 + \mathbf{A}^2 - \mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \mathbf{I}$$

$$\mathbf{A}^{-1} = -\mathbf{A}^2 + \mathbf{A} + \mathbf{I} = \begin{bmatrix} 11 & -6 & -2 \\ 20 & -11 & -4 \\ 0 & 0 & 1 \end{bmatrix} = \mathbf{A}$$

23. $p(\lambda) = -\lambda^3 + \lambda$ so $-\mathbf{A}^3 + \mathbf{A} = \mathbf{0}$

$$\mathbf{A}^2 = \begin{bmatrix} 3 & -3 & 1 \\ 2 & -2 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \qquad \mathbf{A}^3 = \mathbf{A} = \begin{bmatrix} 1 & -1 & 1 \\ 2 & -2 & 1 \\ 4 & -4 & 1 \end{bmatrix}$$

$$\mathbf{A}^4 = \mathbf{A}^2 = \begin{bmatrix} 3 & -3 & 1 \\ 2 & -2 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

Because $\lambda = 0$ is an eigenvalue, $\mathbf{A}$ is singular and $\mathbf{A}^{-1}$ does not exist.

24. $p(\lambda) = -\lambda^3 + \lambda$ so $-\mathbf{A}^3 + \mathbf{A} = \mathbf{0}$

$$\mathbf{A}^2 = \begin{bmatrix} 3 & -3 & -1 \\ 2 & -2 & -1 \\ 0 & 0 & 1 \end{bmatrix}, \qquad \mathbf{A}^3 = \mathbf{A} = \begin{bmatrix} 5 & -5 & -3 \\ 2 & -2 & -1 \\ 4 & -4 & -3 \end{bmatrix}$$

$$\mathbf{A}^4 = \mathbf{A}^2 = \begin{bmatrix} 3 & -3 & -1 \\ 2 & -2 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

Because $\lambda = 0$ is an eigenvalue, $\mathbf{A}$ is singular and $\mathbf{A}^{-1}$ does not exist.

In Problems 25–30 we first find the eigensystem of the given transition matrix $\mathbf{A}$ so as to determine its eigenvector matrix $\mathbf{P}$ and its diagonal eigenvalue matrix $\mathbf{D}$. Then we determine how the matrix power $\mathbf{A}^k = \mathbf{P}\mathbf{D}^k\mathbf{P}^{-1}$ behaves as $k \to \infty$. For simpler calculations of eigenvalues and eigenvectors, we write the entries of $\mathbf{A}$ in fractional rather than decimal form.

25. Characteristic polynomial: $p(\lambda) = \lambda^2 - \frac{9}{5}\lambda + \frac{4}{5} = \frac{1}{5}(\lambda - 1)(5\lambda - 4)$

Eigenvalues: $\lambda_1 = 1, \quad \lambda_2 = \frac{4}{5}$

With $\lambda_1 = 1$: $\left.\begin{aligned} -\frac{1}{10}a+\frac{1}{10}b &= 0 \\ \frac{1}{10}a-\frac{1}{10}b &= 0 \end{aligned}\right\}$ $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

With $\lambda_2 = \frac{4}{5}$: $\left.\begin{aligned} \frac{1}{10}a+\frac{1}{10}b &= 0 \\ \frac{1}{10}a+\frac{1}{10}b &= 0 \end{aligned}\right\}$ $\mathbf{v}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$$\mathbf{P} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}, \quad \mathbf{D} = \begin{bmatrix} 1 & 0 \\ 0 & 4/5 \end{bmatrix}, \quad \mathbf{P}^{-1} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

$$\mathbf{x}_k = \mathbf{A}^k\mathbf{x}_0 = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 4/5 \end{bmatrix}^k \frac{1}{2}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\mathbf{x}_0$$

$$\to \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\frac{1}{2}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\mathbf{x}_0 = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} C_0 \\ S_0 \end{bmatrix} = \left(C_0+S_0\right)\begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}$$

as $k \to \infty$. Thus the long-term distribution of population is 50% city, 50% suburban.

26. Characteristic polynomial: $p(\lambda) = \lambda^2 - \frac{9}{5}\lambda + \frac{4}{5} = \frac{1}{5}(\lambda-1)(5\lambda-4)$

Eigenvalues: $\lambda_1 = 1, \quad \lambda_2 = \frac{4}{5}$

With $\lambda_1 = 1$: $\left.\begin{aligned} -\frac{3}{20}a+\frac{1}{20}b &= 0 \\ \frac{3}{20}a-\frac{1}{20}b &= 0 \end{aligned}\right\}$ $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$

With $\lambda_2 = \frac{4}{5}$: $\left.\begin{aligned} \frac{1}{20}a+\frac{1}{20}b &= 0 \\ \frac{3}{20}a+\frac{3}{20}b &= 0 \end{aligned}\right\}$ $\mathbf{v}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$$\mathbf{P} = \begin{bmatrix} 1 & -1 \\ 3 & 1 \end{bmatrix}, \quad \mathbf{D} = \begin{bmatrix} 1 & 0 \\ 0 & 4/5 \end{bmatrix}, \quad \mathbf{P}^{-1} = \frac{1}{4}\begin{bmatrix} 1 & 1 \\ -3 & 1 \end{bmatrix}$$

$$\mathbf{x}_k = \mathbf{A}^k\mathbf{x}_0 = \begin{bmatrix} 1 & -1 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 4/5 \end{bmatrix}^k \frac{1}{4}\begin{bmatrix} 1 & 1 \\ -3 & 1 \end{bmatrix}\mathbf{x}_0$$

$$\to \begin{bmatrix} 1 & -1 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\frac{1}{4}\begin{bmatrix} 1 & 1 \\ -3 & 1 \end{bmatrix}\mathbf{x}_0 = \frac{1}{4}\begin{bmatrix} 1 & 1 \\ 3 & 3 \end{bmatrix}\begin{bmatrix} C_0 \\ S_0 \end{bmatrix} = \left(C_0+S_0\right)\begin{bmatrix} 1/4 \\ 3/4 \end{bmatrix}$$

as $k \to \infty$. Thus the long-term distribution of population is 25% city, 75% suburban.

27. Characteristic polynomial: $p(\lambda) = \lambda^2 - \frac{8}{5}\lambda + \frac{3}{5} = \frac{1}{5}(\lambda - 1)(5\lambda - 3)$

Eigenvalues: $\lambda_1 = 1, \quad \lambda_2 = \frac{3}{5}$

With $\lambda_1 = 1$: $\left.\begin{aligned} -\frac{1}{4}a + \frac{3}{20}b &= 0 \\ \frac{1}{4}a - \frac{3}{20}b &= 0 \end{aligned}\right\}$ $\quad \mathbf{v}_1 = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$

With $\lambda_2 = \frac{3}{5}$: $\left.\begin{aligned} \frac{3}{20}a + \frac{3}{20}b &= 0 \\ \frac{1}{4}a + \frac{1}{4}b &= 0 \end{aligned}\right\}$ $\quad \mathbf{v}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$$\mathbf{P} = \begin{bmatrix} 3 & -1 \\ 5 & 1 \end{bmatrix}, \quad \mathbf{D} = \begin{bmatrix} 1 & 0 \\ 0 & 3/5 \end{bmatrix}, \quad \mathbf{P}^{-1} = \frac{1}{8}\begin{bmatrix} 1 & 1 \\ -5 & 3 \end{bmatrix}$$

$$\mathbf{x}_k = \mathbf{A}^k\mathbf{x}_0 = \begin{bmatrix} 3 & -1 \\ 5 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 3/5 \end{bmatrix}^k \frac{1}{8}\begin{bmatrix} 1 & 1 \\ -5 & 3 \end{bmatrix}\mathbf{x}_0$$

$$\to \begin{bmatrix} 3 & -1 \\ 5 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\frac{1}{8}\begin{bmatrix} 1 & 1 \\ -5 & 3 \end{bmatrix}\mathbf{x}_0 = \frac{1}{8}\begin{bmatrix} 3 & 3 \\ 5 & 5 \end{bmatrix}\begin{bmatrix} C_0 \\ S_0 \end{bmatrix} = (C_0 + S_0)\begin{bmatrix} 3/8 \\ 5/8 \end{bmatrix}$$

as $k \to \infty$. Thus the long-term distribution of population is 3/8 city, 5/8 suburban.

28. Characteristic polynomial: $p(\lambda) = \lambda^2 - \frac{17}{10}\lambda + \frac{7}{10} = \frac{1}{10}(\lambda - 1)(10\lambda - 7)$

Eigenvalues: $\lambda_1 = 1, \quad \lambda_2 = \frac{7}{10}$

With $\lambda_1 = 1$: $\left.\begin{aligned} -\frac{1}{5}a + \frac{1}{10}b &= 0 \\ \frac{1}{5}a - \frac{1}{10}b &= 0 \end{aligned}\right\}$ $\quad \mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

With $\lambda_2 = \frac{7}{10}$: $\left.\begin{aligned} \frac{1}{10}a + \frac{1}{10}b &= 0 \\ \frac{1}{5}a + \frac{1}{5}b &= 0 \end{aligned}\right\}$ $\quad \mathbf{v}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$$\mathbf{P} = \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix}, \quad \mathbf{D} = \begin{bmatrix} 1 & 0 \\ 0 & 7/10 \end{bmatrix}, \quad \mathbf{P}^{-1} = \frac{1}{3}\begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix}$$

$$\mathbf{x}_k = \mathbf{A}^k\mathbf{x}_0 = \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 7/10 \end{bmatrix}^k \frac{1}{3}\begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix}\mathbf{x}_0$$

$$\to \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\frac{1}{3}\begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix}\mathbf{x}_0 = \frac{1}{3}\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} C_0 \\ S_0 \end{bmatrix} = (C_0 + S_0)\begin{bmatrix} 1/3 \\ 2/3 \end{bmatrix}$$

as $k \to \infty$. Thus the long-term distribution of population is 1/3 city, 2/3 suburban.

29. Characteristic polynomial: $p(\lambda) = \lambda^2 - \frac{37}{20}\lambda + \frac{17}{20} = \frac{1}{20}(\lambda - 1)(20\lambda - 17)$

Eigenvalues: $\lambda_1 = 1, \quad \lambda_2 = \frac{17}{20}$

With $\lambda_1 = 1$: $\left.\begin{aligned} -\frac{1}{10}a + \frac{1}{20}b &= 0 \\ \frac{1}{10}a - \frac{1}{20}b &= 0 \end{aligned}\right\}$ $\quad \mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

With $\lambda_2 = \frac{17}{20}$: $\left.\begin{aligned} \frac{1}{20}a + \frac{1}{20}b &= 0 \\ \frac{1}{10}a + \frac{1}{10}b &= 0 \end{aligned}\right\}$ $\quad \mathbf{v}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$$\mathbf{P} = \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix}, \quad \mathbf{D} = \begin{bmatrix} 1 & 0 \\ 0 & 17/20 \end{bmatrix}, \quad \mathbf{P}^{-1} = \frac{1}{3}\begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix}$$

$$\mathbf{x}_k = \mathbf{A}^k\mathbf{x}_0 = \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 17/20 \end{bmatrix}^k \frac{1}{3}\begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix}\mathbf{x}_0$$

$$\to \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\frac{1}{3}\begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix}\mathbf{x}_0 = \frac{1}{3}\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} C_0 \\ S_0 \end{bmatrix} = (C_0 + S_0)\begin{bmatrix} 1/3 \\ 2/3 \end{bmatrix}$$

as $k \to \infty$. Thus the long-term distribution of population is 1/3 city, 2/3 suburban.

30. Characteristic polynomial: $p(\lambda) = \lambda^2 - \frac{33}{20}\lambda + \frac{13}{20} = \frac{1}{20}(\lambda - 1)(20\lambda - 13)$

Eigenvalues: $\lambda_1 = 1, \quad \lambda_2 = \frac{13}{20}$

With $\lambda_1 = 1$: $\left.\begin{array}{r} -\dfrac{1}{5}a + \dfrac{3}{20}b = 0 \\ \dfrac{1}{5}a - \dfrac{3}{20}b = 0 \end{array}\right\}$ $\mathbf{v}_1 = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$

With $\lambda_2 = \dfrac{13}{20}$: $\left.\begin{array}{r} \dfrac{3}{20}a + \dfrac{3}{20}b = 0 \\ \dfrac{1}{5}a + \dfrac{1}{5}b = 0 \end{array}\right\}$ $\mathbf{v}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$$\mathbf{P} = \begin{bmatrix} 3 & -1 \\ 4 & 1 \end{bmatrix}, \quad \mathbf{D} = \begin{bmatrix} 1 & 0 \\ 0 & 13/20 \end{bmatrix}, \quad \mathbf{P}^{-1} = \frac{1}{7}\begin{bmatrix} 1 & 1 \\ -4 & 3 \end{bmatrix}$$

$$\mathbf{x}_k = \mathbf{A}^k\mathbf{x}_0 = \begin{bmatrix} 3 & -1 \\ 4 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 13/20 \end{bmatrix}^k \frac{1}{7}\begin{bmatrix} 1 & 1 \\ -4 & 3 \end{bmatrix}\mathbf{x}_0$$

$$\to \begin{bmatrix} 3 & -1 \\ 4 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\frac{1}{7}\begin{bmatrix} 1 & 1 \\ -4 & 3 \end{bmatrix}\mathbf{x}_0 = \frac{1}{7}\begin{bmatrix} 3 & 3 \\ 4 & 4 \end{bmatrix}\begin{bmatrix} C_0 \\ S_0 \end{bmatrix} = (C_0 + S_0)\begin{bmatrix} 3/7 \\ 4/7 \end{bmatrix}$$

as $k \to \infty$. Thus the long-term distribution of population is 3/7 city, 4/7 suburban.

In the following three problems, just as in Problems 25–30, we first write the elements of $\mathbf{A}$ in fractional rather than decimal form, with $r = 4/25$ in Problem 31, $r = 7/40$ in Problem 32, and $r = 27/200$ in Problem 33.

31. Characteristic polynomial: $p(\lambda) = \lambda^2 - \dfrac{9}{5}\lambda + \dfrac{4}{5} = \dfrac{1}{5}(\lambda - 1)(5\lambda - 4)$

Eigenvalues: $\lambda_1 = 1, \quad \lambda_2 = \dfrac{4}{5}$

With $\lambda_1 = 1$: $\left.\begin{array}{r} -\dfrac{2}{5}a + \dfrac{1}{2}b = 0 \\ -\dfrac{4}{25}a + \dfrac{1}{5}b = 0 \end{array}\right\}$ $\mathbf{v}_1 = \begin{bmatrix} 5 \\ 4 \end{bmatrix}$

With $\lambda_2 = \dfrac{4}{5}$: $\left.\begin{array}{r} -\dfrac{1}{5}a + \dfrac{1}{2}b = 0 \\ -\dfrac{4}{25}a + \dfrac{2}{5}b = 0 \end{array}\right\}$ $\mathbf{v}_2 = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$

$$\mathbf{P} = \begin{bmatrix} 5 & 5 \\ 4 & 2 \end{bmatrix}, \quad \mathbf{D} = \begin{bmatrix} 1 & 0 \\ 0 & 4/5 \end{bmatrix}, \quad \mathbf{P}^{-1} = \frac{1}{10}\begin{bmatrix} -2 & 5 \\ 4 & -5 \end{bmatrix}$$

$$\mathbf{x}_k = \mathbf{A}^k\mathbf{x}_0 = \begin{bmatrix} 5 & 5 \\ 4 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 4/5 \end{bmatrix}^k \frac{1}{10}\begin{bmatrix} -2 & 5 \\ 4 & -5 \end{bmatrix}\mathbf{x}_0$$

$$\to \begin{bmatrix} 5 & 5 \\ 4 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \frac{1}{10}\begin{bmatrix} -2 & 5 \\ 4 & -5 \end{bmatrix}\mathbf{x}_0 = \frac{1}{10}\begin{bmatrix} -10 & 25 \\ -8 & 20 \end{bmatrix}\begin{bmatrix} F_0 \\ R_0 \end{bmatrix} = \begin{bmatrix} 2.5R_0 - F_0 \\ 2R_0 - 0.8F_0 \end{bmatrix}$$

as $k \to \infty$. Thus the fox-rabbit population approaches a stable situation with $2.5R_0 - F_0$ foxes and $2R_0 - 0.8F_0$ rabbits.

32. Characteristic polynomial: $p(\lambda) = \lambda^2 - \frac{9}{5}\lambda + \frac{63}{80} = \frac{1}{80}(20\lambda - 21)(4\lambda - 3)$

Eigenvalues: $\lambda_1 = \frac{19}{20}, \quad \lambda_2 = \frac{17}{20}$

With $\lambda_1 = \frac{21}{20}$: $\left.\begin{aligned} -\frac{9}{20}a + \frac{1}{2}b &= 0 \\ -\frac{27}{200}a + \frac{3}{20}b &= 0 \end{aligned}\right\}$ $\quad \mathbf{v}_1 = \begin{bmatrix} 10 \\ 9 \end{bmatrix}$

With $\lambda_2 = \frac{3}{4}$: $\left.\begin{aligned} -\frac{3}{20}a + \frac{1}{2}b &= 0 \\ -\frac{27}{200}a + \frac{9}{20}b &= 0 \end{aligned}\right\}$ $\quad \mathbf{v}_2 = \begin{bmatrix} 10 \\ 3 \end{bmatrix}$

$$\mathbf{P} = \begin{bmatrix} 10 & 10 \\ 9 & 3 \end{bmatrix}, \quad \mathbf{D} = \begin{bmatrix} 21/20 & 0 \\ 0 & 3/4 \end{bmatrix}, \quad \mathbf{P}^{-1} = \frac{1}{60}\begin{bmatrix} -3 & 10 \\ 9 & -10 \end{bmatrix}$$

$$\mathbf{x}_k = \mathbf{A}^k\mathbf{x}_0 = \begin{bmatrix} 10 & 10 \\ 9 & 3 \end{bmatrix}\begin{bmatrix} 21/20 & 0 \\ 0 & 3/4 \end{bmatrix}^k \frac{1}{60}\begin{bmatrix} -3 & 10 \\ 9 & -10 \end{bmatrix}\mathbf{x}_0$$

$$\approx \frac{1}{60}\left(\frac{21}{20}\right)^k \begin{bmatrix} 10 & 10 \\ 9 & 3 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} -3 & 10 \\ 9 & -10 \end{bmatrix}\mathbf{x}_0 = \frac{1}{60}(1.05)^k \begin{bmatrix} -30 & 100 \\ -27 & 90 \end{bmatrix}\begin{bmatrix} F_0 \\ R_0 \end{bmatrix}$$

$$= \frac{1}{60}(1.05)^k\,(10R_0 - 3F_0\begin{bmatrix} 10 \\ 9 \end{bmatrix}$$

when k is sufficiently large. Thus the fox and rabbit populations are both increasing at 5% per year, with 10 foxes for each 9 rabbits.

33. Characteristic polynomial: $p(\lambda) = \lambda^2 - \frac{9}{5}\lambda + \frac{323}{400} = \frac{1}{400}(20\lambda - 19)(20\lambda - 17)$

Eigenvalues: $\lambda_1 = \frac{19}{20}, \quad \lambda_2 = \frac{17}{20}$

With $\lambda_1 = \dfrac{19}{20}$: $\left.\begin{aligned} -\frac{7}{20}a + \frac{1}{2}b &= 0 \\ -\frac{7}{40}a + \frac{1}{4}b &= 0 \end{aligned}\right\}$ $\mathbf{v}_1 = \begin{bmatrix} 10 \\ 7 \end{bmatrix}$

With $\lambda_2 = \dfrac{17}{20}$: $\left.\begin{aligned} -\frac{1}{4}a + \frac{1}{2}b &= 0 \\ -\frac{7}{40}a + \frac{7}{20}b &= 0 \end{aligned}\right\}$ $\mathbf{v}_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$$\mathbf{P} = \begin{bmatrix} 10 & 2 \\ 7 & 1 \end{bmatrix}, \quad \mathbf{D} = \begin{bmatrix} 19/20 & 0 \\ 0 & 17/20 \end{bmatrix}, \quad \mathbf{P}^{-1} = \frac{1}{4}\begin{bmatrix} -1 & 2 \\ 7 & -10 \end{bmatrix}$$

$$\mathbf{x}_k = \mathbf{A}^k\mathbf{x}_0 = \begin{bmatrix} 10 & 2 \\ 7 & 1 \end{bmatrix}\begin{bmatrix} 19/20 & 0 \\ 0 & 17/20 \end{bmatrix}^k \frac{1}{4}\begin{bmatrix} -1 & 2 \\ 7 & -10 \end{bmatrix}\mathbf{x}_0$$

$$\to \begin{bmatrix} 10 & 2 \\ 7 & 1 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\frac{1}{4}\begin{bmatrix} -1 & 2 \\ 7 & -10 \end{bmatrix}\mathbf{x}_0 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} F_0 \\ R_0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

as $k \to \infty$. Thus the fox and rabbit population both die out.

34. $\mathbf{A} = \mathbf{PDP}^{-1} = \begin{bmatrix} 3 & 5 \\ 4 & 7 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 7 & -5 \\ -4 & 3 \end{bmatrix} = \begin{bmatrix} 41 & -30 \\ 56 & -41 \end{bmatrix}$

If n is even then $\mathbf{D}^n = \mathbf{I}$ so $\mathbf{A}^n = \mathbf{PD}^n\mathbf{P}^{-1} = \mathbf{PIP}^{-1} = \mathbf{I}$. If n is odd then $\mathbf{A}^n = \mathbf{A}^{n-1}\mathbf{A} = \mathbf{IA} = \mathbf{A}$. Thus $\mathbf{A}^{99} = \mathbf{A}$ and $\mathbf{A}^{100} = \mathbf{I}$.

35. The fact that each $|\lambda| = 1$, so $\lambda = \pm 1$, implies that $\mathbf{D}^n = \mathbf{I}$ if n is even, in which case $\mathbf{A}^n = \mathbf{PD}^n\mathbf{P}^{-1} = \mathbf{PIP}^{-1} = \mathbf{I}$.

36. We find immediately that $\mathbf{A}^2 = \mathbf{I}$, so $\mathbf{A}^3 = \mathbf{A}^2\mathbf{A} = \mathbf{IA} = \mathbf{A}$, $\mathbf{A}^4 = \mathbf{A}^3\mathbf{A} = \mathbf{A}^2 = \mathbf{I}$, and so forth.

37. We find immediately that $\mathbf{A}^2 = -\mathbf{I}$, so $\mathbf{A}^3 = \mathbf{A}^2\mathbf{A} = -\mathbf{IA} = -\mathbf{A}$, $\mathbf{A}^4 = \mathbf{A}^3\mathbf{A} = -\mathbf{A}^2 = \mathbf{I}$, and so forth.

38. If $\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \mathbf{I} + \mathbf{B}$, then $\mathbf{B}^2 = \mathbf{0}$, so it follows that

$$\mathbf{A}^n = (\mathbf{I}+\mathbf{B})^n = \mathbf{I}^n + n\mathbf{I}^{n-1}\mathbf{B} + \tfrac{1}{2}n(n-1)\mathbf{I}^{n-1}\mathbf{B}^2 + \cdots = \mathbf{I} + n\mathbf{B} = \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix}.$$

39. The characteristic equation of $\mathbf{A}$ is

$$\begin{aligned}(p-\lambda)(q-\lambda) = (1-p)(1-q) &= \lambda^2 - (p+q)\lambda + (p+q-1) \\ &= (\lambda-1)[\lambda-(p+q-1)],\end{aligned}$$

so the eigenvalues of A are $\lambda_1 = 1$ and $\lambda_2 = p+q-1$.

40. The fact that the column sums of $\mathbf{A}$ are each 1 implies that the row sums of the transpose matrix $\mathbf{A}^T$ are each 1, so it follows readily that $\mathbf{A}^T\mathbf{v} = \mathbf{v}$. Thus $\lambda = 1$ is an eigenvalue of $\mathbf{A}^T$. But $\mathbf{A}$ and $\mathbf{A}^T$ have the same eigenvalues (by Problem 35 in Section 6.1).

CHAPTER 7

LINEAR SYSTEMS OF DIFFERENTIAL EQUATIONS

SECTION 7.1

FIRST-ORDER SYSTEMS AND APPLICATIONS

1. Let $x_1 = x$ and $x_2 = x_1' = x'$, so $x_2' = x'' = -7x - 3x' + t^2$.

Equivalent system:

$x_1' = x_2, \qquad x_2' = -7x_1 - 3x_2 + t^2$

2. Let $x_1 = x$, $x_2 = x_1' = x'$, $x_3 = x_2' = x''$, $x_4 = x_3' = x'''$, so $x_4' = x^{(4)} = x + 3x' - 6x'' + \cos 3t$.

Equivalent system:

$x_1' = x_2, \qquad x_2' = x_3, \qquad x_3' = x_4, \qquad x_4' = -x_1 + 3x_2 - 6x_3 + \cos 3t$

3. Let $x_1 = x$ and $x_2 = x_1' = x'$, so $x_2' = x'' = \left[\left(1-t^2\right)x - tx'\right]/t^2$.

Equivalent system:

$x_1' = x_2, \qquad t^2 x_2' = (1 - t^2)x_1 - tx_2$

4. Let $x_1 = x$, $x_2 = x_1' = x'$, $x_3 = x_2' = x''$, so $x_3' = x''' = \left(-5x - 3tx' - 2t^2x'' + \ln t\right)/t^3$.

Equivalent system:

$x_1' = x_2, \qquad x_2' = x_3, \qquad t^3 x_3' = -5x_1 - 3tx_2 + 2t^2x_3 + \ln t$

5. Let $x_1 = x$, $x_2 = x_1' = x'$, $x_3 = x_2' = x''$, so $x_3' = x''' = \left(x'\right)^2 + \cos x$.

Equivalent system:

$x_1' = x_2, \qquad x_2' = x_3, \qquad x_3' = x_2^2 + \cos x_1$

6. Let $x_1 = x$, $x_2 = x_1' = x'$, $y_1 = y$, $y_2 = y_1' = y'$ so $x_2' = x'' = 5x - 4y$, $y_2' = y'' = -4x + 5y$.

Equivalent system:

$x_1' = x_2, \qquad x_2' = 5x_1 - 4y_1$

$y_1' = y_2, \qquad y_2' = -4x_1 + 5y_1$

7. Let $x_1 = x,\ x_2 = x_1' = x',\ y_1 = y,\ y_2 = y_1' = y'$ so $x_2' = x'' = -kx/(x^2 + y^2)^{3/2}$, $y_2' = y'' = -ky/(x^2 + y^2)^{3/2}$.

Equivalent system:

$x_1' = x_2, \qquad x_2' = -kx_1/\left(x_1^2 + y_1^2\right)^{3/2}$

$y_1' = y_2, \qquad y_2' = -ky_1/\left(x_1^2 + y_1^2\right)^{3/2}$

8. Let $x_1 = x,\ x_2 = x_1' = x',\ y_1 = y,\ y_2 = y_1' = y'$ so $x_2' = x'' = -4x + 2y - 3x'$, $y_2' = y'' = 3x - y - 2y' + \cos t$.

Equivalent system:

$x_1' = x_2, \qquad x_2' = -4x_1 + 2y_1 - 3x_2$

$y_1' = y_2, \qquad y_2' = 3x_1 - y_1 - 2y_2 + \cos t$

9. Let $x_1 = x,\ x_2 = x_1' = x',\ y_1 = y,\ y_2 = y_1' = y',\ z_1 = z,\ z_2 = z_1' = z'$, so $x_2' = x'' = 3x - y + 2z,\ y_2' = y'' = x + y - 4z,\ z_2' = z'' = 5x - y - z.$

Equivalent system:

$x_1' = x_2, \qquad x_2' = 3x_1 - y_1 + 2z_1$

$y_1' = y_2, \qquad y_2' = x_1 + y_1 - 4z_1$

$z_1' = z_2, \qquad z_2' = 5x_1 - y_1 - z_1$

10. Let $x_1 = x,\ x_2 = x_1' = x',\ y_1 = y,\ y_2 = y_1' = y'$ so $x_2' = x'' = x(1-y),\ \ y_2' = y'' = y(1-x).$

Equivalent system:

$x_1' = x_2, \qquad x_2' = x_1(1 - y_1)$

$y_1' = y_2 \qquad y_2' = y_1(1 - x_1)$

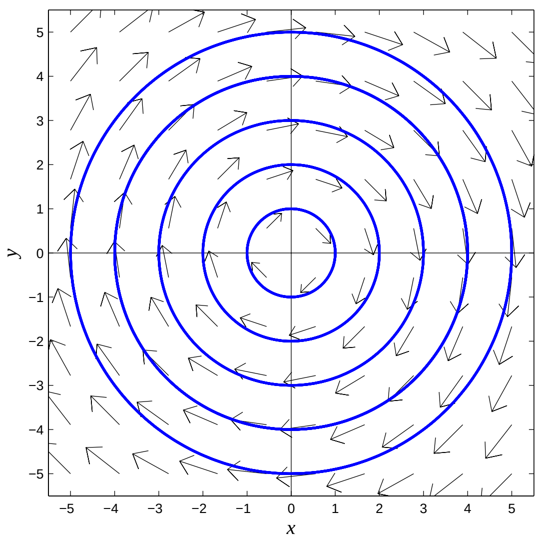
11. The computation $x'' = y' = -x$ yields the single linear second-order equation $x'' + x = 0$ with characteristic equation $r^2 + 1 = 0$ and general solution

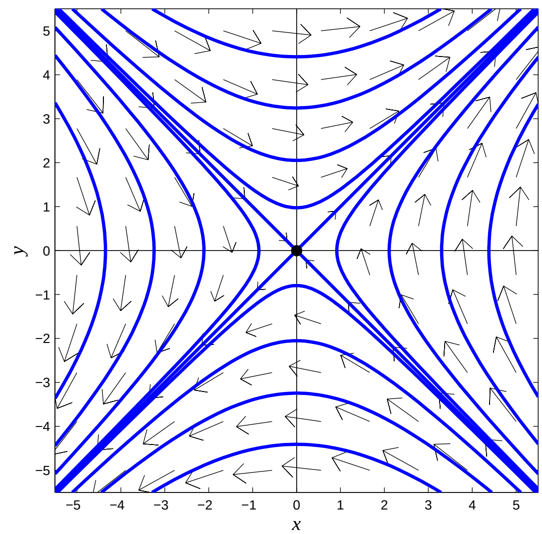
$$x(t) = A\cos t + B\sin t.$$

Then the original first equation $y = x'$ gives

$$y(t) = B\cos t - A\sin t.$$

The figure on the left below shows a direction field and typical solution curves (obviously circles?) for the given system.





12. The computation $x'' = y' = x$ yields the single linear second-order equation $x'' - x = 0$ with characteristic equation $r^2 - 1 = 0$ and general solution

$$x(t) = A\,e^{t} + B\,e^{-t}.$$

Then the original first equation $y = x'$ gives

$$y(t) = A\,e^{t} - B\,e^{-t}.$$

The figure on the right above shows a direction field and some typical solution curves of this system. It appears that the typical solution curve is a branch of a hyperbola.

13. The computation $x'' = -2y' = -4x$ yields the single linear second-order equation $x'' + 4x = 0$ with characteristic equation $r^2 + 4 = 0$ and general solution

$$x(t) = A\cos 2t + B\sin 2t.$$

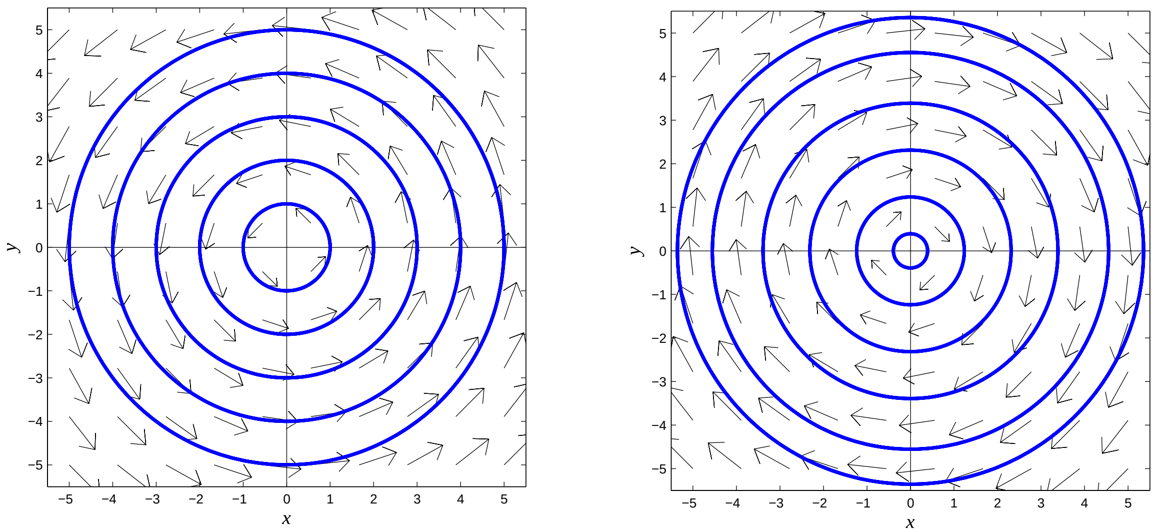
Then the original first equation $y = -x'/2$ gives

$$y(t) = -B\cos 2t + A\sin 2t.$$

Finally, the condition $x(0) = 1$ implies that $A = 1$, and then the condition $y(0) = 0$ gives $B = 0$. Hence the desired particular solution is given by

$$x(t) = \cos 2t, \qquad y(t) = \sin 2t.$$

The figure on the left below shows a direction field and some typical circular solution curves for the given system.



14. The computation $x'' = 10y' = -100x$ yields the single linear second-order equation $x'' + 100x = 0$ with characteristic equation $r^2 + 100 = 0$ and general solution

$$x(t) = A \cos 10t + B \sin 10t.$$

Then the original first equation $y = x'/10$ gives

$$y(t) = B \cos 10t - A \sin 10t.$$

Finally, the condition $x(0) = 3$ implies that $A = 3$, and then the condition $y(0) = 4$ gives $B = 4$. Hence the desired particular solution is given by

$$x(t) = 3 \cos 10t + 4 \sin 10t,$$

$$y(t) = 4 \cos 10t - 3 \sin 10t.$$

The typical solution curve is a circle. See the figure on the right above.

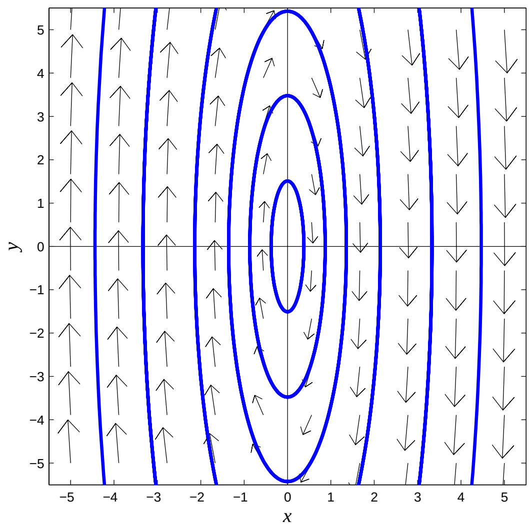
15. The computation $x'' = y'/2 = -4x$ yields the single linear second-order equation $x'' + 4x = 0$ with characteristic equation $r^2 + 4 = 0$ and general solution

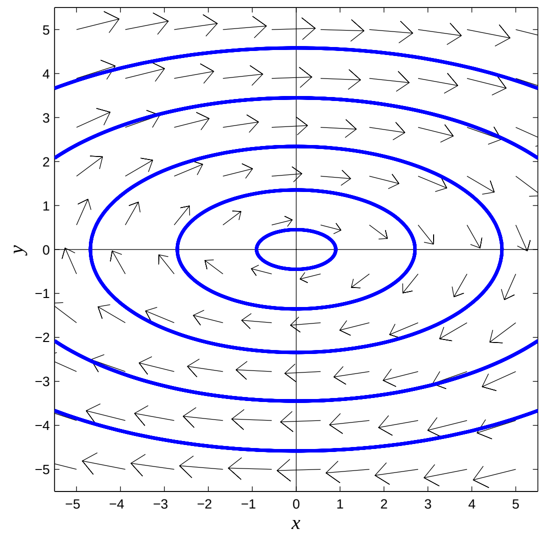
$$x(t) = A \cos 2t + B \sin 2t.$$

Then the original first equation $y = 2x'$ gives

$$y(t) = 4B\cos 2t - 4A\sin 2t.$$

The figure on the left below shows a direction field and some typical elliptical solution curves.





16. The computation $x'' = 8y' = -16x$ yields the single linear second-order equation $x'' + 16x = 0$ with characteristic equation $r^2 + 16 = 0$ and general solution

$$x(t) = A\cos 4t + B\sin 4t.$$

Then the original first equation $y = x'/8$ gives

$$y(t) = (B/2)\cos 4t - (A/2)\sin 4t.$$

The typical solution curve is an ellipse. The figure on the right above shows a direction field and some typical solution curves.

17. The computation $x'' = y' = 6x - y = 6x - x'$ yields the single linear second-order equation $x'' + x' - 6x = 0$ with characteristic equation $r^2 + r - 6 = 0$ and characteristic roots $r = -3$ and 2, so the general solution

$$x(t) = A\,e^{-3t} + B\,e^{2t}.$$

Then the original first equation $y = x'$ gives

$$y(t) = -3A\,e^{-3t} + 2B\,e^{2t}.$$

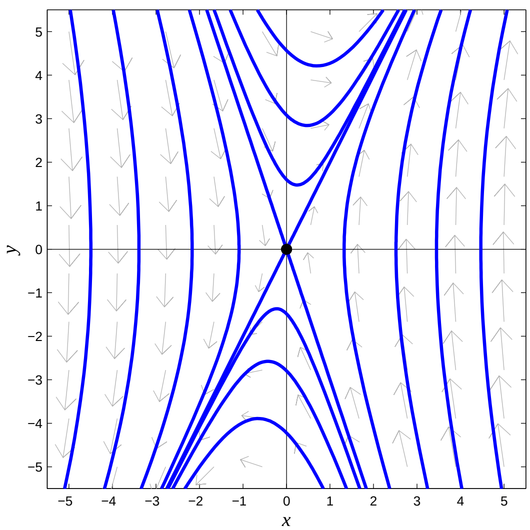
Finally, the initial conditions

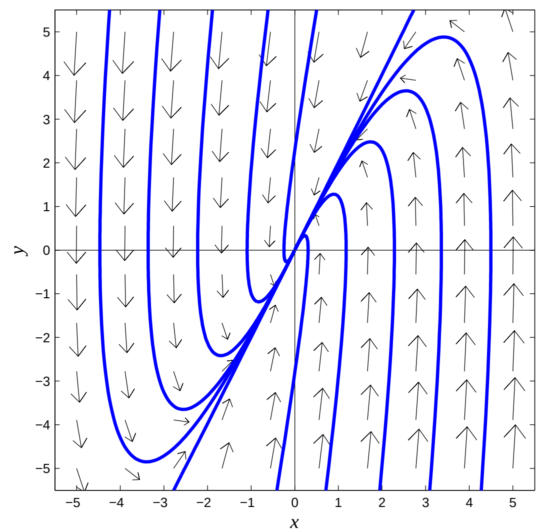
$$x(0) = A + B = 1, \quad y(0) = -3A + 2B = 2$$

imply that $A = 0$ and $B = 1$, so the desired particular solution is given by

$$x(t) = e^{-3t}, \qquad y(t) = 2\,e^{2t}.$$

The figure on the left below shows a direction field and some typical solution curves.





18. The computation $x'' = -y' = -10x + 7y = -10x - 7x'$ yields the single linear second-order equation $x'' + 7x' + 10x = 0$ with characteristic equation $r^2 + 7r + 10 = 0$, characteristic roots $r = -2$ and -5, and general solution

$$x(t) = A\,e^{-2t} + B\,e^{-5t}.$$

Then the original first equation $y = -x'$ gives

$$y(t) = 2A\,e^{-2t} + 5B\,e^{-5t}.$$

Finally, the initial conditions

$$x(0) = A + B = 2, \quad y(0) = 2A + 5B = -7$$

imply that $A = 17/3$, $B = -11/3$, so the desired particular solution is given by

$$x(t) = (17\,e^{-2t} - 11\,e^{-5t})/3, \qquad y(t) = (34e^{-2t} - 55e^{-5t})/3.$$

It appears that the typical solution curve is tangent to the straight line $y = 2x$. See the right-hand figure above for a direction field and typical solution curves.

19. The computation $x'' = -y' = -13x - 4y = -13x + 4x'$ yields the single linear second-order equation $x'' - 4x' + 13x = 0$ with characteristic equation $r^2 - 4r + 13 = 0$ and characteristic roots $r = 2 \pm 3i$, hence the general solution is

$$x(t) = e^{2t}(A\cos 3t + B\sin 3t).$$

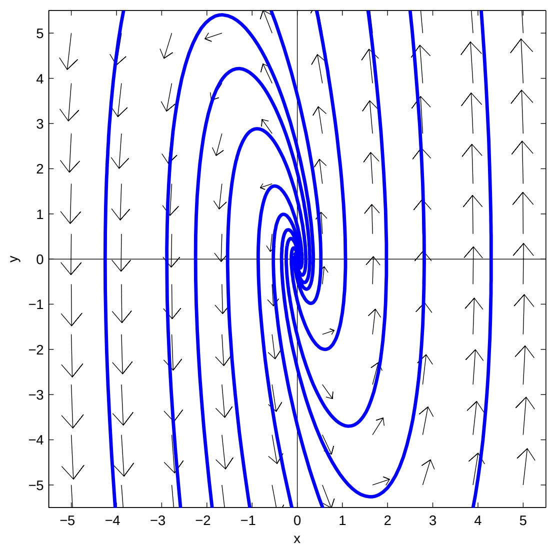
The initial condition $x(0) = 0$ then gives $A = 0$, so $x(t) = B\,e^{2t}\sin 3t$. Then the original first equation $y = -x'$ gives

$$y(t) = -e^{2t}(3B\cos 3t + 2B\sin 3t).$$

Finally, the initial condition $y(0) = 3$ gives $B = -1$, so the desired particular solution is given by

$$x(t) = -e^{2t}\sin 3t, \qquad y(t) = e^{2t}(3\cos 3t + 2\sin 3t).$$

The figure below shows a direction field and some typical solution curves.



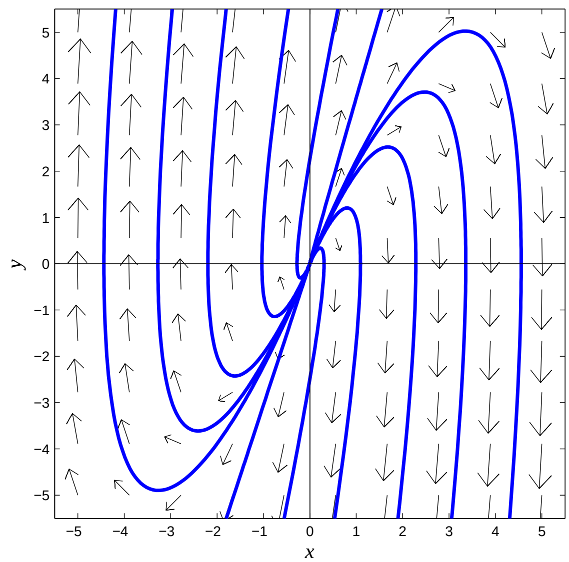
20. The computation $x'' = y' = -9x + 6y = -9x + 6x'$ yields the single linear second-order equation $x'' - 6x' + 9x = 0$ with characteristic equation $r^2 - 6r + 9 = 0$ and repeated characteristic root $r = 3, 3$, so its general solution is given by

$$x(t) = (A + Bt)e^{3t}.$$

Then the original first equation $y = x'$ gives

$$y(t) = (3A + B + 3Bt)e^{3t}.$$

It appears that the typical solution curve is tangent to the straight line $y = 3x$. The figure at the top of the next page shows a direction field and some typical solution curves.



21. **(a)** Substituting the general solution found in Problem 11 we get

$$\begin{aligned} x^2 + y^2 &= (A\cos t + B\sin t)^2 + (B\cos t - A\sin t)^2 \\ &= (A^2 + B^2)(\cos^2 t + \sin^2 t) = A^2 + B^2 \\ x^2 + y^2 &= C^2, \end{aligned}$$

the equation of a circle of radius $C = (A^2 + B^2)^{1/2}$.

(b) Substituting the general solution found in Problem 12 we get

$$x^2 - y^2 = (Ae^t + Be^{-t})^2 - (Ae^t - Be^{-t})^2 = 4AB,$$

the equation of a hyperbola.

22. **(a)** Substituting the general solution found in Problem 13 we get

$$\begin{aligned} x^2 + y^2 &= (A\cos 2t + B\sin 2t)^2 + (-B\cos 2t + A\sin 2t)^2 \\ &= (A^2 + B^2)(\cos^2 2t + \sin^2 2t) = A^2 + B^2 \\ x^2 + y^2 &= C^2, \end{aligned}$$

the equation of a circle of radius $C = (A^2 + B^2)^{1/2}$.

(b) Substituting the general solution found in Problem 15 we get

$$\begin{aligned} 16x^2 + y^2 &= 16(A\cos 2t + B\sin 2t)^2 + (4B\cos 2t - 4A\sin 2t)^2 \\ &= 16(A^2 + B^2)(\cos^2 2t + \sin^2 2t) = 16(A^2 + B^2) \end{aligned}$$

$$16x^2 + y^2 = C^2,$$

the equation of an ellipse with semi-axes 1 and 4.

23. When we solve Equations (20) and (21) in the text for e^{-t} and e^{2t} we get

$$2x - y = 3Ae^{-t} \text{and} \qquad x + y = 3Be^{2t}.$$

Hence

$$(2x - y)^2(x + y) = (3Ae^{-t})^2(3Be^{2t}) = 27A^2B = C.$$

Clearly $y = 2x$ or $y = -x$ if $C = 0$, and expansion gives the equation $4x^3 - 3xy^2 + y^3 = C$.

24. Looking at Fig. 7.1.9 in the text, we see that the first spring is stretched by x_1, the second spring is stretched by $x_2 - x_1$, and the third spring is compressed by x_2. Hence Newton's second law gives $m_1 x_1'' = -k_1(x_1) + k_2(x_2 - x_1)$ and $m_2 x_2'' = -k_2(x_2 - x_1) - k_3(x_2)$.

25. Looking at Fig. 7.1.10 in the text, we see that

$$my_1'' = -T\sin\theta_1 + T\sin\theta_2 \approx -T\tan\theta_1 + T\tan\theta_2 = -Ty_1/L + T(y_2 - y_1)/L,$$
$$my_2'' = -T\sin\theta_2 - T\sin\theta_3 \approx -T\tan\theta_2 - T\tan\theta_3 = -T(y_2 - y_1)/L - Ty_2/L.$$

We get the desired equations when we multiply each of these equations by L/T and set $k = mL/T$.

26. The concentration of salt in tank i is $c_i = x_i/100$ for $i = 1, 2, 3$ and each inflow-outflow rate is $r = 10$. Hence

$$x_1' = -rc_1 + rc_3 = \tfrac{1}{10}(-x_1 + x_3),$$

$$x_2' = +rc_1 - rc_2 = \tfrac{1}{10}(x_1 - x_2),$$

$$x_3' = +rc_2 - rc_3 = \tfrac{1}{10}(x_2 - x_3).$$

27. If θ is the polar angular coordinate of the point (x, y) and we write $F = k/(x^2 + y^2) = k/r^2$, then Newton's second law gives

$$mx'' = -F\cos\theta = -(k/r^2)(x/r) = -kx/r^3,$$
$$my'' = -F\sin\theta = -(k/r^2)(y/r) = -ky/r^3,$$

28. If we write (x', y') for the velocity vector and $v = \sqrt{(x')^2 + (y')^2}$ for the speed, then $(x'/v, y'/v)$ is a unit vector pointing in the direction of the velocity vector, and so the components of the air resistance force F_r are given by

$$F_r = -kv^2(x'/v, y'/v) = (-kvx', -kvy').$$

29. If $\mathbf{r} = (x, y, z)$ is the particle's position vector, then Newton's law $m\mathbf{r}'' = \mathbf{F}$ gives

$$m\mathbf{r}'' = q\,\mathbf{v}\times\mathbf{B} = q\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x' & y' & z' \\ 0 & 0 & B \end{vmatrix} = +qBy'\mathbf{i} - qBx'\mathbf{j} = qB(-y', x', 0).$$

and the characteristic equation $(r^2+2)[(r^2+2)^2 - 2] = 0$ has roots $\pm i\sqrt{2}$ and $\pm i\sqrt{2\pm\sqrt{2}}$.

SECTION 7.2

MATRICES AND LINEAR SYSTEMS

1. $$(\mathbf{AB})' = \begin{bmatrix} t-4t^2+6t^3 & t+t^2-4t^3+8t^4 \\ 3t+t^3-t^4 & 4t^2+t^3+t^4 \end{bmatrix}' = \begin{bmatrix} 1-8t+18t^2 & 1+2t-12t^2+32t^3 \\ 3+3t^2-4t^3 & 8t+3t^2+4t^3 \end{bmatrix}$$

$$\begin{aligned}\mathbf{A}'\mathbf{B} + \mathbf{A}\mathbf{B}' &= \begin{bmatrix} 1 & 2 \\ 3t^2 & -\dfrac{1}{t^2} \end{bmatrix}\begin{bmatrix} 1-t & 1+t \\ 3t^2 & 4t^3 \end{bmatrix} + \begin{bmatrix} t & 2t-1 \\ t^3 & \dfrac{1}{t} \end{bmatrix}\begin{bmatrix} -1 & 1 \\ 6t & 12t^2 \end{bmatrix} \\ &= \begin{bmatrix} 1-t+6t^2 & 1+t+8t^3 \\ -3+3t^2-3t^3 & -4t+3t^2+3t^3 \end{bmatrix} + \begin{bmatrix} -7t+12t^2 & t-12t^2+24t^3 \\ 6-t^3 & 12t+t^3 \end{bmatrix} \\ &= \begin{bmatrix} 1-8t+18t^2 & 1+2t-12t^2+32t^3 \\ 3+3t^2-4t^3 & 8t+3t^2+4t^3 \end{bmatrix}\end{aligned}$$

2. $$(\mathbf{AB})' = \begin{bmatrix} 3t^3+3e^t+2te^{-t} \\ 3t \\ 3t^4+24t-2e^{-t} \end{bmatrix}' = \begin{bmatrix} 9t^2+3e^t+2e^{-t}-2te^{-t} \\ 3 \\ 12t^3+24+2e^{-t} \end{bmatrix}$$

$$\mathbf{A'B}+\mathbf{AB'} = \begin{bmatrix} e^t & 1 & 2t \\ -1 & 0 & 0 \\ 8 & 0 & 3t^2 \end{bmatrix}\begin{bmatrix} 3 \\ 2e^{-t} \\ 3t \end{bmatrix} + \begin{bmatrix} e^t & t & t^2 \\ -t & 0 & 2 \\ 8t & -1 & t^3 \end{bmatrix}\begin{bmatrix} 0 \\ -2e^{-t} \\ 3 \end{bmatrix}$$

$$= \begin{bmatrix} 6t^2+3e^t+2e^{-t} \\ -3 \\ 24+9t^3 \end{bmatrix} + \begin{bmatrix} 3t^2-2t\,e^{-t} \\ 6 \\ 3t^3+2e^{-t} \end{bmatrix} = \begin{bmatrix} 9t^2+3e^t+2e^{-t}-2t\,e^{-t} \\ 3 \\ 12t^3+24+2e^{-t} \end{bmatrix}$$

3. $\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}, \quad \mathbf{P}(t) = \begin{bmatrix} 0 & -3 \\ 3 & 0 \end{bmatrix}, \quad \mathbf{f}(t) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

4. $\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}, \quad \mathbf{P}(t) = \begin{bmatrix} 3 & -2 \\ 2 & 1 \end{bmatrix}, \quad \mathbf{f}(t) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

5. $\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}, \quad \mathbf{P}(t) = \begin{bmatrix} 2 & 4 \\ 5 & -1 \end{bmatrix}, \quad \mathbf{f}(t) = \begin{bmatrix} 3e^t \\ -t^2 \end{bmatrix}$

6. $\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}, \quad \mathbf{P}(t) = \begin{bmatrix} t & -e^t \\ e^{-t} & t^2 \end{bmatrix}, \quad \mathbf{f}(t) = \begin{bmatrix} \cos t \\ -\sin t \end{bmatrix}$

7. $\mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \quad \mathbf{P}(t) = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}, \quad \mathbf{f}(t) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

8. $\mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \quad \mathbf{P}(t) = \begin{bmatrix} 2 & -3 & 0 \\ 1 & 1 & 2 \\ 0 & 5 & -7 \end{bmatrix}, \quad \mathbf{f}(t) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

9. $\mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \quad \mathbf{P}(t) = \begin{bmatrix} 3 & -4 & 1 \\ 1 & 0 & -3 \\ 0 & 6 & -7 \end{bmatrix}, \quad \mathbf{f}(t) = \begin{bmatrix} t \\ t^2 \\ t^3 \end{bmatrix}$

10. $\mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \quad \mathbf{P}(t) = \begin{bmatrix} t & -1 & e^t \\ 2 & t^2 & -1 \\ e^{-t} & 3t & t^3 \end{bmatrix}, \quad \mathbf{f}(t) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

11. $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}, \quad \mathbf{P}(t) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 4 & 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{f}(t) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

12. $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}, \quad \mathbf{P}(t) = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{bmatrix}, \quad \mathbf{f}(t) = \begin{bmatrix} 0 \\ t \\ t^2 \\ t^3 \end{bmatrix}$

13. $W(t) = \begin{vmatrix} 2e^t & e^{2t} \\ -3e^t & -e^{2t} \end{vmatrix} = e^{3t} \neq 0$

$$\mathbf{x}_1' = \begin{bmatrix} 2e^t \\ -3e^t \end{bmatrix}' = \begin{bmatrix} 2e^t \\ -3e^t \end{bmatrix} = \begin{bmatrix} 4 & 2 \\ -3 & -1 \end{bmatrix} \begin{bmatrix} 2e^t \\ -3e^t \end{bmatrix} = \mathbf{A}\mathbf{x}_1$$

$$\mathbf{x}_2' = \begin{bmatrix} e^{2t} \\ -e^{2t} \end{bmatrix}' = \begin{bmatrix} 2e^{2t} \\ -2e^{2t} \end{bmatrix} = \begin{bmatrix} 4 & 2 \\ -3 & -1 \end{bmatrix} \begin{bmatrix} e^{2t} \\ -e^{2t} \end{bmatrix} = \mathbf{A}\mathbf{x}_2$$

$$\mathbf{x}(t) = c_1\mathbf{x}_1 + c_2\mathbf{x}_2 = c_1 \begin{bmatrix} 2e^t \\ -3e^t \end{bmatrix} + c_2 \begin{bmatrix} e^{2t} \\ -e^{2t} \end{bmatrix} = \begin{bmatrix} 2c_1e^t + c_2e^{2t} \\ -3c_1e^t - c_2e^{2t} \end{bmatrix}$$

In most of Problems 14-22, we omit the verifications of the given solutions. In each case, this is simply a matter of calculating both the derivative $\mathbf{x}_i'$ of the given solution vector and the product $\mathbf{A}\mathbf{x}_i$ (where $\mathbf{A}$ is the coefficient matrix in the given differential equation) to verify that $\mathbf{x}_i' = \mathbf{A}\mathbf{x}_i$ (just as in the verification of the solutions $\mathbf{x}_1$ and $\mathbf{x}_2$ in Problem 13 above).

14. $W(t) = \begin{vmatrix} e^{3t} & 2e^{-2t} \\ 3e^{3t} & e^{-2t} \end{vmatrix} = -5e^{3t} \neq 0$

$$\mathbf{x}(t) = c_1\mathbf{x}_1 + c_2\mathbf{x}_2 = c_1 \begin{bmatrix} e^{3t} \\ 3e^{3t} \end{bmatrix} + c_2 \begin{bmatrix} 2e^{-2t} \\ e^{-2t} \end{bmatrix} = \begin{bmatrix} c_1e^{3t} + 2c_2e^{-2t} \\ 3c_1e^{3t} + c_2e^{-2t} \end{bmatrix}$$

15. $W(t) = \begin{vmatrix} e^{2t} & e^{-2t} \\ e^{2t} & 5e^{-2t} \end{vmatrix} = 4 \neq 0$

$$\mathbf{x}(t) = c_1\mathbf{x}_1 + c_2\mathbf{x}_2 = c_1\begin{bmatrix}1\\1\end{bmatrix}e^{2t} + c_2\begin{bmatrix}1\\5\end{bmatrix}e^{-2t} = \begin{bmatrix} c_1e^{2t} + c_2e^{-2t} \\ c_1e^{2t} + 5c_2e^{-2t} \end{bmatrix}$$

16. $W(t) = \begin{vmatrix} e^{3t} & e^{2t} \\ -e^{3t} & -2e^{2t} \end{vmatrix} = e^{5t} \neq 0$

$$\mathbf{x}(t) = c_1\mathbf{x}_1 + c_2\mathbf{x}_2 = c_1\begin{bmatrix}1\\-1\end{bmatrix}e^{3t} + c_2\begin{bmatrix}1\\-2\end{bmatrix}e^{2t} = \begin{bmatrix} c_1e^{3t} + c_2e^{2t} \\ -c_1e^{3t} - 2c_2e^{2t} \end{bmatrix}$$

17. $W(t) = \begin{vmatrix} 3e^{2t} & e^{-5t} \\ 2e^{2t} & 3e^{-5t} \end{vmatrix} = 7e^{-3t} \neq 0$

$$\mathbf{x}(t) = c_1\mathbf{x}_1 + c_2\mathbf{x}_2 = c_1\begin{bmatrix}3e^{2t}\\2e^{2t}\end{bmatrix} + c_2\begin{bmatrix}e^{-5t}\\3e^{-5t}\end{bmatrix} = \begin{bmatrix} 3c_1e^{2t} + c_2e^{-5t} \\ 2c_1e^{2t} + 3c_2e^{-5t} \end{bmatrix}$$

18. $W(t) = \begin{vmatrix} 2e^{t} & -2e^{3t} & 2e^{5t} \\ 2e^{t} & 0 & -2e^{5t} \\ e^{t} & e^{3t} & e^{5t} \end{vmatrix} = 16e^{9t} \neq 0$

$$\mathbf{x}(t) = c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + c_3\mathbf{x}_3 = c_1\begin{bmatrix}2\\2\\1\end{bmatrix}e^{t} + c_2\begin{bmatrix}-2\\0\\1\end{bmatrix}e^{3t} + c_3\begin{bmatrix}2\\-2\\1\end{bmatrix}e^{5t} = \begin{bmatrix} 2c_1e^{t} - 2c_2e^{3t} + 2c_3e^{5t} \\ 2c_1e^{t} - 2c_3e^{5t} \\ c_1e^{t} + c_2e^{3t} + c_3e^{5t} \end{bmatrix}$$

19. $W(t) = \begin{vmatrix} e^{2t} & e^{-t} & 0 \\ e^{2t} & 0 & e^{-t} \\ e^{2t} & -e^{-t} & -e^{-t} \end{vmatrix} = 3 \neq 0$

$$\mathbf{x}(t) = c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + c_3\mathbf{x}_3 = c_1\begin{bmatrix}1\\1\\1\end{bmatrix}e^{2t} + c_2\begin{bmatrix}1\\0\\-1\end{bmatrix}e^{-t} + c_3\begin{bmatrix}0\\1\\-1\end{bmatrix}e^{-t} = \begin{bmatrix} c_1e^{2t} + c_2e^{-t} \\ c_1e^{2t} + c_3e^{-t} \\ c_1e^{2t} - c_2e^{-t} - c_3e^{-t} \end{bmatrix}$$

$$\mathbf{x}_1' = \begin{bmatrix}2\\2\\2\end{bmatrix}e^{t} = \begin{bmatrix}0&1&1\\1&0&1\\1&1&0\end{bmatrix}\begin{bmatrix}1\\1\\1\end{bmatrix}e^{t} = \mathbf{A}\mathbf{x}_1$$

$$\mathbf{x}_2' = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} e^{-t} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} e^{-t} = \mathbf{A}\mathbf{x}_2$$

$$\mathbf{x}_3' = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} e^{t} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} e^{-t} = \mathbf{A}\mathbf{x}_3$$

20. $$W(t) = \begin{vmatrix} 1 & 2e^{3t} & -e^{4t} \\ 6 & 3e^{3t} & 2e^{4t} \\ -13 & -2e^{3t} & e^{4t} \end{vmatrix} = -84e^{7t} \neq 0$$

$$\mathbf{x}(t) = c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + c_3\mathbf{x}_3 = c_1 \begin{bmatrix} 1 \\ 6 \\ -13 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ 3 \\ -2 \end{bmatrix} e^{3t} + c_3 \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} e^{4t} = \begin{bmatrix} c_1 + 2c_2e^{3t} - c_3e^{4t} \\ 6c_1 + 3c_2e^{3t} + 2c_3e^{4t} \\ -13c_1 - 2c_2e^{3t} + c_3e^{4t} \end{bmatrix}$$

21. $$W(t) = \begin{vmatrix} 3e^{-2t} & e^{t} & e^{3t} \\ -2e^{-2t} & -e^{t} & -e^{3t} \\ 2e^{-2t} & e^{t} & 0 \end{vmatrix} = e^{2t} \neq 0$$

$$\mathbf{x}(t) = c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + c_3\mathbf{x}_3 = c_1 \begin{bmatrix} 3 \\ -2 \\ 2 \end{bmatrix} e^{-2t} + c_2 \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} e^{t} + c_3 \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} e^{3t} = \begin{bmatrix} 3c_1e^{-2t} + c_2e^{t} + c_3e^{3t} \\ -2c_1e^{-2t} - c_2e^{t} - c_3e^{3t} \\ 2c_1e^{-2t} + c_2e^{t} \end{bmatrix}$$

22. $$W(t) = \begin{vmatrix} e^{-t} & 0 & 0 & e^{t} \\ 0 & 0 & e^{t} & 0 \\ 0 & e^{-t} & 0 & 3e^{t} \\ e^{-t} & 0 & -2e^{t} & 0 \end{vmatrix} = -e^{-t} \begin{vmatrix} e^{-t} & 0 & e^{t} \\ 0 & e^{t} & 0 \\ e^{-t} & -2e^{t} & 0 \end{vmatrix} = -\begin{vmatrix} 0 & e^{t} \\ e^{1t} & -2e^{t} \end{vmatrix} = 1 \neq 0$$

$$\mathbf{x}(t) = c_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} e^{-t} + c_2 \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} e^{-t} + c_3 \begin{bmatrix} 0 \\ 1 \\ 0 \\ -2 \end{bmatrix} e^{t} + c_4 \begin{bmatrix} 1 \\ 0 \\ 3 \\ 0 \end{bmatrix} e^{t} = \begin{bmatrix} c_1e^{-t} + c_4e^{t} \\ c_3e^{t} \\ c_2e^{-t} + 3c_4e^{t} \\ c_1e^{-t} - 2c_3e^{t} \end{bmatrix}$$

In Problems 23–26 (and similarly in Problems 27–32) we give first the scalar components $x_1(t)$ and $x_2(t)$ of a general solution, then the equations in the coefficients c_1 and c_2 that are obtained when the given initial conditions are imposed, and finally the resulting particular solution of the given system.

23. $x_1(t) = c_1e^{3t} + 2c_2e^{-2t}, \quad x_2(t) = 3c_1e^{3t} + c_2e^{-2t}$

$c_1 + 2c_2 = 0, \qquad 3c_1 + c_2 = 5$

$x_1(t) = 2e^{3t} - 2e^{-2t}, \quad x_2(t) = 6e^{3t} - e^{-2t}$

24. $x_1(t) = c_1e^{2t} + c_2e^{-2t}, \quad x_2(t) = c_1e^{2t} + 5c_2e^{-2t}$

$c_1 + c_2 = 5, \qquad c_1 + 5c_2 = -3$

$x_1(t) = 7e^{2t} - 2e^{-2t}, \quad x_2(t) = 7e^{2t} - 10e^{-2t}$

25. $x_1(t) = c_1e^{3t} + c_2e^{2t}, \quad x_2(t) = -c_1e^{3t} - 2c_2e^{2t}$

$c_1 + c_2 = 11, \qquad -c_1 - 2c_2 = -7$

$x_1(t) = 15e^{3t} - 4e^{2t}, \quad x_2(t) = -15e^{3t} + 8e^{2t}$

26. $x_1(t) = 3c_1e^{2t} + c_2e^{-5t}, \quad x_2(t) = 2c_1e^{2t} + 3c_2e^{-5t}$

$3c_1 + c_2 = 8, \qquad 2c_1 + 3c_2 = 0$

$x_1(t) = \frac{8}{7}\left(9e^{2t} - 2e^{-5t}\right), \quad x_2(t) = \frac{48}{7}\left(e^{2t} - e^{-5t}\right)$

27. $x_1(t) = 2c_1e^{t} - 2c_2e^{3t} + 2c_3e^{5t}, \quad x_2(t) = 2c_1e^{t} - 2c_3e^{5t}, \quad x_3(t) = c_1e^{t} + c_2e^{3t} + c_3e^{5t}$

$2c_1 - 2c_2 + 2c_3 = 0, \qquad 2c_1 - 2c_3 = 0, \qquad c_1 + c_2 + c_3 = 4$

$x_1(t) = 2e^{t} - 4e^{3t} + 2e^{5t}, \quad x_2(t) = 2e^{t} - 2e^{5t}, \quad x_3(t) = e^{t} + 2e^{3t} + e^{5t}$

28. $x_1(t) = c_1e^{2t} + c_2e^{-t}, \quad x_2(t) = c_1e^{2t} + c_3e^{-t}, \quad x_3(t) = c_1e^{2t} - c_2e^{-t} - c_3e^{-t}$

$c_1 + c_2 = 10, \qquad c_1 + c_3 = 12, \qquad c_1 - c_2 - c_3 = -1$

$x_1(t) = 7e^{2t} + 3e^{-t}, \quad x_2(t) = 7e^{2t} + 5e^{-t}, \quad x_3(t) = 7e^{2t} - 8e^{-t}$

29. $x_1(t) = 3c_1e^{-2t} + c_2e^{t} + c_3e^{3t}, \quad x_2(t) = -2c_1e^{-2t} - c_2e^{t} - c_3e^{3t}, \quad x_3(t) = 2c_1e^{-2t} + c_2e^{t}$

$3c_1 + c_2 + c_3 = 1, \qquad -2c_1 - c_2 - c_3 = 2, \qquad 2c_1 + c_2 = 3$

$x_1(t) = 9e^{-2t} - 3e^{t} - 5e^{3t}, \quad x_2(t) = -6e^{-2t} + 3e^{t} + 5e^{3t}, \quad x_3(t) = 6e^{-2t} - 3e^{t}$

30. $x_1(t) = 3c_1e^{-2t} + c_2e^t + c_3e^{3t}, \quad x_2(t) = -2c_1e^{-2t} - c_2e^t - c_3e^{3t}, \quad x_3(t) = 2c_1e^{-2t} + c_2e^t$

$3c_1 + c_2 + c_3 = 5, \quad -2c_1 - c_2 - c_3 = -7, \quad 2c_1 + c_2 = 11$

$x_1(t) = -6e^{-2t} + 15e^t - 4e^{3t}, \quad x_2(t) = 4e^{-2t} - 15e^t + 4e^{3t}, \quad x_3(t) = -4e^{-2t} + 15e^t$

31. $x_1(t) = c_1e^{-t} + c_4e^t, \quad x_2(t) = c_3e^t, \quad x_3(t) = c_2e^{-t} + 3c_4e^t, \quad x_4(t) = c_1e^{-t} - 2c_3e^t$

$c_1 + c_4 = 1, \quad c_3 = 1, \quad c_2 + 3c_4 = 1, \quad c_1 - 2c_3 = 1$

$x_1(t) = 3e^{-t} - 2e^t, \quad x_2(t) = e^t, \quad x_3(t) = 7e^{-t} - 6e^t, \quad x_4(t) = 3e^{-t} - 2e^t$

32. $x_1(t) = c_1e^{-t} + c_4e^t, \quad x_2(t) = c_3e^t, \quad x_3(t) = c_2e^{-t} + 3c_4e^t, \quad x_4(t) = c_1e^{-t} - 2c_3e^t$

$c_1 + c_4 = 1, \quad c_3 = 3, \quad c_2 + 3c_4 = 4, \quad c_1 - 2c_3 = 7$

$x_1(t) = 13e^{-t} - 12e^t, \quad x_2(t) = 3e^t, \quad x_3(t) = 40e^{-t} - 36e^t, \quad x_4(t) = 13e^{-t} - 6e^t$

33. **(a)** $\mathbf{x}_2 = t\mathbf{x}_1$, so neither is a constant multiple of the other.

(b) $W(\mathbf{x}_1, \mathbf{x}_2) = 0$, whereas Theorem 2 would imply that $W \neq 0$ if $\mathbf{x}_1$ and $\mathbf{x}_2$ were independent solutions of a system of the indicated form.

34. If $x_{12}(t) = c\,x_{11}(t)$ and $x_{22}(t) = c\,x_{21}(t)$ then

$$W(t) = x_{11}(t)x_{22}(t) - x_{12}(t)x_{21}(t) = c\,x_{11}(t)x_{21}(t) - c\,x_{11}(t)x_{21}(t) = 0.$$

35. Suppose $W(a) = x_{11}(a)x_{22}(a) - x_{12}(a)x_{21}(a) = 0$. Then the coefficient determinant of the homogeneous linear system $c_1x_{11}(a) + c_2x_{12}(a) = 0, \; c_1x_{21}(a) + c_2x_{22}(a) = 0$ vanishes. The system therefore has a non-trivial solution $\{c_1, c_2\}$ such that $c_1\mathbf{x}_1(a) + c_2\mathbf{x}_2(a) = \mathbf{0}$. Then $\mathbf{x}(t) = c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t)$ is a solution of $\mathbf{x}' = \mathbf{P}\mathbf{x}$ such that $\mathbf{x}(a) = \mathbf{0}$. It therefore follows (by uniqueness of solutions) that $\mathbf{x}(t) \equiv \mathbf{0}$, that is, $c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t) \equiv 0$ with c_1 and c_2 not both zero. Thus the solution vectors $\mathbf{x}_1$ and $\mathbf{x}_2$ are linearly dependent.

36. The argument is precisely the same, except with n solution vectors each having n component functions (rather than 2 solution vectors each having 2 component functions).

37. Suppose that $c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t) + \cdots + c_n\mathbf{x}_n(t) \equiv \mathbf{0}$. Then the ith scalar component of this vector equation is $c_1x_{i1}(t) + c_2x_{i2}(t) + \cdots + c_nx_{in}(t) \equiv 0$. Hence the fact that the scalar functions $x_{i1}(t), \; x_{i2}(t), \; \cdots, \; x_{in}(t)$ are linear linearly independent implies that

$c_1 = c_2 = \cdots\ c_n = 0$. Consequently the vector functions $\mathbf{x}_1(t), \mathbf{x}_2(t), \cdots, \mathbf{x}_n(t)$ are linearly independent.

SECTION 7.3

THE EIGENVALUE METHOD FOR LINEAR SYSTEMS

In each of Problems 1–16 we give the characteristic equation, the eigenvalues λ_1 and λ_2 of the coefficient matrix of the given system, the corresponding equations determining the associated eigenvectors $\mathbf{v}_1 = [a_1 \quad b_1]^{\mathrm{T}}$ and $\mathbf{v}_2 = [a_2 \quad b_2]^{\mathrm{T}}$, these eigenvectors, and the resulting scalar components $x_1(t)$ and $x_2(t)$ of a general solution $\mathbf{x}(t) = c_1\mathbf{v}_1 e^{\lambda_1 t} + c_2\mathbf{v}_2 e^{\lambda_2 t}$ of the system.

1. Characteristic equation $\quad \lambda^2 - 2\lambda - 3 = 0$

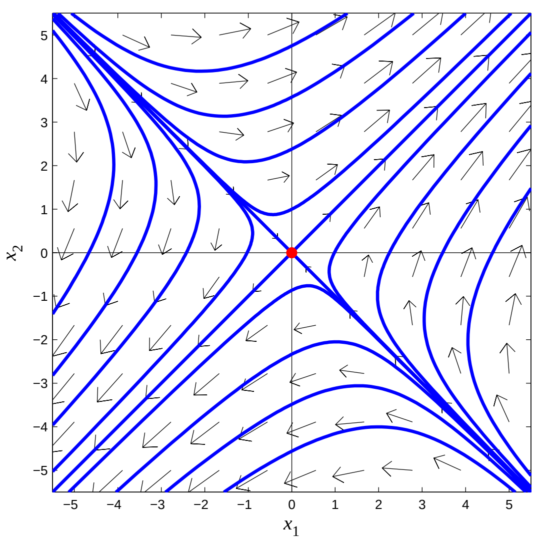
Eigenvalues $\quad \lambda_1 = -1$ and $\lambda_2 = 3$

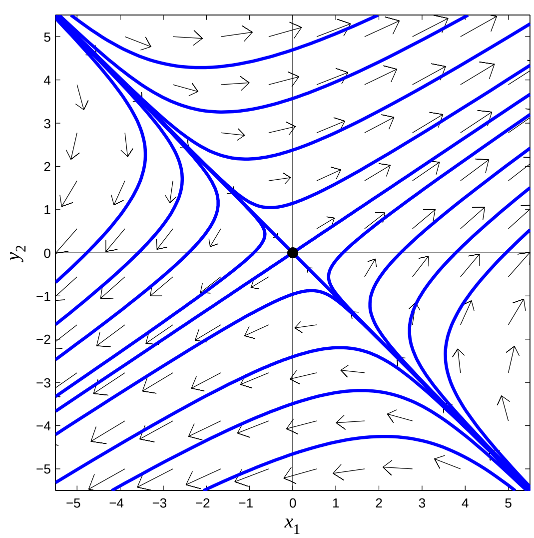
Eigenvector equations $\begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} a_1 \\ b_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} -2 & 2 \\ 2 & -2 \end{bmatrix}\begin{bmatrix} a_2 \\ b_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

Eigenvectors $\mathbf{v}_1 = [1 \quad -1]^{\mathrm{T}}$ and $\mathbf{v}_2 = [1 \quad 1]^{\mathrm{T}}$

$x_1(t) = c_1 e^{-t} + c_2 e^{3t}, \quad x_2(t) = -c_1 e^{-t} + c_2 e^{3t}$

The left-hand figure below shows a direction field and some typical solution curves for the system in Problem 1.





2. Characteristic equation $\lambda^2 - 3\lambda - 4 = 0$

Eigenvalues $\lambda_1 = -1$ and $\lambda_2 = 4$

Eigenvector equations $\begin{bmatrix} 3 & 3 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} a_1 \\ b_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} -2 & 3 \\ 2 & -3 \end{bmatrix}\begin{bmatrix} a_2 \\ b_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

Eigenvectors $\mathbf{v}_1 = [1 \quad -1]^T$ and $\mathbf{v}_2 = [3 \quad 2]^T$

$x_1(t) = c_1e^{-t} + 3c_2e^{4t}, \quad x_2(t) = -c_1e^{-t} + 2c_2e^{4t}$

The right-hand figure at the bottom of the preceding page shows a direction field and some typical solution curves.

3. Characteristic equation $\lambda^2 - 5\lambda - 6 = 0$

Eigenvalues $\lambda_1 = -1$ and $\lambda_2 = 6$

Eigenvector equations $\begin{bmatrix} 4 & 4 \\ 3 & 3 \end{bmatrix}\begin{bmatrix} a_1 \\ b_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} -3 & 4 \\ 3 & -4 \end{bmatrix}\begin{bmatrix} a_2 \\ b_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

Eigenvectors $\mathbf{v}_1 = [1 \quad -1]^T$ and $\mathbf{v}_2 = [4 \quad 3]^T$

$x_1(t) = c_1e^{-t} + 4c_2e^{6t}, \quad x_2(t) = -c_1e^{-t} + 3c_2e^{6t}$

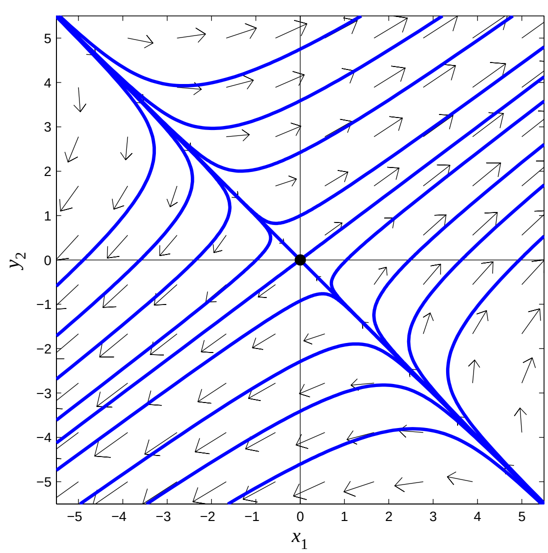
The equations

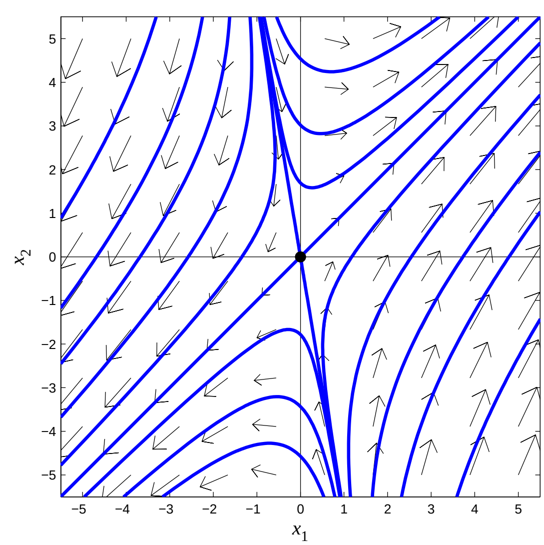
$$x_1(0) = c_1 + 4c_2 = 1$$
$$x_2(0) = -c_1 + 3c_2 = 1$$

yield $c_1 = -1/7$ and $c_2 = 2/7$, so the desired particular solution is given by

$$x_1(t) = \tfrac{1}{7}(-e^{-t} + 8e^{6t}), \quad x_2(t) = \tfrac{1}{7}(e^{-t} + 6e^{6t}).$$

The left-hand figure below shows a direction field and some typical solution curves.





4. Characteristic equation $\lambda^2 - 3\lambda - 10 = 0$

Eigenvalues $\lambda_1 = -2$ and $\lambda_2 = 5$

Eigenvector equations $\begin{bmatrix} 6 & 1 \\ 6 & 1 \end{bmatrix}\begin{bmatrix} a_1 \\ b_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} -1 & 1 \\ 6 & -6 \end{bmatrix}\begin{bmatrix} a_2 \\ b_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

Eigenvectors $\mathbf{v}_1 = [1 \ \ -6]^T$ and $\mathbf{v}_2 = [1 \quad 1]^T$

$x_1(t) = c_1 e^{-2t} + c_2 e^{5t}, \quad x_2(t) = -6c_1 e^{-2t} + c_2 e^{5t}$

The right-hand figure at the bottom of the preceding page shows a direction field and some typical solution curves.

5. Characteristic equation $\lambda^2 - 4\lambda - 5 = 0$

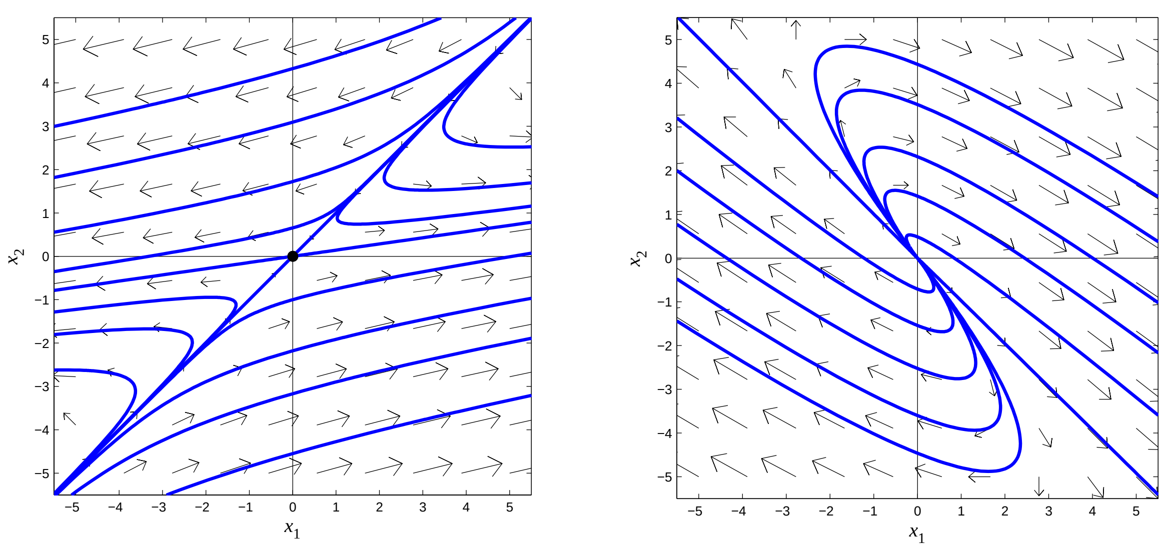
Eigenvalues $\lambda_1 = -1$ and $\lambda_2 = 5$

Eigenvector equations $\begin{bmatrix} 7 & -7 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} a_1 \\ b_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 1 & -7 \\ 1 & -7 \end{bmatrix}\begin{bmatrix} a_2 \\ b_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

Eigenvectors $\mathbf{v}_1 = [1 \quad 1]^T$ and $\mathbf{v}_2 = [7 \quad 1]^T$

$x_1(t) = c_1 e^{-t} + 7c_2 e^{5t}, \quad x_2(t) = c_1 e^{-t} + c_2 e^{5t}$

The left-hand figure below shows a direction field and some typical solution curves.



6. Characteristic equation $\lambda^2 - 7\lambda + 12 = 0$

Eigenvalues $\lambda_1 = 3$ and $\lambda_2 = 4$

Eigenvector equations $\begin{bmatrix} 6 & 5 \\ -6 & -5 \end{bmatrix}\begin{bmatrix} a_1 \\ b_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 5 & 5 \\ -6 & -6 \end{bmatrix}\begin{bmatrix} a_2 \\ b_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

Eigenvectors $\mathbf{v}_1 = [5 \quad -6]^T$ and $\mathbf{v}_2 = [1 \quad -1]^T$

$x_1(t) = 5c_1e^{3t} + c_2e^{4t}, \quad x_2(t) = -6c_1e^{3t} - c_2e^{4t}$

The initial conditions yield $c_1 = -1$ and $c_2 = 6$, so

$$x_1(t) = -5e^{3t} + 6e^{4t}, \quad x_2(t) = 6e^{3t} - 6e^{4t}.$$

The right-hand figure at the bottom of the preceding page shows a direction field and some typical solution curves.

7. Characteristic equation $\lambda^2 + 8\lambda - 9 = 0$

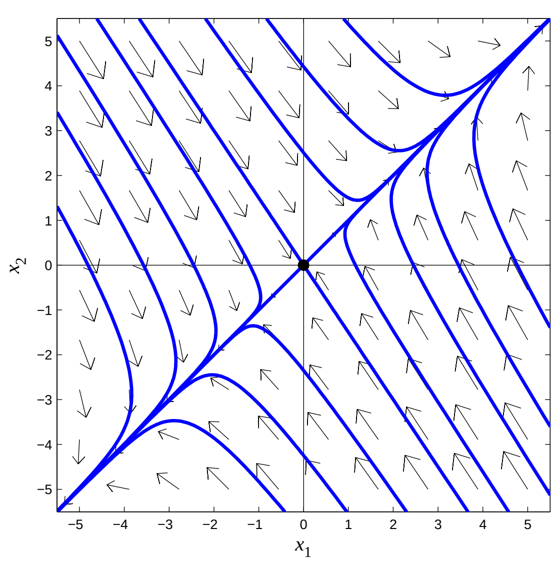
Eigenvalues $\lambda_1 = 1$ and $\lambda_2 = -9$

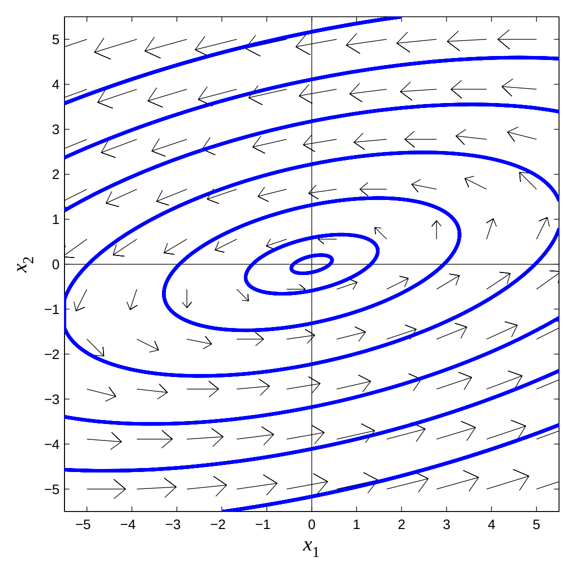
Eigenvector equations $\begin{bmatrix} -4 & 4 \\ 6 & -6 \end{bmatrix}\begin{bmatrix} a_1 \\ b_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 6 & 4 \\ 6 & 4 \end{bmatrix}\begin{bmatrix} a_2 \\ b_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

Eigenvectors $\mathbf{v}_1 = [1 \quad 1]^T$ and $\mathbf{v}_2 = [2 \quad -3]^T$

$x_1(t) = c_1e^{t} + 2c_2e^{-9t}, \quad x_2(t) = c_1e^{t} - 3c_2e^{-9t}$

The left-hand figure below shows a direction field and some typical solution curves.





8. Characteristic equation $\lambda^2 + 4 = 0$

Eigenvalue $\lambda = 2i$

Eigenvector equation $\begin{bmatrix} 1-2i & -5 \\ 1 & -1-2i \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

Eigenvector $\mathbf{v} = [5 \quad 1-2i]^T$

$$\mathbf{x}(t) = \mathbf{v}e^{2it} = \begin{bmatrix} 5\cos 2t + 5i\sin 2t \\ (\cos 2t + 2\sin 2t) + i(\sin 2t - 2\cos 2t) \end{bmatrix}$$

$x_1(t) = 5c_1 \cos 2t + 5c_2 \sin 2t$

$x_2(t) = c_1(\cos 2t + 2\sin 2t) + c_2(\sin 2t - 2\cos 2t)$

$\quad = (c_1 - 2c_2)\cos 2t + (2c_1 + c_2)\sin 2t$

The right-hand figure at the bottom of the preceding page shows a direction field and some typical solution curves.

9. Characteristic equation $\lambda^2 + 16 = 0$

Eigenvalue $\lambda = 4i$

Eigenvector equation $\begin{bmatrix} 2-4i & -5 \\ 4 & -2-4i \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

Eigenvector $\mathbf{v} = [5 \quad 2-4i]^T$

The real and imaginary parts of

$$\mathbf{x}(t) = \mathbf{v}e^{4it} = \begin{bmatrix} 5\cos 4t + 5i\sin 4t \\ (2\cos 4t + 4\sin 4t) + i(2\sin 4t - 4\cos 4t) \end{bmatrix}$$

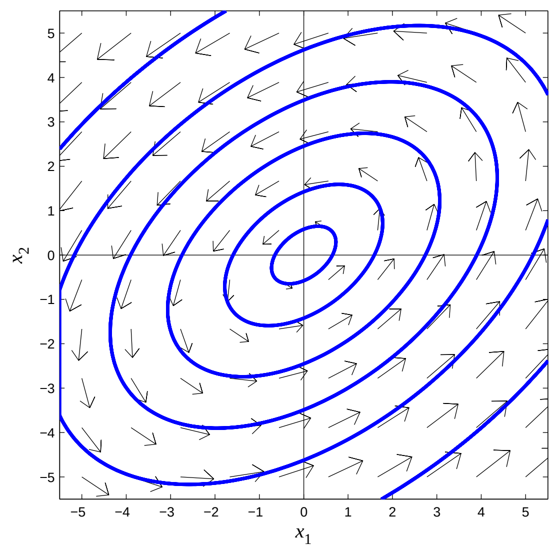
yield the general solution

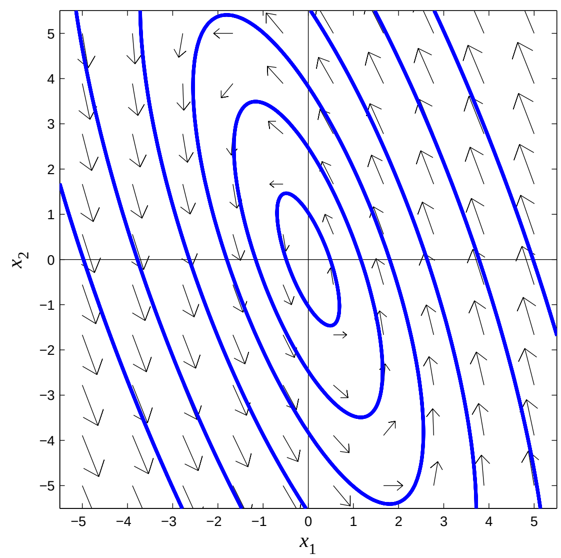
$$x_1(t) = 5c_1 \cos 4t + 5c_2 \sin 4t$$
$$x_2(t) = c_1(2\cos 4t + 4\sin 4t) + c_2(2\sin 4t - 4\cos 4t).$$

The initial conditions $x_1(0) = 2$ and $x_2(0) = 3$ give $c_1 = 2/5$ and $c_2 = -11/20$, so the desired particular solution is

$$x_1(t) = 2\cos 4t - \tfrac{11}{4}\sin 4t$$
$$x_2(t) = 3\cos 4t + \tfrac{1}{2}\sin 4t.$$

The left-hand figure at the top of the next page shows a direction field and some typical solution curves.





10. Characteristic equation $\lambda^2 + 9 = 0$

Eigenvalue $\lambda = 3i$

Eigenvector equation $\begin{bmatrix} -3-3i & -2 \\ 9 & 3-3i \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

Eigenvector $\mathbf{v} = [-2 \quad 3+3i]^T$

$$\mathbf{x}(t) = \mathbf{v}e^{3it} = \begin{bmatrix} -2\cos 3t - 2i\sin 3t \\ (3\cos 3t - 3\sin 3t) + i(3\sin 3t + 3\cos 3t) \end{bmatrix}$$

$x_1(t) = -2c_1\cos 3t - 2c_2\sin 3t$

$x_2(t) = c_1(3\cos 3t - 3\sin 3t) + c_2(3\cos 3t + 3\sin 3t)$

$= (3c_1 + 3c_2)\cos 3t + (3c_2 - 3c_1)\sin 3t$

The right-hand figure above shows a direction field and some typical solution curves for this system.

11. Characteristic equation $\lambda^2 - 2\lambda + 5 = 0$

Eigenvalue $\lambda = 1 - 2i$

Eigenvector equation $\begin{bmatrix} 2i & -2 \\ 2 & 2i \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

Eigenvector $\mathbf{v} = [1 \quad i]^T$

The real and imaginary parts of

$$\mathbf{x}(t) = [1 \quad i]^T e^t(\cos 2t - i\sin 2t)$$
$$= e^t[\cos 2t \quad \sin 2t]^T + i e^t[-\sin 2t \quad \cos 2t]^T$$

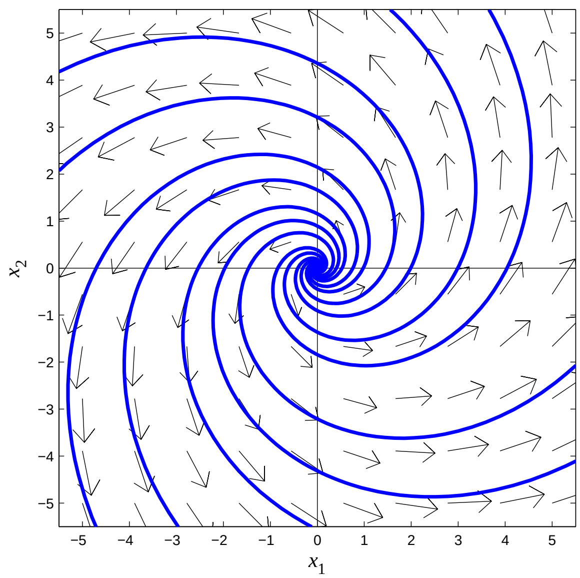
yield the general solution

$$x_1(t) = e^t(c_1\cos 2t - c_2\sin 2t)$$
$$x_2(t) = e^t(c_1\sin 2t + c_2\cos 2t).$$

The particular solution with $x_1(0) = 0$ and $x_2(0) = 4$ is obtained with $c_1 = 0$ and $c_2 = 4$, so

$$x_1(t) = -4e^t\sin 2t, \qquad x_2(t) = 4e^t\cos 2t.$$

The figure below shows a direction field and some typical solution curves.



12. Characteristic equation $\lambda^2 - 4\lambda + 8 = 0$

Eigenvalue $\lambda = 2 + 2i$

Eigenvector equation $\begin{bmatrix} -1-2i & -5 \\ 1 & 1-2i \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

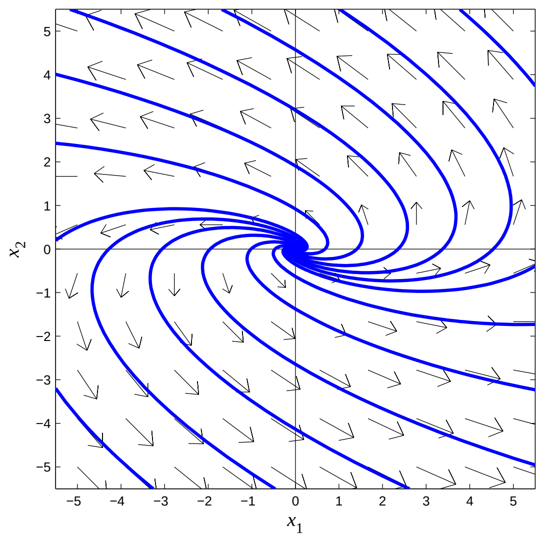
Eigenvector $\mathbf{v} = [-5 \quad 1+2i]^T$

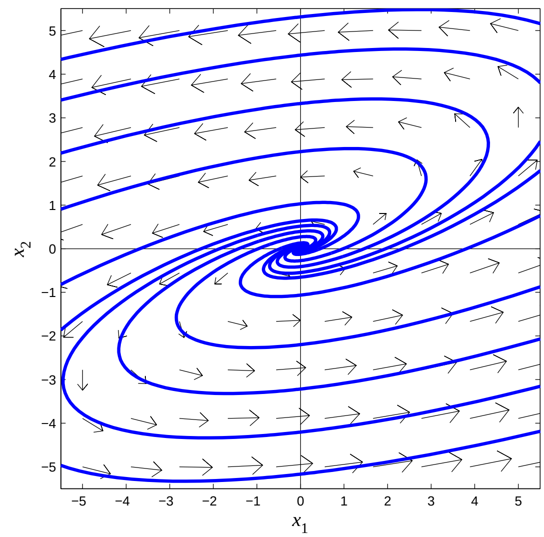
$$\mathbf{x}(t) = \mathbf{v}\, e^{(2+2i)\,t} = e^{2t}\begin{bmatrix} -5\cos 2t - 5i\sin 2t \\ (\cos 2t - 2\sin 2t) + i(\sin 2t + 2\cos 2t) \end{bmatrix}$$

$$x_1(t) = e^{2t}(-5c_1\cos 2t - 5c_2\sin 2t)$$

$$x_2(t) = e^{2t}\,[c_1(\cos 2t - 2\sin 2t) + c_2(2\cos 2t + \sin 2t)]$$
$$= e^{2t}\,[(c_1 + 2c_2)\cos 2t + (-2c_1 + c_2)\sin 2t]$$

The left-hand figure below shows a direction field and some typical solution curves.





13. Characteristic equation $\lambda^2 - 4\lambda + 13 = 0$

Eigenvalue $\lambda = 2 - 3i$

Eigenvector equation $\begin{bmatrix} 3+3i & -9 \\ 2 & -3+3i \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

Eigenvector $\mathbf{v} = [3 \quad 1+i]^T$

$$\mathbf{x}(t) = \mathbf{v}\,e^{(2-3i)\,t} = e^{2t}\begin{bmatrix} 3\cos 3t - 3i\sin 3t \\ (\cos 3t + \sin 3t) + i(\cos 3t - \sin 3t) \end{bmatrix}$$

$$x_1(t) = 3e^{2t}(c_1\cos 3t - c_2\sin 3t)$$

$$x_2(t) = e^{2t}\,[(c_1 + c_2)\cos 3t + (c_1 - c_2)\sin 3t)].$$

The right-hand figure above shows a direction field and some typical solution curves.

14. Characteristic equation $\lambda^2 - 2\lambda + 5 = 0$

Eigenvalue $\lambda = 3 + 4i$

Eigenvector equation $\begin{bmatrix} -4i & -4 \\ 4 & -4i \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

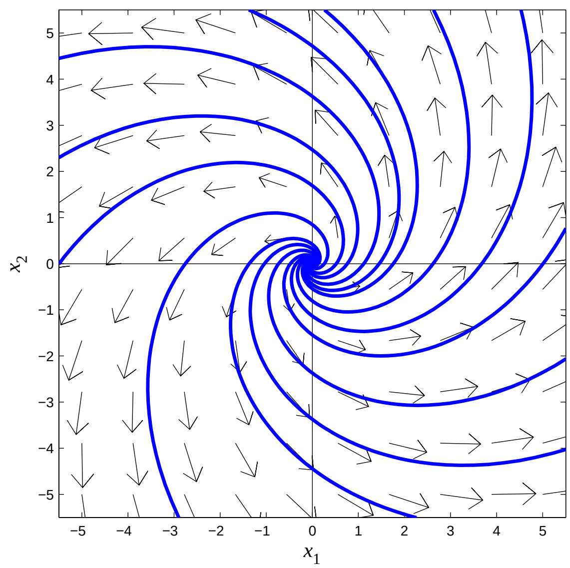
Eigenvector $\mathbf{v} = [1 \quad -i]^T$

$$\mathbf{x}(t) = \mathbf{v}\,e^{(3+4i)\,t} = e^{3t}\begin{bmatrix} \cos 4t + i\sin 4t \\ \sin 4t - i\cos 4t \end{bmatrix}$$

$$x_1(t) = e^{3t}(c_1\cos 4t + c_2\sin 4t)$$

$$x_2(t) = e^{3t}(c_1\sin 4t - c_2\cos 4t)$$

The figure below shows a direction field and some typical solution curves.



15. Characteristic equation $\lambda^2 - 10\lambda + 41 = 0$

Eigenvalue $\lambda = 5 - 4i$

Eigenvector equation $\begin{bmatrix} 2+4i & -5 \\ 4 & -2+4i \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

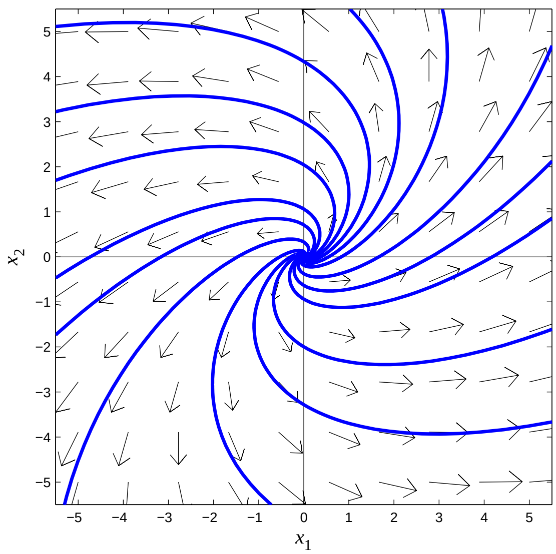
Eigenvector $\mathbf{v} = [5 \quad 2+4i]^T$

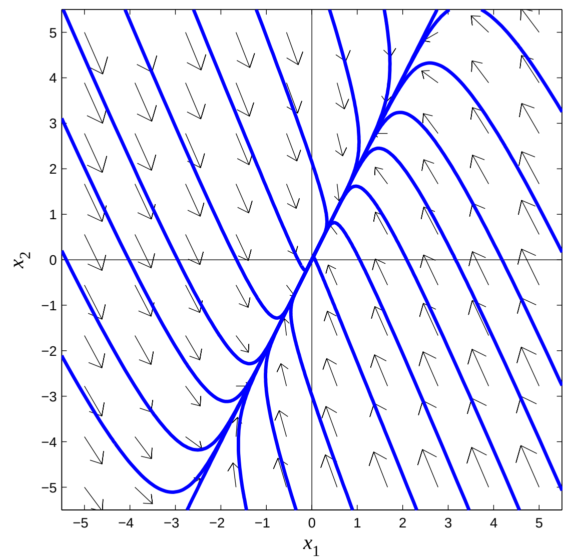
$$\mathbf{x}(t) = \mathbf{v}\,e^{(5-4i)\,t} = e^{5t}\begin{bmatrix} 5\cos 4t - 5i\sin 4t \\ (2\cos 4t + 4\sin 4t) + i\,(4\cos 4t - 2\sin 4t) \end{bmatrix}$$

$$x_1(t) = 5e^{5t}(c_1\cos 4t - c_2\sin 4t)$$

$$x_2(t) = e^{5t}\,[(2c_1 + 4c_2)\cos 4t + (4c_1 - 2c_2)\sin 4t)]$$

The left-hand figure at the top of the next page shows a direction field and some typical solution curves.





16. Characteristic equation $\lambda^2 + 110\lambda + 1000 = 0$

Eigenvalues $\lambda_1 = -10$ and $\lambda_2 = -100$

Eigenvector equations $\begin{bmatrix} -40 & 20 \\ 100 & -50 \end{bmatrix}\begin{bmatrix} a_1 \\ b_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 50 & 20 \\ 100 & 40 \end{bmatrix}\begin{bmatrix} a_2 \\ b_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

Eigenvectors $\mathbf{v}_1 = [1 \quad 2]^T$ and $v_2 = [2 \quad -5]^T$

$$x_1(t) = c_1 e^{-10t} + 2c_2 e^{-100t}$$
$$x_2(t) = 2c_1 e^{-10t} - 5c_2 e^{-100t}$$

The right-hand figure above shows a direction field and some typical solution curves.

17. Characteristic equation $-\lambda^3 + 15\lambda^2 - 54\lambda = 0$

Eigenvalues $\lambda_1 = 9, \quad \lambda_2 = 6, \quad \lambda_3 = 0$

Eigenvector equations

$$\begin{bmatrix} -5 & 1 & 4 \\ 1 & -2 & 1 \\ 4 & 1 & -5 \end{bmatrix}\begin{bmatrix} a_1 \\ b_1 \\ c_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} -2 & 1 & 4 \\ 1 & 1 & 1 \\ 4 & 1 & -2 \end{bmatrix}\begin{bmatrix} a_2 \\ b_2 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 4 & 1 & 4 \\ 1 & 7 & 1 \\ 4 & 1 & 4 \end{bmatrix}\begin{bmatrix} a_3 \\ b_3 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Eigenvectors $\mathbf{v}_1 = [1 \quad 1 \quad 1]^T, \qquad \mathbf{v}_2 = [1 \quad -2 \quad 1]^T, \qquad \mathbf{v}_3 = [1 \quad 0 \quad -1]^T$

$$x_1(t) = c_1 e^{9t} + c_2 e^{6t} + c_3$$
$$x_2(t) = c_1 e^{9t} - 2c_2 e^{6t}$$
$$x_3(t) = c_1 e^{9t} + c_2 e^{6t} - c_3$$

18. Characteristic equation $-\lambda^3 + 15\lambda^2 - 54\lambda = 0$

Eigenvalues $\lambda_1 = 9, \quad \lambda_2 = 6, \quad \lambda_3 = 0$

Eigenvector equations

$$\begin{bmatrix} -8 & 2 & 2 \\ 2 & -2 & 1 \\ 2 & 1 & -2 \end{bmatrix}\begin{bmatrix} a_1 \\ b_1 \\ c_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} -5 & 2 & 2 \\ 2 & 1 & 1 \\ 2 & 1 & 1 \end{bmatrix}\begin{bmatrix} a_2 \\ b_2 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & 2 & 2 \\ 2 & 7 & 1 \\ 2 & 1 & 7 \end{bmatrix}\begin{bmatrix} a_3 \\ b_3 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Eigenvectors $\mathbf{v}_1 = [1 \ \ 2 \ \ 2]^T, \quad \mathbf{v}_2 = [0 \ \ 1 \ \ -1]^T, \quad \mathbf{v}_3 = [4 \ \ -1 \ \ -1]^T$

$x_1(t) = c_1 e^{9t} + 4c_3$

$x_2(t) = 2c_1 e^{9t} + c_2 e^{6t} - c_3$

$x_3(t) = 2c_1 e^{9t} - c_2 e^{6t} - c_3$

19. Characteristic equation $-\lambda^3 + 12\lambda^2 - 45\lambda + 54 = 0$

Eigenvalues $\lambda_1 = 6, \quad \lambda_2 = 3, \quad \lambda_3 = 3$

Eigenvector equations

$$\begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix}\begin{bmatrix} a_1 \\ b_1 \\ c_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} a_2 \\ b_2 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} a_3 \\ b_3 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Eigenvectors $\mathbf{v}_1 = [1 \ \ 1 \ \ 1]^T, \quad \mathbf{v}_2 = [1 \ \ -2 \ \ 1]^T, \quad \mathbf{v}_3 = [1 \ \ 0 \ \ -1]^T$

$x_1(t) = c_1 e^{6t} + c_2 e^{3t} + c_3 e^{3t}$

$x_2(t) = c_1 e^{6t} - 2c_2 e^{3t}$

$x_3(t) = c_1 e^{6t} + c_2 e^{3t} - c_3 e^{3t}$

20. Characteristic equation $-\lambda^3 + 17\lambda^2 - 84\lambda + 108 = 0$

Eigenvalues $\lambda_1 = 9, \quad \lambda_2 = 6, \quad \lambda_3 = 2$

Eigenvector equations

$$\begin{bmatrix} -4 & 1 & 3 \\ 1 & -2 & 1 \\ 3 & 1 & -4 \end{bmatrix}\begin{bmatrix} a_1 \\ b_1 \\ c_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} -1 & 1 & 3 \\ 1 & 1 & 1 \\ 3 & 1 & -1 \end{bmatrix}\begin{bmatrix} a_2 \\ b_2 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 3 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 3 \end{bmatrix}\begin{bmatrix} a_3 \\ b_3 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Eigenvectors $\mathbf{v}_1 = [1 \ \ 1 \ \ 1]^T, \quad \mathbf{v}_2 = [1 \ \ -2 \ \ 1]^T, \quad \mathbf{v}_3 = [1 \ \ 0 \ \ -1]^T$

$x_1(t) = c_1e^{9t} + c_2e^{6t} + c_3e^{2t}$

$x_2(t) = c_1e^{9t} - 2c_2e^{6t}$

$x_3(t) = c_1e^{9t} + c_2e^{6t} - c_3e^{2t}$

21. Characteristic equation $-\lambda^3 + \lambda = 0$

Eigenvalues $\lambda_1 = 0,\ \lambda_2 = 1,\ \lambda_3 = -1$

Eigenvector equations

$$\begin{bmatrix} 5 & 0 & -6 \\ 2 & -1 & -2 \\ 4 & -2 & -4 \end{bmatrix}\begin{bmatrix} a_1 \\ b_1 \\ c_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 4 & 0 & -6 \\ 2 & -2 & -2 \\ 4 & -2 & -5 \end{bmatrix}\begin{bmatrix} a_2 \\ b_2 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 6 & 0 & -6 \\ 2 & 0 & -2 \\ 4 & -2 & -3 \end{bmatrix}\begin{bmatrix} a_3 \\ b_3 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Eigenvectors $\mathbf{v}_1 = [6 \quad 2 \quad 5]^T$, $\quad \mathbf{v}_2 = [3 \quad 1 \quad 2]^T$, $\quad \mathbf{v}_3 = [2 \quad 1 \quad 2]^T$

$x_1(t) = 6c_1 + 3c_2e^{t} + 2c_3e^{-t}$

$x_2(t) = 2c_1 + c_2e^{t} + c_3e^{-t}$

$x_3(t) = 5c_1 + 2c_2e^{t} + 2c_3e^{-t}$

22. Characteristic equation $-\lambda^3 + 2\lambda^2 + 5\lambda - 6 = 0$

Distinct eigenvalues $\lambda_1 = -2,\ \lambda_2 = 1,\ \lambda_3 = 3$

Eigenvector equations

$$\begin{bmatrix} 5 & 2 & 2 \\ -5 & -2 & -2 \\ 5 & 5 & 5 \end{bmatrix}\begin{bmatrix} a_1 \\ b_1 \\ c_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 2 & 2 & 2 \\ -5 & -5 & -2 \\ 5 & 5 & 2 \end{bmatrix}\begin{bmatrix} a_2 \\ b_2 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 2 & 2 \\ -5 & -7 & -2 \\ 5 & 5 & 0 \end{bmatrix}\begin{bmatrix} a_3 \\ b_3 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Eigenvectors $\mathbf{v}_1 = [0 \quad 1 \quad -1]^T$, $\quad \mathbf{v}_2 = [1 \quad -1 \quad 0]^T$, $\quad \mathbf{v}_3 = [1 \quad -1 \quad 1]^T$

$x_1(t) = c_2e^{t} + c_3e^{3t}$

$x_2(t) = c_1e^{-2t} - c_2e^{t} - c_3e^{3t}$

$x_3(t) = -c_1e^{-2t} + c_3e^{3t}$

23. Characteristic equation $-\lambda^3 + 3\lambda^2 + 4\lambda - 12 = 0$

Eigenvalues $\lambda_1 = 2,\ \lambda_2 = -2,\ \lambda_3 = 3$

Eigenvector equations

$$\begin{bmatrix} 1 & 1 & 1 \\ -5 & -5 & -1 \\ 5 & 5 & 1 \end{bmatrix}\begin{bmatrix} a_1 \\ b_1 \\ c_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 5 & 1 & 1 \\ -5 & -1 & -1 \\ 5 & 5 & 5 \end{bmatrix}\begin{bmatrix} a_2 \\ b_2 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 & 1 \\ -5 & -6 & -1 \\ 5 & 5 & 0 \end{bmatrix}\begin{bmatrix} a_3 \\ b_3 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Eigenvectors $\mathbf{v}_1 = [1 \;\; -1 \;\; 0]^T$, $\mathbf{v}_2 = [0 \;\; 1 \;\; -1]^T$, $\mathbf{v}_3 = [1 \;\; -1 \;\; 1]^T$

$$\begin{aligned} x_1(t) &= c_1 e^{2t} + c_3 e^{3t} \\ x_2(t) &= -c_1 e^{2t} + c_2 e^{-2t} - c_3 e^{3t} \\ x_3(t) &= -c_2 e^{-2t} + c_3 e^{3t} \end{aligned}$$

24. Characteristic equation $-\lambda^3 + \lambda^2 - 4\lambda + 4 = 0$

Eigenvalues $\lambda = 1$ and $\lambda = \pm 2i$

With $\lambda = 1$ the eigenvector equation

$$\begin{bmatrix} 1 & 1 & -1 \\ -4 & -4 & -1 \\ 4 & 4 & 1 \end{bmatrix}\begin{bmatrix} a_1 \\ b_1 \\ c_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{gives eigenvector } \mathbf{v}_1 = [1 \;\; -1 \;\; 0]^T.$$

To find an eigenvector $\mathbf{v} = [a \;\; b \;\; c]^T$ associated with $\lambda = 2i$ we must find a nontrivial solution of the equations

$$\begin{aligned} (2-2i)a + b - c &= 0 \\ -4a + (-3-2i)b - c &= 0 \\ 4a + 4b + (2-2i)c &= 0. \end{aligned}$$

Subtraction of the first two equations yields

$$(6-2i)a + (4+2i)b = 0,$$

so we take $a = 2+i$ and $b = -3+i$. Then the first equation gives $c = 3-i$. Thus $\mathbf{v} = [2+i \;\; -3+i \;\; 3-i]^T$. Finally

$$\begin{aligned} (2+i)e^{2it} &= (2\cos 2t - \sin 2t) + i(\cos 2t + 2\sin 2t) \\ (3-i)e^{2it} &= (3\cos 2t + \sin 2t) + i(3\sin 2t - \cos 2t), \end{aligned}$$

so the solution is

$$
\begin{aligned}
x_1(t) &= c_1e^t + c_2(2\cos 2t - \sin 2t) + c_3(\cos 2t + 2\sin 2t) \\
x_2(t) &= -c_1e^t - c_2(3\cos 2t + \sin 2t) + c_3(\cos 2t - 3\sin 2t) \\
x_3(t) &= c_2(3\cos 2t + \sin 2t) + c_3(3\sin 2t - \cos 2t).
\end{aligned}
$$

25. Characteristic equation $-\lambda^3 + 4\lambda^2 - 13\lambda = 0$

Eigenvalues $\lambda = 0$ and $2 \pm 3i$

With $\lambda = 1$ the eigenvector equation

$$
\begin{bmatrix} 5 & 5 & 2 \\ -6 & -6 & -5 \\ 6 & 6 & 5 \end{bmatrix}
\begin{bmatrix} a_1 \\ b_1 \\ c_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{gives eigenvector } \mathbf{v}_1 = [1 \ \ -1 \ \ 0]^T.
$$

With $\lambda = 2 + 3i$ we solve the eigenvector equation

$$
\begin{bmatrix} 3-3i & 5 & 2 \\ -6 & -8-3i & -5 \\ 6 & 6 & 3-3i \end{bmatrix}
\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}
$$

to find the complex-valued eigenvector $\mathbf{v} = [1+i \ \ -2 \ \ 2]^T$. The corresponding complex-valued solution is

$$
\mathbf{x}(t) = \mathbf{v}e^{(2+3i)t} = e^{2t}\begin{bmatrix} (\cos 3t - \sin 3t) + i(\cos 3t + \sin 3t \\ -2\cos 3t - 2i\sin 3t \\ 2\cos 3t + 2i\sin 3t \end{bmatrix}.
$$

The scalar components of the resulting general solution are

$$
\begin{aligned}
x_1(t) &= c_1 + e^{2t}[(c_2 + c_3)\cos 3t + (-c_2 + c_3)\sin 3t] \\
x_2(t) &= -c_1 + 2e^{2t}(-c_2\cos 3t - c_3\sin 3t) \\
x_3(t) &= 2e^{2t}(c_2\cos 3t + c_3\sin 3t)
\end{aligned}
$$

26. Characteristic equation $-\lambda^3 + \lambda^2 + 4\lambda + 6 = 0$

Eigenvalues $\lambda = 3$ and $\lambda = -1 \pm i$

With $\lambda = 3$ the eigenvector equation

$$\begin{bmatrix} 0 & 0 & 1 \\ 9 & -4 & 2 \\ -9 & 4 & -4 \end{bmatrix} \begin{bmatrix} a_1 \\ b_1 \\ c_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{gives eigenvector } \mathbf{v}_1 = [4 \quad 9 \quad 0]^T.$$

With $\lambda = -1 + i$ we solve the eigenvector equation

$$\begin{bmatrix} 4-i & 0 & 1 \\ 9 & i & 2 \\ -9 & 4 & i \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

to find the complex-valued eigenvector $\mathbf{v} = [1 \quad 2-i \quad -4+i\,]^T$. The corresponding complex-valued solution is

$$\mathbf{x}(t) = \mathbf{v}e^{(-1+i)\,t} = e^{-t}\begin{bmatrix} \cos t + i\sin t \\ (2\cos t + \sin t) + i(-\cos t + 2\sin t) \\ (-4\cos t - \sin t) + i(\cos t - 4\sin t) \end{bmatrix}$$

with real and imaginary parts $\mathbf{x}_2(t)$ and $\mathbf{x}_3(t)$. Assembling the general solution $\mathbf{x} = c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + c_3\mathbf{x}_3$, we get the scalar equations

$$\begin{aligned} x_1(t) &= 4c_1e^{3t} + e^{-t}\,[c_2\cos t + c_3\sin t] \\ x_2(t) &= 9c_1e^{3t} + e^{-t}\,[(2c_2 - c_3)\cos t + (c_2 + 2c_3)\sin t] \\ x_3(t) &= \qquad\qquad e^{-t}\,[(-4c_2 + c_3)\cos t + (-c_2 - 4c_3)\sin t]. \end{aligned}$$

Finally, the given initial conditions yield the values $c_1 = 1$, $c_2 = -4$, $c_3 = 1$, so the desired particular solution is

$$\begin{aligned} x_1(t) &= 4e^{3t} - e^{-t}(4\cos t - \sin t) \\ x_2(t) &= 9e^{3t} - e^{-t}(9\cos t + 2\sin t) \\ x_3(t) &= 17e^{-t}\cos t. \end{aligned}$$

27. The coefficient matrix

$$\mathbf{A} = \begin{bmatrix} -0.2 & 0 \\ 0.2 & -0.4 \end{bmatrix}$$

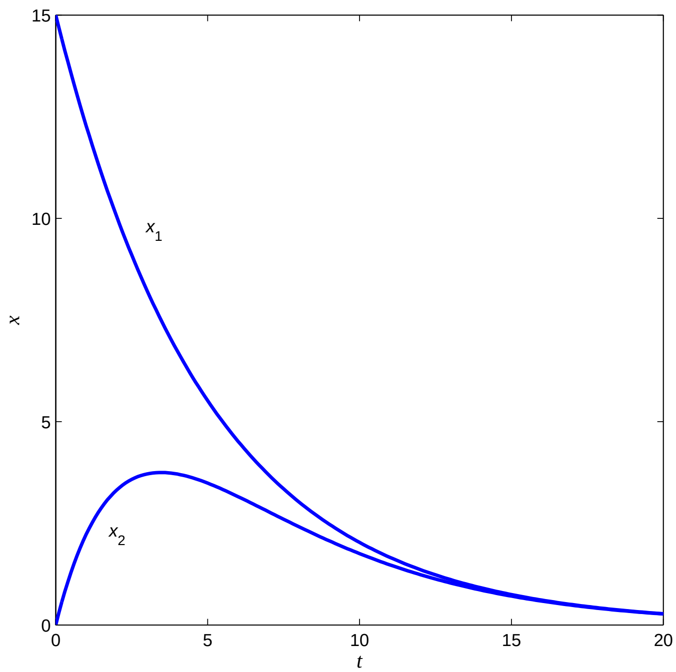
has characteristic equation $\lambda^2 + 0.6\lambda + 0.08 = 0$ with eigenvalues $\lambda_1 = -0.2$ and $\lambda_2 = -0.4$. We find easily that the associated eigenvectors are $\mathbf{v}_1 = [1 \quad 1]^T$ and $\mathbf{v}_2 = [0 \quad 1]^T$, so we get the general solution

$$x_1(t) = c_1e^{-0.2t}, \qquad x_2(t) = c_1e^{-0.2t} + c_2e^{-0.4t}.$$

The initial conditions $x_1(0) = 15,\ x_2(0) = 0$ give $c_1 = 15$ and $c_2 = -15,$ so we get

$$x_1(t) = 15e^{-0.2t}, \quad x_2(t) = 15e^{-0.2t} - 15e^{-0.4t}.$$

To find the maximum value of $x_2(t)$, we solve the equation $x_2'(t) = 0$ for $t = 5 \ln 2$, which gives the maximum value $x_2(5 \ln 2) =$ 3.75 lb. The following figure shows the graphs of $x_1(t)$ and $x_2(t)$.



28. The coefficient matrix

$$\mathbf{A} = \begin{bmatrix} -0.4 & 0 \\ 0.4 & -0.25 \end{bmatrix}$$

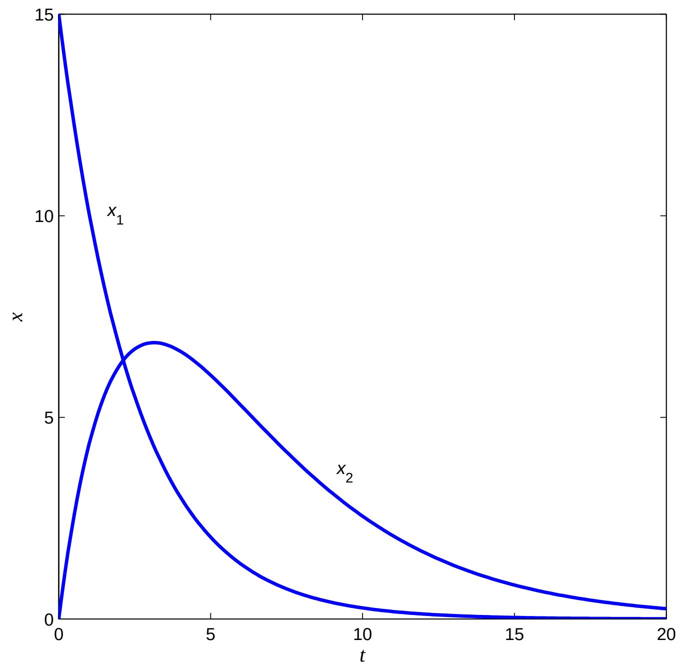
has characteristic equation $\lambda^2 + 0.65\lambda + 0.10 = 0$ with eigenvalues $\lambda_1 = -0.4$ and $\lambda_2 = -0.25$. We find easily that the associated eigenvectors are $\mathbf{v}_1 = [3 \quad -8]^T$ and $\mathbf{v}_2 = [0 \quad 1]^T$, so we get the general solution

$$x_1(t) = 3c_1e^{-0.2t}, \quad x_2(t) = -8c_1e^{-0.2t} + c_2e^{-0.4t}.$$

The initial conditions $x_1(0) = 15,\ x_2(0) = 0$ give $c_1 = 5$ and $c_2 = 40,$ so we get

$$x_1(t) = 15e^{-0.4t}, \qquad x_2(t) = -40e^{-0.4t} + 40e^{-0.25t}.$$

To find the maximum value of $x_2(t)$, we solve the equation $x_2'(t) = 0$ for $t_m = \frac{20}{3}\ln\frac{8}{5}$, which gives the maximum value $x_2(t_m) \approx 6.85$ lb. The following figure shows the graphs of $x_1(t)$ and $x_2(t)$.



29. The coefficient matrix

$$\mathbf{A} = \begin{bmatrix} -0.2 & 0.4 \\ 0.2 & -0.4 \end{bmatrix}$$

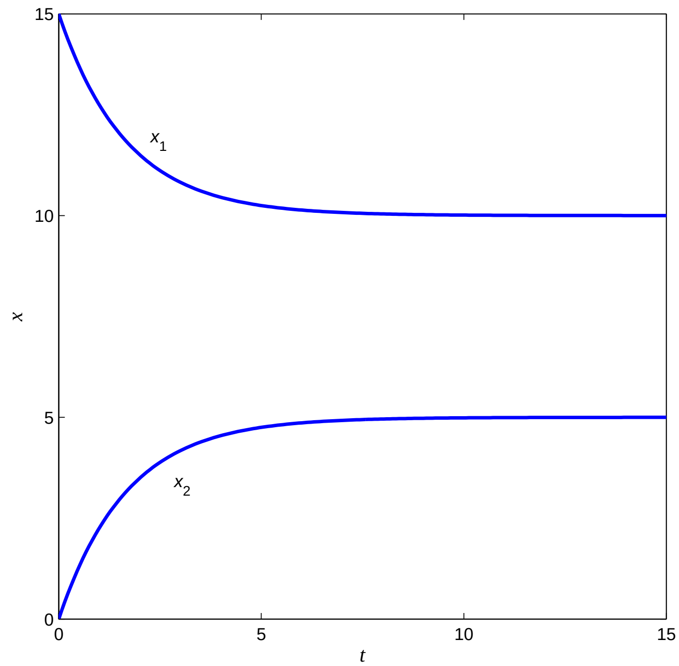
has eigenvalues $\lambda_1 = 0$ and $\lambda_2 = -0.6$, with eigenvectors $\mathbf{v}_1 = [2 \quad 1]^T$ and $\mathbf{v}_2 = [1 \quad -1]^T$ that yield the general solution

$$x_1(t) = 2c_1 + c_2 e^{-0.6t}, \qquad x_2(t) = c_1 - c_2 e^{-0.6t}.$$

The initial conditions $x_1(0) = 15$, $x_2(0) = 0$ give $c_1 = c_2 = 5$, so we get

$$x_1(t) = 10 + 5e^{-0.6t}, \qquad x_2(t) = 5 - 5e^{-0.6t}.$$

The figure at the top of the next page shows the graphs of $x_1(t)$ and $x_2(t)$.



30. The coefficient matrix

$$\mathbf{A} = \begin{bmatrix} -0.4 & 0.25 \\ 0.4 & -0.25 \end{bmatrix}$$

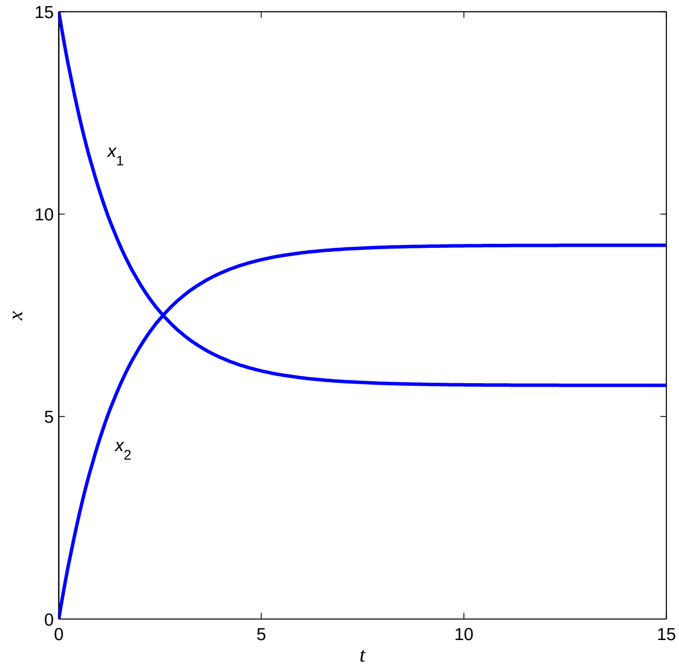
has eigenvalues $\lambda_1 = 0$ and $\lambda_2 = -0.65$, with eigenvectors $\mathbf{v}_1 = [5 \quad 8]^T$ and $\mathbf{v}_2 = [1 \quad -1]^T$ that yield the general solution

$$x_1(t) = 5c_1 + c_2 e^{-0.65t}, \qquad x_2(t) = 8c_1 - c_2 e^{-0.65t}.$$

The initial conditions $x_1(0) = 15$, $x_2(0) = 0$ give $c_1 = 15/13$, $c_2 = 120/13$, so we get

$$x_1(t) = (\ 75 + 120e^{-0.65t})/13$$
$$x_2(t) = (120 - 120e^{-0.65t})/13.$$

The figure at the top of the next page shows the graphs of $x_1(t)$ and $x_2(t)$.



31. The coefficient matrix

$$\mathbf{A} = \begin{bmatrix} -1 & 0 & 0 \\ 1 & -2 & 0 \\ 0 & 2 & -3 \end{bmatrix}$$

has as eigenvalues its diagonal elements $\lambda_1 = -1$, $\lambda_2 = -2$, and $\lambda_3 = -3$. We find readily that the associated eigenvectors are $\mathbf{v}_1 = [1 \quad 1 \quad 1]^T$, $\mathbf{v}_2 = [0 \quad 1 \quad 2]^T$, and $\mathbf{v}_3 = [0 \quad 0 \quad 1]^T$. The resulting general solution is solution is given by

$$\begin{aligned} x_1(t) &= c_1 e^{-t} \\ x_2(t) &= c_1 e^{-t} + c_2 e^{-2t} \\ x_3(t) &= c_1 e^{-t} + 2c_2 e^{-2t} + c_3 e^{-3t}. \end{aligned}$$

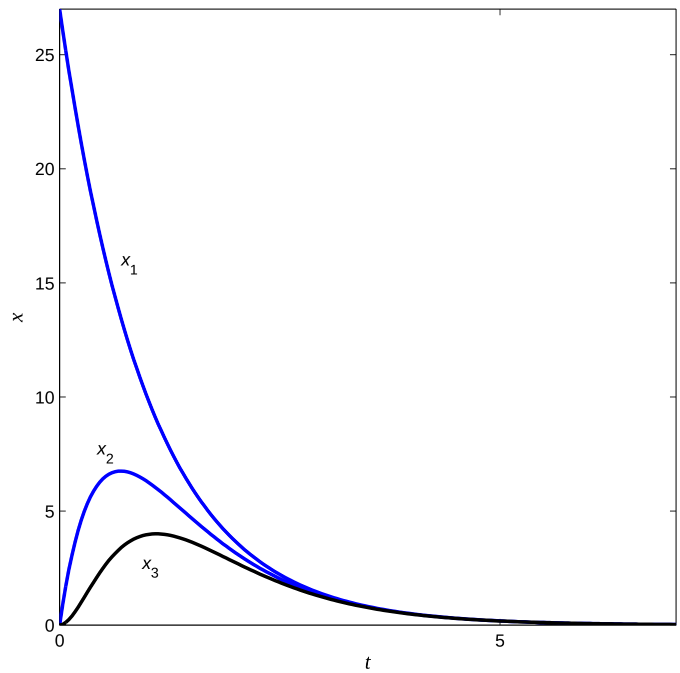
The initial conditions $x_1(0) = 27,\ x_2(0) = x_2(0) = 0$ give $c_1 = c_3 = 27,\ c_2 = -27,$ so we get

$$\begin{aligned} x_1(t) &= 27 e^{-t} \\ x_2(t) &= 27 e^{-t} - 27 e^{-2t} \\ x_3(t) &= 27 e^{-t} - 54 e^{-2t} + 27 e^{-3t}. \end{aligned}$$

The equation $x_3'(t) = 0$ simplifies to the equation

$$3e^{-2t} - 4e^{-t} + 1 = \left(3e^{-t} - 1\right)\left(e^{-t} - 1\right) = 0$$

with positive solution $t_m = \ln 3$. Thus the maximum amount of salt ever in tank 3 is $x_3(\ln 3) = 4$ pounds. The figure below shows the graphs of $x_1(t), x_2(t)$, and $x_3(t)$.



32. The coefficient matrix

$$\mathbf{A} = \begin{bmatrix} -3 & 0 & 0 \\ 3 & -2 & 0 \\ 0 & 2 & -1 \end{bmatrix}$$

has as eigenvalues its diagonal elements $\lambda_1 = -3$, $\lambda_2 = -2$, and $\lambda_3 = -1$. We find readily that the associated eigenvectors are $\mathbf{v}_1 = [1 \;\; -3 \;\; 3]^T$, $\mathbf{v}_2 = [0 \;\; -1 \;\; 2]^T$, and $\mathbf{v}_3 = [0 \;\; 0 \;\; 1]^T$. The resulting general solution is solution is given by

$$\begin{aligned} x_1(t) &= c_1 e^{-3t} \\ x_2(t) &= -3c_1 e^{-3t} - c_2 e^{-2t} \\ x_3(t) &= 3c_1 e^{-3t} + 2c_2 e^{-2t} + c_3 e^{-t}. \end{aligned}$$

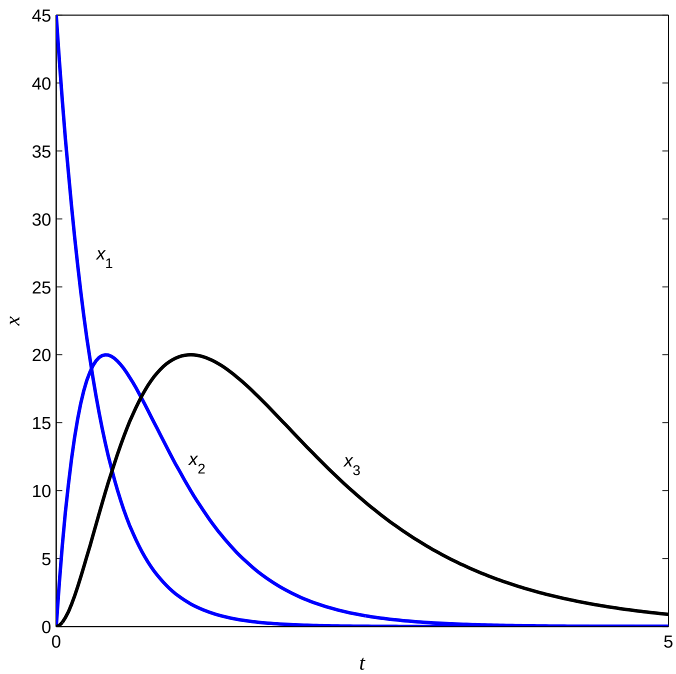
The initial conditions $x_1(0) = 45$, $x_2(0) = x_2(0) = 0$ give $c_1 = 45$, $c_2 = -135$, $c_3 = 135$, so we get

$$\begin{aligned} x_1(t) &= 45e^{-3t} \\ x_2(t) &= -135e^{-3t} + 135e^{-2t} \\ x_3(t) &= 135e^{-3t} - 270e^{-2t} + 135e^{-t}. \end{aligned}$$

The equation $x_3'(t) = 0$ simplifies to the equation

$$3e^{-2t} - 4e^{-t} + 1 = \left(3e^{-t} - 1\right)\left(e^{-t} - 1\right) = 0$$

with positive solution $t_m = \ln 3$. Thus the maximum amount of salt ever in tank 3 is $x_3(\ln 3) = 20$ pounds. The figure below shows the graphs of $x_1(t)$, $x_2(t)$, and $x_3(t)$.



33. The coefficient matrix

$$\mathbf{A} = \begin{bmatrix} -4 & 0 & 0 \\ 4 & -6 & 0 \\ 0 & 6 & -2 \end{bmatrix}$$

has as eigenvalues its diagonal elements $\lambda_1 = -4$, $\lambda_2 = -6$, and $\lambda_3 = -2$. We find readily that the associated eigenvectors are $\mathbf{v}_1 = [-1 \quad -2 \quad 6]^T$, $\mathbf{v}_2 = [0 \quad -2 \quad 3]^T$, and $\mathbf{v}_3 = [0 \quad 0 \quad 1]^T$. The resulting general solution is solution is given by

$$
\begin{aligned}
x_1(t) &= -c_1 e^{-4t} \\
x_2(t) &= -2c_1 e^{-4t} - 2c_2 e^{-6t} \\
x_3(t) &= 6c_1 e^{-4t} + 3c_2 e^{-6t} + c_3 e^{-2t}.
\end{aligned}
$$

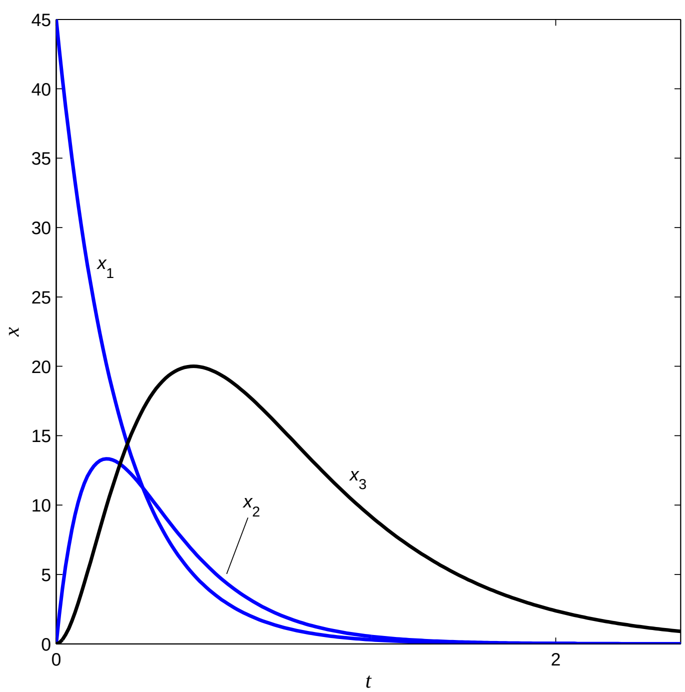
The initial conditions $x_1(0) = 45,\ x_2(0) = x_2(0) = 0$ give $c_1 = -45,\ c_2 = 45,\ c_3 = 135,$ so we get

$$
\begin{aligned}
x_1(t) &= 45e^{-4t} \\
x_2(t) &= 90e^{-4t} - 90e^{-6t} \\
x_3(t) &= -270e^{-4t} + 135e^{-6t} + 135e^{-2t}.
\end{aligned}
$$

The equation $x_3'(t) = 0$ simplifies to the equation

$$3e^{-4t} - 4e^{-2t} + 1 = \left(3e^{-2t} - 1\right)\left(e^{-2t} - 1\right) = 0$$

with positive solution $t_m = \frac{1}{2}\ln 3$. Thus the maximum amount of salt ever in tank 3 is $x_3(\frac{1}{2}\ln 3) = 20$ pounds. The figure below shows the graphs of $x_1(t), x_2(t),$ and $x_3(t)$.



34. The coefficient matrix

$$\mathbf{A} = \begin{bmatrix} -3 & 0 & 0 \\ 3 & -5 & 0 \\ 0 & 5 & -1 \end{bmatrix}$$

has as eigenvalues its diagonal elements $\lambda_1 = -3$, $\lambda_2 = -5$, and $\lambda_3 = -1$. We find readily that the associated eigenvectors are $\mathbf{v}_1 = [-4 \quad -6 \quad 15]^T$, $\mathbf{v}_2 = [0 \quad -4 \quad 5]^T$, and $\mathbf{v}_3 = [0 \quad 0 \quad 1]^T$. The resulting general solution is solution is given by

$$\begin{aligned} x_1(t) &= -4c_1 e^{-3t} \\ x_2(t) &= -6c_1 e^{-3t} - 4c_2 e^{-5t} \\ x_3(t) &= 15c_1 e^{-3t} + 5c_2 e^{-5t} + c_3 e^{-t}. \end{aligned}$$

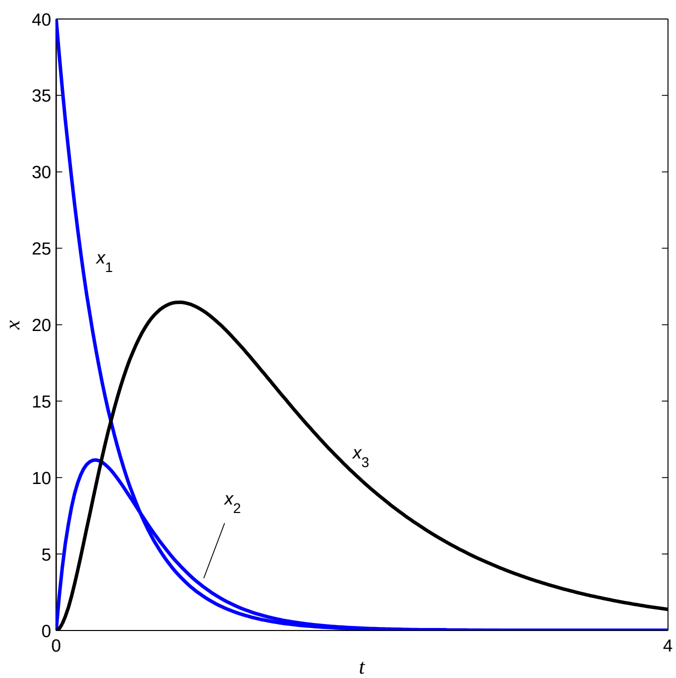
The initial conditions $x_1(0) = 40$, $x_2(0) = x_2(0) = 0$ give $c_1 = -10$, $c_2 = 15$, $c_3 = 75$, so we get

$$\begin{aligned} x_1(t) &= 40 e^{-3t} \\ x_2(t) &= 60 e^{-3t} - 60 e^{-5t} \\ x_3(t) &= -150 e^{-3t} + 75 e^{-5t} + 75 e^{-t}. \end{aligned}$$

The equation $x_3'(t) = 0$ simplifies to the equation

$$5e^{-4t} - 6e^{-2t} + 1 = \left(5e^{-2t} - 1\right)\left(e^{-2t} - 1\right) = 0$$

with positive solution $t_m = \frac{1}{2}\ln 5$. Thus the maximum amount of salt ever in tank 3 is $x_3(\frac{1}{2}\ln 5) \approx 21.4663$ pounds. The figure below shows the graphs of $x_1(t)$, $x_2(t)$, and $x_3(t)$.



35. The coefficient matrix

$$\mathbf{A} = \begin{bmatrix} -6 & 0 & 3 \\ 6 & -20 & 0 \\ 0 & 20 & -3 \end{bmatrix}$$

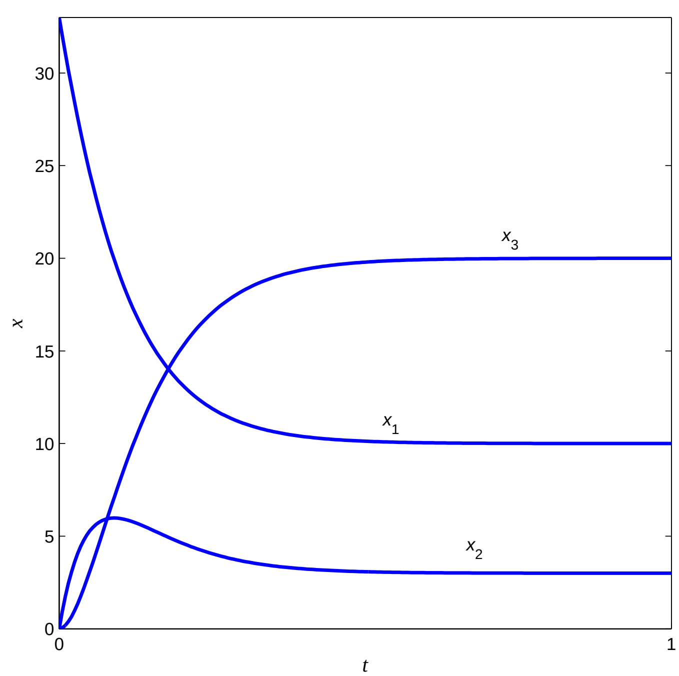
has characteristic equation $-\lambda^3 - 29\lambda^2 - 198\lambda = -\lambda(\lambda-18)(\lambda-11) = 0$ with eigenvalues $\lambda_0 = 0$, $\lambda_1 = -18$, and $\lambda_2 = -11$. We find that associated eigenvectors are $\mathbf{v}_0 = [10 \quad 3 \quad 20]^T$, $\mathbf{v}_1 = [-1 \quad -3 \quad 4]^T$, and $\mathbf{v}_2 = [-3 \quad -2 \quad 5]^T$. The resulting general solution is solution is given by

$$\begin{aligned} x_1(t) &= 10c_0 - c_1 e^{-18t} - 3c_2 e^{-11t} \\ x_2(t) &= 3c_0 - 3c_1 e^{-18t} - 2c_2 e^{-11t} \\ x_3(t) &= 20c_0 + 4c_1 e^{-18t} + 5c_2 e^{-11t}. \end{aligned}$$

The initial conditions $x_1(0) = 33,\ x_2(0) = x_2(0) = 0$ give $c_1 = 1,\ c_2 = 55/7,\ c_3 = -72/7$, so we get

$$\begin{aligned} x_1(t) &= 10 - \tfrac{1}{7}\left(55e^{-18t} - 216e^{-11t}\right) \\ x_2(t) &= 3 - \tfrac{1}{7}\left(165e^{-18t} - 144e^{-11t}\right) \\ x_3(t) &= 20 + \tfrac{1}{7}\left(220e^{-18t} - 360e^{-11t}\right). \end{aligned}$$

Thus the limiting amounts of salt in tanks 1, 2, and 3 are 10 lb, 3 lb, and 20 lb. The figure below shows the graphs of $x_1(t),\ x_2(t),$ and $x_3(t)$.



36. The coefficient matrix

$$\mathbf{A} = \begin{bmatrix} -\frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{5} & 0 \\ 0 & \frac{1}{5} & -\frac{1}{2} \end{bmatrix}$$

has characteristic equation $-\lambda^3 - (6/5)\lambda^2 - (9/20)\lambda = 0$ with eigenvalues $\lambda_0 = 0$, $\lambda_1 = -3(2+i)/10$, and $\lambda_2 = -3(2-i)/10$. The eigenvector equation

$$\begin{bmatrix} -\frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{5} & 0 \\ 0 & \frac{1}{5} & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

associated with the eigenvalue $\lambda_0 = 0$ yields the associated eigenvector $\mathbf{v}_0 = [1 \quad 5/2 \quad 1]^T$ and consequently the constant solution $\mathbf{x}_0(t) \equiv \mathbf{v}_0$. Then the eigenvector equation

$$\begin{bmatrix} \frac{1}{10}(1+3i) & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{10}(4+3i) & 0 \\ 0 & \frac{1}{5} & \frac{1}{10}(1+3i) \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

associated with $\lambda_1 = -3(2+i)/10$ yields the complex-valued eigenvector $\mathbf{v}_1 = [-(1-3i)/2 \quad -(1+3i)/2 \quad 1]^T$. The corresponding complex-valued solution is

$$\begin{aligned} \mathbf{x}_1(t) &= \mathbf{v}_1 e^{(-6-3i)\,t/10} \\ &= \frac{1}{2}e^{-3t/5} \begin{bmatrix} \left(-\cos(3t/10) + 3\sin(3t/10)\right) + i\left(3\cos(3t/10) + \sin(3t/10)\right) \\ \left(-\cos(3t/10) - 3\sin(3t/10)\right) + i\left(-3\cos(3t/10) + \sin(3t/10)\right) \\ 2\cos(3t/10) - 2i\sin(3t/10) \end{bmatrix}. \end{aligned}$$

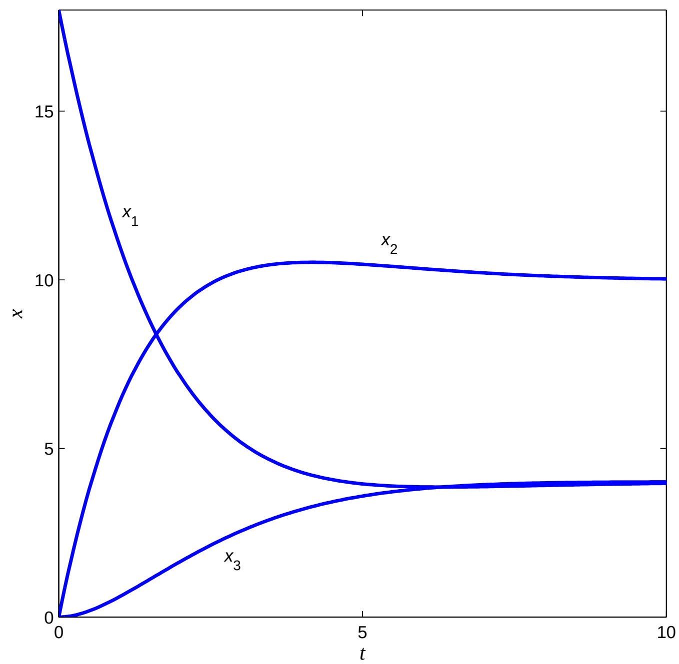
The scalar components of resulting general solution $\mathbf{x} = c_0\mathbf{x}_0 + c_1\,\mathrm{Re}(\mathbf{x}_1) + c_2\,\mathrm{Im}(\mathbf{x}_1)$ are given by

$$\begin{aligned} x_1(t) &= c_0 + \tfrac{1}{2}e^{-3t/5}\left[\left(-c_1 + 3c_2\right)\cos(3t/10) + \left(3c_1 + c_2\right)\sin(3t/10)\right] \\ x_2(t) &= \tfrac{5}{2}c_0 + \tfrac{1}{2}e^{-3t/5}\left[\left(-c_1 - 3c_2\right)\cos(3t/10) + \left(-3c_1 + c_2\right)\sin(3t/10)\right] \\ x_3(t) &= c_0 + e^{-3t/5}\left[c_1\cos(3t/10) - c_2\sin(3t/10)\right]. \end{aligned}$$

When we impose the initial conditions $x_1(0) = 18$, $x_2(0) = x_2(0) = 0$ we find that $c_0 = 4$, $c_1 = -4$, and $c_2 = 8$. This finally gives the particular solution

$$\begin{aligned} x_1(t) &= 4 + e^{-3t/5}\left[14\cos(3t/10) - 2\sin(3t/10)\right] \\ x_2(t) &= 10 - e^{-3t/5}\left[10\cos(3t/10) - 10\sin(3t/10)\right] \\ x_3(t) &= 4 - e^{-3t/5}\left[\ 4\cos(3t/10) + 8\sin(3t/10)\right]. \end{aligned}$$

Thus the limiting amounts of salt in tanks 1, 2, and 3 are 4 lb, 10 lb, and 4 lb. The figure below shows the graphs of $x_1(t),\ x_2(t),$ and $x_3(t)$.



37. The coefficient matrix

$$\mathbf{A} = \begin{bmatrix} -1 & 0 & 2 \\ 1 & -3 & 0 \\ 0 & 3 & -2 \end{bmatrix}$$

has characteristic equation $-\lambda^3 - 6\lambda^2 - 11\lambda = 0$ with eigenvalues $\lambda_0 = 0$, $\lambda_1 = -3 - i\sqrt{2}$, and $\lambda_2 = -3 + i\sqrt{2}$. The eigenvector equation

$$\begin{bmatrix} -1 & 0 & 2 \\ 1 & -3 & 0 \\ 0 & 3 & -2 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

associated with the eigenvalue $\lambda_0 = 0$ yields the associated eigenvector

$\mathbf{v}_0 = [6 \quad 2 \quad 3]^T$ and consequently the constant solution $\mathbf{x}_0(t) \equiv \mathbf{v}_0$. Then the eigenvector equation

$$\begin{bmatrix} 2+i\sqrt{2} & 0 & 2 \\ 1 & i\sqrt{2} & 0 \\ 0 & 3 & 1+i\sqrt{2} \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

associated with $\lambda_1 = -3 - i\sqrt{2}$ yields the complex-valued eigenvector $\mathbf{v}_1 = \begin{bmatrix}(-2+i\sqrt{2})/3 & (-1-i\sqrt{2})/3 & 1\end{bmatrix}^T$. The corresponding complex-valued solution is

$$\mathbf{x}_1(t) = \mathbf{v}_1 e^{(-3-i\sqrt{2})t}$$

$$= \frac{1}{3}e^{-3t}\begin{bmatrix} \left(-2\cos(t\sqrt{2}) + \sqrt{2}\sin(t\sqrt{2})\right) + i\left(\sqrt{2}\cos(t\sqrt{2}) + 2\sin(t\sqrt{2})\right) \\ \left(-\cos(t\sqrt{2}) - \sqrt{2}\sin(t\sqrt{2})\right) + i\left(-\sqrt{2}\cos(t\sqrt{2}) + \sin(t\sqrt{2})\right) \\ 3\cos(t\sqrt{2}) - 3i\sin(t\sqrt{2}) \end{bmatrix}.$$

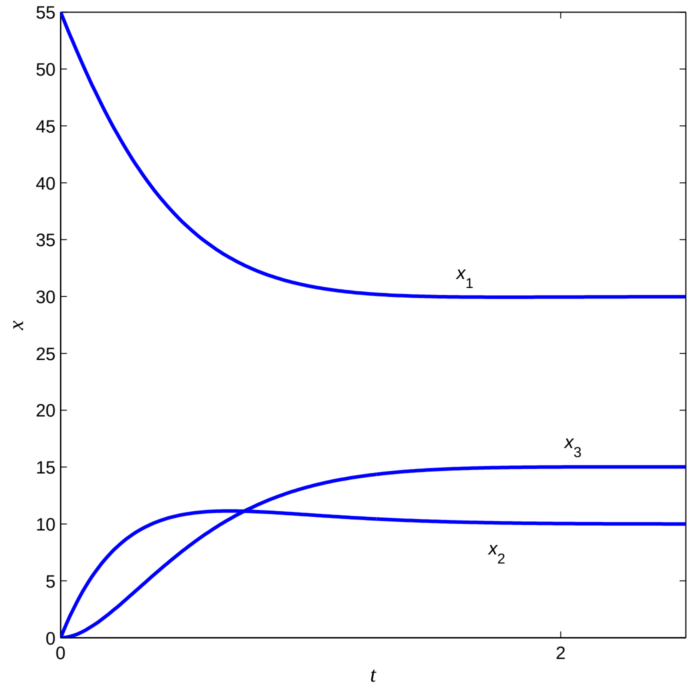
The scalar components of resulting general solution $\mathbf{x} = c_0\mathbf{x}_0 + c_1 \operatorname{Re}(\mathbf{x}_1) + c_2 \operatorname{Im}(\mathbf{x}_1)$ are given by

$$\begin{aligned}
x_1(t) &= 6c_0 + \tfrac{1}{3}e^{-3t}\left[\left(-2c_1 + \sqrt{2}\,c_2\right)\cos(t\sqrt{2}) + \left(\sqrt{2}\,c_1 + 2c_2\right)\sin(t\sqrt{2})\right] \\
x_2(t) &= 2c_0 + \tfrac{1}{3}e^{-3t}\left[\left(-c_1 - \sqrt{2}\,c_2\right)\cos(t\sqrt{2}) + \left(-\sqrt{2}\,c_1 + c_2\right)\sin(t\sqrt{2})\right] \\
x_3(t) &= 3c_0 + e^{-3t}\left[c_1\cos(t\sqrt{2}) - c_2\sin(t\sqrt{2})\right].
\end{aligned}$$

When we impose the initial conditions $x_1(0) = 55,\ x_2(0) = x_2(0) = 0$ we find that $c_0 = 5,\ c_1 = -15,$ and $c_2 = 45/\sqrt{2}$. This finally gives the particular solution

$$\begin{aligned}
x_1(t) &= 30 + e^{-3t}\left[25\cos(t\sqrt{2}) + 10\sqrt{2}\sin(t\sqrt{2})\right] \\
x_2(t) &= 10 - e^{-3t}\left[10\cos(t\sqrt{2}) - \tfrac{25}{2}\sqrt{2}\sin(t\sqrt{2})\right] \\
x_3(t) &= 15 - e^{-3t}\left[15\cos(t\sqrt{2}) + \tfrac{45}{2}\sqrt{2}\sin(t\sqrt{2})\right].
\end{aligned}$$

Thus the limiting amounts of salt in tanks 1, 2, and 3 are 30 lb, 10 lb, and 15 lb. The figure at the top of the next page shows the graphs of $x_1(t),\ x_2(t),$ and $x_3(t)$.



In Problems 38–41 the Maple command **`with(linalg):eigenvects(A)`**, the Mathematica command **`Eigensystem[A]`**, or the MATLAB command **`[V,D] = eig(A)`** can be used to find the eigenvalues and associated eigenvectors of the given coefficient matrix **A**.

38. Characteristic equation: $(\lambda - 1)(\lambda - 2)(\lambda - 3)(\lambda - 4) = 0$

Eigenvalues and associated eigenvectors:

$$\begin{aligned}
\lambda &= 1, & \mathbf{v} &= [1 \quad -2 \quad 3 \quad -4]^T \\
\lambda &= 2, & \mathbf{v} &= [0 \quad 1 \quad -3 \quad 6]^T \\
\lambda &= 3, & \mathbf{v} &= [0 \quad 0 \quad 1 \quad -4]^T \\
\lambda &= 4, & \mathbf{v} &= [0 \quad 0 \quad 0 \quad 1]^T
\end{aligned}$$

Scalar solution equations:

$$x_1(t) = c_1 e^t$$

$$x_2(t) = -2c_1 e^t + c_2 e^{2t}$$

$$x_3(t) = 3c_1 e^t - 3c_2 e^{2t} + c_3 e^{3t}$$

$$x_4(t) = -4c_1 e^t + 6c_2 e^{2t} - 4c_3 e^{3t} + c_4 e^{4t}$$

39. Characteristic equation: $(\lambda^2 - 1)(\lambda^2 - 4) = 0$

Eigenvalues and associated eigenvectors:

$$\begin{aligned}
\lambda &= 1, & \mathbf{v} &= [3 \quad -2 \quad 4 \quad 1]^T \\
\lambda &= -1, & \mathbf{v} &= [0 \quad 0 \quad 1 \quad 0]^T \\
\lambda &= 2, & \mathbf{v} &= [0 \quad 1 \quad 0 \quad 0]^T \\
\lambda &= -2, & \mathbf{v} &= [1 \quad -1 \quad 0 \quad 0]^T
\end{aligned}$$

Scalar solution equations:

$$\begin{aligned}
x_1(t) &= 3c_1e^t + c_4e^{-2t} \\
x_2(t) &= -2c_1e^t + c_3e^{2t} - c_4e^{-2t} \\
x_3(t) &= 4c_1e^t + c_2e^{-t} \\
x_4(t) &= c_1e^t
\end{aligned}$$

40. Characteristic equation: $(\lambda^2 - 4)(\lambda^2 - 25) = 0$

Eigenvalues and associated eigenvectors:

$$\begin{aligned}
\lambda &= 2, & \mathbf{v} &= [1 \quad -3 \quad 0 \quad 0]^T \\
\lambda &= -2, & \mathbf{v} &= [0 \quad 3 \quad 0 \quad -1]^T \\
\lambda &= 5, & \mathbf{v} &= [0 \quad 0 \quad 1 \quad -3]^T \\
\lambda &= -5, & \mathbf{v} &= [0 \quad 1 \quad 0 \quad 0]^T
\end{aligned}$$

Scalar solution equations:

$$\begin{aligned}
x_1(t) &= c_1e^{2t} \\
x_2(t) &= -3c_1e^{2t} + 3c_2e^{-2t} - c_4e^{-5t} \\
x_3(t) &= c_3e^{5t} \\
x_4(t) &= -c_2e^{-2t} - 3c_3e^{5t}
\end{aligned}$$

41. The eigenvectors associated with the respective eigenvalues $\lambda_1 = -3$, $\lambda_2 = -6$, $\lambda_3 = 10$, and $\lambda_4 = 15$ are

$$\begin{aligned}
\mathbf{v}_1 &= [\ 1 \quad 0 \quad 0 \quad -1]^T \\
\mathbf{v}_2 &= [\ 0 \quad 1 \quad -1 \quad 0]^T \\
\mathbf{v}_3 &= [-2 \quad 1 \quad 1 \quad -2]^T \\
\mathbf{v}_4 &= [\ 1 \quad 2 \quad 2 \quad 1]^T.
\end{aligned}$$

Hence the general solution has scalar component functions

$$\begin{aligned}
x_1(t) &= c_1e^{-3t} - 2c_3e^{10t} + c_4e^{15t} \\
x_2(t) &= c_2e^{-6t} + c_3e^{10t} + 2c_4e^{15t} \\
x_3(t) &= -c_2e^{-6t} + c_3e^{10t} + 2c_4e^{15t} \\
x_4(t) &= -c_1e^{-3t} - 2c_3e^{10t} + c_4e^{15t}.
\end{aligned}$$

The given initial conditions are satisfied by choosing $c_1 = c_2 = 0$, $c_3 = -1$, and $c_4 = 1$, so the desired particular solution is given by

$$\begin{aligned}
x_1(t) &= 2e^{10t} + e^{15t} = x_4(t) \\
x_2(t) &= -e^{10t} + 2e^{15t} = x_3(t) .
\end{aligned}$$

In Problems 42–50 we give a general solution in the form $\mathbf{x}(t) = c_1\mathbf{v}_1e^{\lambda_1 t} + c_2\mathbf{v}_2e^{\lambda_2 t} + \cdots$ that exhibits explicitly the eigenvalues $\lambda_1, \lambda_2, \ldots$ and corresponding eigenvectors $\mathbf{v}_1, \mathbf{v}_2, \ldots$ of the given coefficient matrix $\mathbf{A}$.

42. $$\mathbf{x}(t) = c_1\begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix} + c_2\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}e^{2t} + c_3\begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix}e^{5t}$$

43. $$\mathbf{x}(t) = c_1\begin{bmatrix} 3 \\ -1 \\ 5 \end{bmatrix}e^{-2t} + c_2\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}e^{4t} + c_3\begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}e^{8t}$$

44. $$\mathbf{x}(t) = c_1\begin{bmatrix} 3 \\ -2 \\ 2 \end{bmatrix}e^{-3t} + c_2\begin{bmatrix} 7 \\ 1 \\ 5 \end{bmatrix}e^{6t} + c_3\begin{bmatrix} 5 \\ -3 \\ 3 \end{bmatrix}e^{12t}$$

45. $$\mathbf{x}(t) = c_1\begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix}e^{-3t} + c_2\begin{bmatrix} 1 \\ 2 \\ -1 \\ 1 \end{bmatrix} + c_3\begin{bmatrix} 2 \\ 1 \\ 1 \\ 1 \end{bmatrix}e^{3t} + c_4\begin{bmatrix} 1 \\ -1 \\ 2 \\ -1 \end{bmatrix}e^{6t}$$

46. $\mathbf{x}(t) = c_1\begin{bmatrix}3\\2\\-1\\1\end{bmatrix}e^{-4t} + c_2\begin{bmatrix}1\\2\\2\\-1\end{bmatrix}e^{2t} + c_3\begin{bmatrix}1\\1\\-1\\1\end{bmatrix}e^{4t} + c_4\begin{bmatrix}3\\-2\\3\\-3\end{bmatrix}e^{8t}$

47. $\mathbf{x}(t) = c_1\begin{bmatrix}2\\2\\1\\-1\end{bmatrix}e^{-3t} + c_2\begin{bmatrix}1\\2\\-1\\1\end{bmatrix}e^{3t} + c_3\begin{bmatrix}2\\1\\1\\1\end{bmatrix}e^{6t} + c_4\begin{bmatrix}1\\-1\\2\\-1\end{bmatrix}e^{9t}$

48. $\mathbf{x}(t) = c_1\begin{bmatrix}1\\2\\-1\\2\end{bmatrix}e^{16t} + c_2\begin{bmatrix}2\\5\\1\\-1\end{bmatrix}e^{32t} + c_3\begin{bmatrix}3\\-1\\1\\2\end{bmatrix}e^{48t} + c_4\begin{bmatrix}1\\1\\2\\-3\end{bmatrix}e^{64t}$

49. $\mathbf{x}(t) = c_1\begin{bmatrix}1\\0\\3\\1\\1\end{bmatrix}e^{-3t} + c_2\begin{bmatrix}0\\3\\0\\-1\\1\end{bmatrix} + c_3\begin{bmatrix}1\\7\\1\\1\\1\end{bmatrix}e^{3t} + c_4\begin{bmatrix}0\\1\\0\\1\\1\end{bmatrix}e^{6t} + c_5\begin{bmatrix}2\\0\\5\\2\\1\end{bmatrix}e^{9t}$

50. $\mathbf{x}(t) = c_1\begin{bmatrix}0\\1\\1\\1\\0\\1\end{bmatrix}e^{-7t} + c_2\begin{bmatrix}1\\0\\0\\0\\1\\1\end{bmatrix}e^{-4t} + c_3\begin{bmatrix}0\\1\\0\\1\\0\\1\end{bmatrix}e^{3t} + c_4\begin{bmatrix}0\\0\\1\\0\\1\\0\end{bmatrix}e^{5t} + c_5\begin{bmatrix}1\\1\\0\\0\\0\\1\end{bmatrix}e^{9t} + c_6\begin{bmatrix}0\\0\\1\\-1\\-1\\0\end{bmatrix}e^{11t}$

SECTION 7.4

SECOND-ORDER SYSTEMS AND MECHANICAL APPLICATIONS

This section uses the eigenvalue method to exhibit realistic applications of linear systems. If a computer system like Maple, Mathematica, MATLAB, or even a TI-85/86/89/92 calculator is available, then a system of more than three railway cars, or a multistory building with four or more floors (as in the project), can be investigated. However, the problems in the text are intended for manual solution.

Problems 1–7 involve the system

$$\begin{aligned} m_1x_1'' &= -(k_1+k_2)x_1 + \qquad k_2x_2 \\ m_2x_2'' &= \qquad k_2x_1 - (k_2+k_3)x_2 \end{aligned}$$

with various values of m_1, m_2 and k_1, k_2, k_3. In each problem we divide the first equation by m_1 and the second one by m_2 to obtain a second-order linear system $\mathbf{x}'' = \mathbf{A}\,\mathbf{x}$ in the standard form of Theorem 1 in this section. If the eigenvalues λ_1 and λ_2 are both negative, then the natural (circular) frequencies of the system are $\omega_1 = \sqrt{-\lambda_1}$ and $\omega_2 = \sqrt{-\lambda_2}$, and — according to Eq. (11) in Theorem 1 of this section — the eigenvalues $\mathbf{v}_1$ and $\mathbf{v}_2$ associated with λ_1 and λ_2 determine the natural modes of oscillations at these frequencies.

1. The matrix $\mathbf{A} = \begin{bmatrix} -2 & 2 \\ 2 & -2 \end{bmatrix}$ has eigenvalues $\lambda_0 = 0$ and $\lambda_1 = -4$ with associated eigenvalues $\mathbf{v}_0 = [1 \quad 1]^T$ and $\mathbf{v}_1 = [1 \quad -1]^T$. Thus we have the special case described in Eq. (12) of Theorem 1, and a general solution is given by

$$\begin{aligned} x_1(t) &= a_1 + a_2\,t + b_1\cos 2t + b_2\sin 2t, \\ x_2(t) &= a_1 + a_2\,t - b_1\cos 2t - b_2\sin 2t. \end{aligned}$$

The natural frequencies are $\omega_1 = 0$ and $\omega_2 = 2$. In the degenerate natural mode with "frequency" $\omega_1 = 0$ the two masses move by translation without oscillating. At frequency $\omega_2 = 2$ they oscillate in opposite directions with equal amplitudes.

2. The matrix $\mathbf{A} = \begin{bmatrix} -5 & 4 \\ 5 & -5 \end{bmatrix}$ has eigenvalues $\lambda_1 = -1$ and $\lambda_2 = -9$ with associated eigenvalues $\mathbf{v}_1 = [1 \quad 1]^T$ and $\mathbf{v}_2 = [1 \quad -1]^T$. Hence a general solution is given by

$$\begin{aligned} x_1(t) &= a_1\cos t + a_2\sin t + b_1\cos 3t + b_2\sin 3t, \\ x_2(t) &= a_1\cos t + a_2\sin t - b_1\cos 3t - b_2\sin 3t. \end{aligned}$$

3. The matrix $\mathbf{A} = \begin{bmatrix} -3 & 2 \\ 1 & -2 \end{bmatrix}$ has eigenvalues $\lambda_1 = -1$ and $\lambda_2 = -4$ with associated eigenvalues $\mathbf{v}_1 = [1 \quad 1]^T$ and $\mathbf{v}_2 = [2 \quad -1]^T$. Hence a general solution is given by

$$\begin{aligned} x_1(t) &= a_1 \cos t + a_2 \sin t + 2b_1 \cos 2t + 2b_2 \sin 2t, \\ x_2(t) &= a_1 \cos t + a_2 \sin t \; - b_1 \cos 2t \; - b_2 \sin 2t. \end{aligned}$$

The natural frequencies are $\omega_1 = 1$ and $\omega_2 = 2$. In the natural mode with frequency ω_1, the two masses m_1 and m_2 move in the same direction with equal amplitudes of oscillation. In the natural mode with frequency ω_2 they move in opposite directions with the amplitude of oscillation of m_1 twice that of m_2.

4. The matrix $\mathbf{A} = \begin{bmatrix} -3 & 2 \\ 2 & -3 \end{bmatrix}$ has eigenvalues $\lambda_1 = -1$ and $\lambda_2 = -5$ with associated eigenvalues $\mathbf{v}_1 = [1 \quad 1]^{\mathrm{T}}$ and $\mathbf{v}_2 = [1 \quad -1]^{\mathrm{T}}$. Hence a general solution is given by

$$\begin{aligned} x_1(t) &= a_1 \cos t + a_2 \sin t + b_1 \cos t\sqrt{5} + b_2 \sin t\sqrt{5}, \\ x_2(t) &= a_1 \cos t + a_2 \sin t - b_1 \cos t\sqrt{5} - b_2 \sin t\sqrt{5}. \end{aligned}$$

The natural frequencies are $\omega_1 = 1$ and $\omega_2 = \sqrt{5}$. In the natural mode with frequency ω_1, the two masses m_1 and m_2 move in the same direction with equal amplitudes of oscillation. At frequency ω_2 they move in opposite directions with equal amplitudes.

5. The matrix $\mathbf{A} = \begin{bmatrix} -3 & 1 \\ 1 & -3 \end{bmatrix}$ has eigenvalues $\lambda_1 = -2$ and $\lambda_2 = -4$ with associated eigenvalues $\mathbf{v}_1 = [1 \quad 1]^{\mathrm{T}}$ and $\mathbf{v}_2 = [1 \quad -1]^{\mathrm{T}}$. Hence a general solution is given by

$$\begin{aligned} x_1(t) &= a_1 \cos t\sqrt{2} + a_2 \sin t\sqrt{2} + b_1 \cos 2t + b_2 \sin 2t, \\ x_2(t) &= a_1 \cos t\sqrt{2} + a_2 \sin t\sqrt{2} - b_1 \cos 2t - b_2 \sin 2t. \end{aligned}$$

The natural frequencies are $\omega_1 = \sqrt{2}$ and $\omega_2 = 2$. In the natural mode with frequency ω_1, the two masses m_1 and m_2 move in the same direction with equal amplitudes of oscillation. At frequency ω_2 they move in opposite directions with equal amplitudes.

6. The matrix $\mathbf{A} = \begin{bmatrix} -6 & 4 \\ 2 & -4 \end{bmatrix}$ has eigenvalues $\lambda_1 = -2$ and $\lambda_2 = -8$ with associated eigenvalues $\mathbf{v}_1 = [1 \quad 1]^{\mathrm{T}}$ and $\mathbf{v}_2 = [2 \quad -1]^{\mathrm{T}}$. Hence a general solution is given by

$$\begin{aligned} x_1(t) &= a_1 \cos t\sqrt{2} + a_2 \sin t\sqrt{2} + 2b_1 \cos t\sqrt{8} + 2b_2 \sin t\sqrt{8}, \\ x_2(t) &= a_1 \cos t\sqrt{2} + a_2 \sin t\sqrt{2} \; - b_1 \cos t\sqrt{8} \; - b_2 \sin t\sqrt{8}. \end{aligned}$$

The natural frequencies are $\omega_1 = \sqrt{2}$ and $\omega_2 = \sqrt{8}$. In the natural mode with frequency ω_1, the two masses m_1 and m_2 move in the same direction with equal amplitudes of oscillation. In the natural mode with frequency ω_2 they move in opposite directions with the amplitude of oscillation of m_1 twice that of m_2.

7. The matrix $\mathbf{A} = \begin{bmatrix} -10 & 6 \\ 6 & -10 \end{bmatrix}$ has eigenvalues $\lambda_1 = -4$ and $\lambda_2 = -16$ with associated eigenvalues $\mathbf{v}_1 = [1 \quad 1]^{\mathrm{T}}$ and $\mathbf{v}_2 = [1 \quad -1]^{\mathrm{T}}$. Hence a general solution is given by

$$\begin{aligned} x_1(t) &= a_1 \cos 2t + a_2 \sin 2t + b_1 \cos 4t + b_2 \sin 4t, \\ x_2(t) &= a_1 \cos 2t + a_2 \sin 2t - b_1 \cos 4t - b_2 \sin 4t. \end{aligned}$$

The natural frequencies are $\omega_1 = 2$ and $\omega_2 = 4$. In the natural mode with frequency ω_1, the two masses m_1 and m_2 move in the same direction with equal amplitudes of oscillation. At frequency ω_2 they move in opposite directions with equal amplitudes.

8. Substitution of the trial solution $x_1 = c_1 \cos 5t$, $x_2 = c_2 \cos 5t$ in the system

$$x_1'' = -5x_1 + 4x_2 + 96\cos 5t, \qquad x_2'' = 4x_1 - 5x_2$$

yields $c_1 = -5$, $c_2 = 1$, so a general solution is given by

$$\begin{aligned} x_1(t) &= a_1 \cos t + a_2 \sin t + b_1 \cos 3t + b_2 \sin 3t - 5\cos 5t, \\ x_2(t) &= a_1 \cos t + a_2 \sin t - b_1 \cos 3t - b_2 \sin 3t + \cos 5t. \end{aligned}$$

Imposition of the initial conditions $x_1(0) = x_2(0) = x_1'(0) = x_2'(0) = 0$ now yields $a_1 = 2$, $a_2 = 0$, $b_1 = 3$, $b_2 = 0$. The resulting particular solution is

$$\begin{aligned} x_1(t) &= 2\cos t + 3\cos 3t - 5\cos 5t, \\ x_2(t) &= 2\cos t - 3\cos 3t + \cos 5t. \end{aligned}$$

We have a superposition of three oscillations, in which the two masses move

- in the same direction with frequency $\omega_1 = 1$ and equal amplitudes;
- in opposite directions with frequency $\omega_2 = 3$ and equal amplitudes;
- in opposite directions with frequency $\omega_3 = 5$ and with the amplitude of motion of m_1 being 5 times that of m_2.

9. Substitution of the trial solution $x_1 = c_1 \cos 3t,\ x_2 = c_2 \cos 3t$ in the system

$$x_1'' = -3x_1 + 2x_2, \qquad 2x_2'' = 2x_1 - 4x_2 + 120\cos 3t$$

yields $c_1 = 3,\ c_2 = -9,$ so a general solution is given by

$$\begin{aligned} x_1(t) &= a_1 \cos t + a_2 \sin t + 2b_1 \cos 2t + 2b_2 \sin 2t + 3\cos 3t, \\ x_2(t) &= a_1 \cos t + a_2 \sin t - b_1 \cos 2t - b_2 \sin 2t - 9\cos 3t. \end{aligned}$$

Imposition of the initial conditions $x_1(0) = x_2(0) = x_1'(0) = x_2'(0) = 0$ now yields $a_1 = 5,\ a_2 = 0,\ b_1 = -4,\ b_2 = 0.$ The resulting particular solution is

$$\begin{aligned} x_1(t) &= 5\cos t - 8\cos 2t + 3\cos 3t, \\ x_2(t) &= 5\cos t + 4\cos 2t - 9\cos 3t. \end{aligned}$$

We have a superposition of three oscillations, in which the two masses move

- in the same direction with frequency $\omega_1 = 1$ and equal amplitudes;
- in opposite directions with frequency $\omega_2 = 2$ and with the amplitude of motion of m_1 being twice that of m_2;
- in opposite directions with frequency $\omega_3 = 3$ and with the amplitude of motion of m_2 being 3 times that of m_1.

10. Substitution of the trial solution $x_1 = c_1 \cos t,\ x_2 = c_2 \cos t$ in the system

$$x_1'' = -10x_1 + 6x_2 + 30\cos t, \qquad x_2'' = 6x_1 - 10x_2 + 60\cos t$$

yields $c_1 = 14,\ c_2 = 16,$ so a general solution is given by

$$\begin{aligned} x_1(t) &= a_1 \cos 2t + a_2 \sin 2t + b_1 \cos 4t + b_2 \sin 4t + 14\cos t, \\ x_2(t) &= a_1 \cos 2t + a_2 \sin 2t - b_1 \cos 4t - b_2 \sin 4t + 16\cos t. \end{aligned}$$

Imposition of the initial conditions $x_1(0) = x_2(0) = x_1'(0) = x_2'(0) = 0$ now yields $a_1 = 1,\ a_2 = 0,\ b_1 = -15,\ b_2 = 0.$ The resulting particular solution is

$$\begin{aligned} x_1(t) &= \cos 2t - 15\cos 4t + 14\cos t, \\ x_2(t) &= \cos 2t + 15\cos 4t + 16\cos t. \end{aligned}$$

We have a superposition of three oscillations, in which the two masses move

- in the same direction with frequency $\omega_1 = 1$ and with the amplitude of motion of m_2 being 8/7 times that of m_1;
- in the same direction with frequency $\omega_2 = 2$ and equal amplitudes;
- in opposite directions with frequency $\omega_3 = 4$ and equal amplitudes.

11. **(a)** The matrix $\mathbf{A} = \begin{bmatrix} -40 & 8 \\ 12 & -60 \end{bmatrix}$ has eigenvalues $\lambda_1 = -36$ and $\lambda_2 = -64$ with associated eigenvalues $\mathbf{v}_2 = [2 \quad 1]^{\mathrm{T}}$ and $\mathbf{v}_2 = [1 \quad -3]^{\mathrm{T}}$. Hence a general solution is given by

$$\begin{aligned} x(t) &= 2a_1 \cos 6t + 2a_2 \sin 6t + b_1 \cos 8t + b_2 \sin 8t, \\ y(t) &= a_1 \cos 6t + a_2 \sin 6t - 3b_1 \cos 8t - 3b_2 \sin 8t. \end{aligned}$$

The natural frequencies are $\omega_1 = 6$ and $\omega_2 = 8$. In mode 1 the two masses oscillate in the same direction with frequency $\omega_1 = 6$ and with the amplitude of motion of m_1 being twice that of m_2. In mode 2 the two masses oscillate in opposite directions with frequency $\omega_2 = 8$ and with the amplitude of motion of m_2 being 3 times that of m_1.

(b) Substitution of the trial solution $x = c_1 \cos 7t$, $y = c_2 \cos 7t$ in the system

$$x'' = -40x + 8y - 195\cos 7t, \qquad y'' = 12x - 60y - 195\cos 7t$$

yields $c_1 = 19$, $c_2 = 3$, so a general solution is given by

$$\begin{aligned} x(t) &= 2a_1 \cos 6t + 2a_2 \sin 6t + b_1 \cos 8t + b_2 \sin 8t + 19\cos 7t, \\ y(t) &= a_1 \cos 6t + a_2 \sin 6t - 3b_1 \cos 8t - 3b_2 \sin 8t + 3\cos 7t. \end{aligned}$$

Imposition of the initial conditions $x(0) = 19$, $x'(0) = 12$, $y(0) = 3$, $y'(0) = 6$ now yields $a_1 = 0$, $a_2 = 1$, $b_1 = 0$, $b_2 = 0$. The resulting particular solution is

$$\begin{aligned} x(t) &= 2\sin 6t + 19\cos 7t, \\ y(t) &= \sin 6t + 3\cos 7t. \end{aligned}$$

Thus the expected oscillation with frequency $\omega_2 = 8$ is missing, and we have a superposition of (only two) oscillations, in which the two masses move

- in the same direction with frequency $\omega_1 = 6$ and with the amplitude of motion of m_1 being twice that of m_2;
- in the same direction with frequency $\omega_3 = 7$ and with the amplitude of motion of m_1 being 19/3 times that of m_2.

12. The coefficient matrix $\mathbf{A} = \begin{bmatrix} -2 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -2 \end{bmatrix}$ has characteristic polynomial

$$\lambda^3 + 6\lambda^2 + 10\lambda + 4 = (\lambda + 2)(\lambda^2 + 4\lambda + 2).$$

Its eigenvalues $\lambda_1 = -2,\ \lambda_2 = -2-\sqrt{2},\ \lambda_3 = -2+\sqrt{2}$ have associated eigenvectors $\mathbf{v}_1 = [1 \quad 0 \quad -1]^T,\ \mathbf{v}_2 = [1 \quad -\sqrt{2} \quad 1]^T,\ \mathbf{v}_3 = [1 \quad \sqrt{2} \quad 1]^T$. Hence the system's three natural modes of oscillation have

- Natural frequency $\omega_1 = \sqrt{2}$ with amplitude ratios $1 : 0 : -1$.
- Natural frequency $\omega_2 = \sqrt{2+\sqrt{2}}$ with amplitude ratios $1 : -\sqrt{2} : 1$.
- Natural frequency $\omega_2 = \sqrt{2-\sqrt{2}}$ with amplitude ratios $1 : \sqrt{2} : 1$.

13. The coefficient matrix $\mathbf{A} = \begin{bmatrix} -4 & 2 & 0 \\ 2 & -4 & 2 \\ 0 & 2 & -4 \end{bmatrix}$ has characteristic polynomial

$$-\lambda^3 - 12\lambda^2 - 40\lambda - 32 = -(\lambda + 4)(\lambda^2 + 8\lambda + 8).$$

Its eigenvalues $\lambda_1 = -4,\ \lambda_2 = -4-2\sqrt{2},\ \lambda_3 = -4+2\sqrt{2}$ have associated eigenvectors $\mathbf{v}_1 = [1 \quad 0 \quad -1]^T,\ \mathbf{v}_2 = [1 \quad -\sqrt{2} \quad 1]^T,\ \mathbf{v}_3 = [1 \quad \sqrt{2} \quad 1]^T$. Hence the system's three natural modes of oscillation have

- Natural frequency $\omega_1 = 2$ with amplitude ratios $1 : 0 : -1$.
- Natural frequency $\omega_2 = \sqrt{4+2\sqrt{2}}$ with amplitude ratios $1 : -\sqrt{2} : 1$.
- Natural frequency $\omega_2 = \sqrt{4-2\sqrt{2}}$ with amplitude ratios $1 : \sqrt{2} : 1$.

14. The equations of motion of the given system are

$$\begin{aligned} x_1'' &= -50x_1 + 10(x_2 - x_1) + 5\cos 10t \\ m_2 x_2'' &= -10(x_2 - x_1). \end{aligned}$$

When we substitute $x_1 = A\cos 10t$, $x_2 = B\cos 10t$ and cancel $\cos 10t$ throughout we get the equations

$$\begin{aligned} -40A - \qquad\quad 10B &= 5 \\ -10A + (10 - 100m_2)B &= 0. \end{aligned}$$

If $m_2 = 0.1$ (slug) then it follows that $A = 0$, so the mass m_1 remains at rest.

15. First we need the general solution of the homogeneous system $\mathbf{x}'' = \mathbf{A}\mathbf{x}$ with

$$\mathbf{A} = \begin{bmatrix} -50 & 25/2 \\ 50 & -50 \end{bmatrix}$$

The eigenvalues of $\mathbf{A}$ are $\lambda_1 = -25$ and $\lambda_2 = -75$, so the natural frequencies of the system are $\omega_1 = 5$ and $\omega_2 = 5\sqrt{3}$. The associated eigenvectors are $\mathbf{v}_1 = [1 \quad 2]^T$ and $\mathbf{v}_2 = [1 \quad -2]^T$, so the complementary solution $\mathbf{x}_c(t)$ is given by

$$\begin{aligned} x_1(t) &= a_1\cos 5t + a_2 \sin 5t \;+\; b_1\cos 5\sqrt{3}t + b_2 \sin 5\sqrt{3}t, \\ x_2(t) &= 2a_1\cos 5t + 2a_2 \sin 5t - 2b_1\cos 5\sqrt{3}t - 2b_2 \sin 5\sqrt{3}t. \end{aligned}$$

When we substitute the trial solution $\mathbf{x}_p(t) = [c_1 \quad c_2]^T \cos 10t$ in the nonhomogeneous system, we find that $c_1 = 4/3$ and $c_2 = -16/3$, so a particular solution $\mathbf{x}_p(t)$ is described by

$$x_1(t) = (4/3)\cos 10t, \qquad x_2(t) = -(16/3)\cos 10t.$$

Finally, when we impose the zero initial conditions on the solution $\mathbf{x}(t) = \mathbf{x}_c(t) + \mathbf{x}_p(t)$ we find that $a_1 = 2/3$, $a_2 = 0$, $b_1 = -2$, and $b_2 = 0$. Thus the solution we seek is described by

$$x_1(t) = \tfrac{2}{3}\cos 5t - 2\cos 5\sqrt{3}t + \tfrac{4}{3}\cos 10t$$

$$x_2(t) = \tfrac{4}{3}\cos 5t + 4\cos 5\sqrt{3}t + \tfrac{16}{3}\cos 10t.$$

We have a superposition of two oscillations with the natural frequencies $\omega_1 = 5$ and $\omega_2 = 5\sqrt{3}$ and a forced oscillation with frequency $\omega = 10$. In each of the two natural oscillations the amplitude of motion of m_2 is twice that of m_1, while in the forced oscillation the amplitude of motion of m_2 is four times that of m_1.

16. The characteristic equation of $\mathbf{A}$ is

$$(-c_1 - \lambda)(-c_2 - \lambda) - c_1c_2 = \lambda^2 + (c_1 + c_2)\lambda = 0,$$

whence the given eigenvalues and eigenvectors follow readily.

17. With $c_1 = c_2 = 2$, it follows from Problem 16 that the natural frequencies and associated eigenvectors are $\omega_1 = 0$, $\mathbf{v}_1 = [1 \quad 1]^T$ and $\omega_2 = 2$, $\mathbf{v}_2 = [1 \quad -1]^T$. Hence Theorem 1 gives the general solution

$$x_1(t) = a_1 + b_1 t + a_2 \cos 2t + b_2 \sin 2t$$
$$x_2(t) = a_1 + b_1 t - a_2 \cos 2t - b_2 \sin 2t.$$

The initial conditions $x_1'(0) = v_0$, $x_1(0) = x_2(0) = x_2'\,0) = 0$ yield $a_1 = a_2 = 0$ and $b_1 = v_0/2$, $b_2 = v_0/4$, so

$$x_1(t) = (v_0/4)(2t + \sin 2t)$$
$$x_2(t) = (v_0/4)(2t - \sin 2t)$$

while $x_2 - x_1 = (v_0/4)(-2 \sin 2t) < 0$, that is, until $t = \pi/2$. Finally, $x_1'(\pi/2) = 0$ and $x_2'(\pi/2) = v_0$.

18. With $c_1 = 6$ and $c_2 = 3$, it follows from Problem 16 that the natural frequencies and associated eigenvectors are $\omega_1 = 0$, $\mathbf{v}_1 = [1 \quad 1]^T$ and $\omega_2 = 3$, $\mathbf{v}_2 = [2 \quad -1]^T$. Hence Theorem 1 gives the general solution

$$x_1(t) = a_1 + b_1 t + 2a_2 \cos 3t + 2b_2 \sin 3t$$
$$x_2(t) = a_1 + b_1 t - a_2 \cos 3t - b_2 \sin 3t.$$

The initial conditions $x_1'(0) = v_0$, $x_1(0) = x_2(0) = x_2'(0) = 0$ yield $a_1 = a_2 = 0$ and $b_1 = v_0/3$, $b_2 = v_0/9$, so

$$x_1(t) = (v_0/9)(3t + 2 \sin 3t)$$
$$x_2(t) = (v_0/9)(3t - \sin 3t)$$

while $x_2 - x_1 = (v_0/9)(-3 \sin 3t) < 0$; that is, until $t = \pi/3$. Finally, $x_1'(\pi/3) = -v_0/3$ and $x_2'(\pi/3) = 2v_0/3$.

19. With $c_1 = 1$ and $c_2 = 3$, it follows from Problem 16 that the natural frequencies and associated eigenvectors are $\omega_1 = 0$, $\mathbf{v}_1 = [1 \quad 1]^T$ and $\omega_2 = 2$, $\mathbf{v}_2 = [1 \quad -3]^T$. Hence Theorem 1 gives the general solution

$$x_1(t) = a_1 + b_1 t + a_2 \cos 2t + b_2 \sin 2t$$
$$x_2(t) = a_1 + b_1 t - 3a_2 \cos 2t - 3b_2 \sin 2t.$$

The initial conditions $x_1'(0) = v_0$, $x_1(0) = x_2(0) = x_2'(0) = 0$ yield $a_1 = a_2 = 0$ and $b_1 = 3v_0/4$, $b_2 = v_0/8$, so

$$x_1(t) = (v_0/8)(6t + \sin 2t)$$
$$x_2(t) = (v_0/8)(6t - 3 \sin 2t)$$

while $x_2 - x_1 = (v_0/8)(-4 \sin 2t) < 0$; that is, until $t = \pi/2$. Finally, $x_1'(\pi/2) = v_0/2$ and $x_2'(\pi/2) = 3v_0/2$.

20. With $c_1 = c_3 = 4$ and $c_2 = 16$ the characteristic equation of the matrix

$$\mathbf{A} = \begin{bmatrix} -4 & 4 & 0 \\ 16 & -32 & 16 \\ 0 & 4 & -4 \end{bmatrix}$$

is

$$\lambda^3 + 40\lambda^2 + 144\lambda = \lambda(\lambda+4)(\lambda+36) = 0.$$

The resulting eigenvalues, natural frequencies, and associated eigenvectors are

$$\begin{aligned} \lambda_1 &= 0, & \omega_1 &= 0, & \mathbf{v}_1 &= [1 \quad 1 \quad 1]^T \\ \lambda_2 &= -4, & \omega_2 &= 2, & \mathbf{v}_2 &= [1 \quad 0 \quad -1]^T \\ \lambda_3 &= -36, & \omega_3 &= 6, & \mathbf{v}_3 &= [1 \quad -8 \quad 1]^T. \end{aligned}$$

Theorem 1 then gives the general solution

$$\begin{aligned} x_1(t) &= a_1 + b_1 t + a_2\cos 2t + b_2\sin 2t + a_3\cos 6t + b_3\sin 6t \\ x_2(t) &= a_1 + b_1 t \qquad\qquad\qquad\qquad\quad - 8a_3\cos 6t - 8b_3\sin 6t \\ x_3(t) &= a_1 + b_1 t - a_2\cos 2t - b_2\sin 2t + a_3\cos 6t + b_3\sin 6t. \end{aligned}$$

The initial conditons yield $a_1 = a_2 = a_3 = 0$ and $b_1 = 4v_0/9$, $b_2 = v_0/4$, $b_3 = v_0/108$, so

$$\begin{aligned} x_1(t) &= (v_0/108)(48t + 27\sin 2t + \sin 6t) \\ x_2(t) &= (v_0/108)(48t \qquad\qquad\quad - 8\sin 6t) \\ x_3(t) &= (v_o/108)(48t - 27\sin 2t + \sin 6t) \end{aligned}$$

while

$$\begin{aligned} x_2 - x_1 &= -18(\sin 2t)(3 - 2\sin^3 2t) < 0, \\ x_3 - x_2 &= -9(4\sin^3 2t) < 0; \end{aligned}$$

that is, until $t = \pi/2$. Finally

$$x_1'(\pi/2) = -v_0/9, \quad x_2'(\pi/2) = 8v_0/9, \quad x_3'(\pi/2) = 8v_0/9.$$

21. **(a)** The matrix

$$\mathbf{A} = \begin{bmatrix} -160/3 & 320/3 \\ 8 & -116 \end{bmatrix}$$

has eigenvalues $\lambda_1 \approx -41.8285$ and $\lambda_2 \approx -127.5049$, so the natural frequencies are

$$\omega_1 \approx 6.4675 \text{ rad/sec} \approx 1.0293 \text{ Hz}$$
$$\omega_2 \approx 11.2918 \text{ rad/sec} \approx 1.7971 \text{ Hz.}$$

(b) Resonance occurs at the two critical speeds

$$v_1 = 20\omega_1/\pi \approx 41 \text{ ft/sec} \approx 28 \text{ mi/h}$$
$$v_2 = 20\omega_2/\pi \approx 72 \text{ ft/sec} \approx 49 \text{ mi/h.}$$

22. With $k_1 = k_2 = k$ and $L_1 = L_2 = L/2$ the equations in (42) reduce to

$$mx'' = -2kx \quad \text{and} \quad I\theta'' = -kL^2\theta/2.$$

The first equation yields $\omega_1 = \sqrt{2k/m}$ and the second one yields $\omega_2 = \sqrt{kL^2/2I}.$

In Problems 23–25 we substitute the given physical parameters into the equations in (42):

$$mx'' = -(k_1 + k_2)x + (k_1L_1 - k_2L_2)\theta$$

$$I\theta'' = (k_1L_1 - k_2L_2)x - (k_1L_1^2 + k_2L_2^2)\theta$$

As in Problem 21, a critical frequency of ω rad/sec yields a critical velocity of $v = 20\omega/\pi$ ft/sec.

23. $100x'' = -4000x, \quad 800\theta'' = 100000\theta$

Obviously the matrix $\mathbf{A} = \begin{bmatrix} -40 & 0 \\ 0 & -125 \end{bmatrix}$ has eigenvalues $\lambda_1 = -40$ and $\lambda_2 = -125$.

Up-and-down: $\omega_1 = \sqrt{40}$, $v_1 \approx 40.26$ ft/sec ≈ 27 mph

Angular: $\omega_2 = \sqrt{125}$, $v_2 \approx 71.18$ ft/sec ≈ 49 mph

24.
$$100x'' = -4000x + 4000\theta$$
$$1000\theta'' = 4000x - 104000\theta$$

The matrix $\mathbf{A} = \begin{bmatrix} -40 & 40 \\ 4 & -104 \end{bmatrix}$ has eigenvalues $\lambda_1, \lambda_2 = 4\left(-18 \pm \sqrt{74}\right)$.

$$\omega_1 \approx 6.1311, \qquad v_1 \approx 39.03 \text{ ft/sec} \approx 27 \text{ mph}$$
$$\omega_2 \approx 10.3155, \qquad v_2 \approx 65.67 \text{ ft/sec} \approx 45 \text{ mph}$$

25.
$$\begin{aligned} 100x'' &= -3000x - 5000\theta \\ 800\theta'' &= -5000x - 75000\theta \end{aligned}$$

The matrix $\mathbf{A} = \begin{bmatrix} -30 & -50 \\ -25/4 & -375/4 \end{bmatrix}$ has eigenvalues $\lambda_1, \lambda_2 = \frac{5}{8}\left(-99 \pm \sqrt{3401}\right)$.

$$\omega_1 \approx 5.0424, \qquad v_1 \approx 32.10 \text{ ft/sec} \approx 22 \text{ mph}$$
$$\omega_2 \approx 9.9158, \qquad v_2 \approx 63.13 \text{ ft/sec} \approx 43 \text{ mph}$$

SECTION 7.5

MULTIPLE EIGENVALUE SOLUTIONS

In each of Problems 1–6 we give first the characteristic equation with repeated (multiplicity 2) eigenvalue λ. In each case we find that $(\mathbf{A} - \lambda\mathbf{I})^2 = \mathbf{0}$. Then $\mathbf{w} = [1 \quad 0]^T$ is a generalized eigenvector and $\mathbf{v} = (\mathbf{A} - \lambda\mathbf{I})\mathbf{w} \neq \mathbf{0}$ is an ordinary eigenvector associated with λ. We give finally the scalar component functions $x_1(t)$, $x_2(t)$ of the general solution

$$\mathbf{x}(t) = c_1\mathbf{v}e^{\lambda t} + c_2(\mathbf{v}t + \mathbf{w})e^{\lambda t}$$

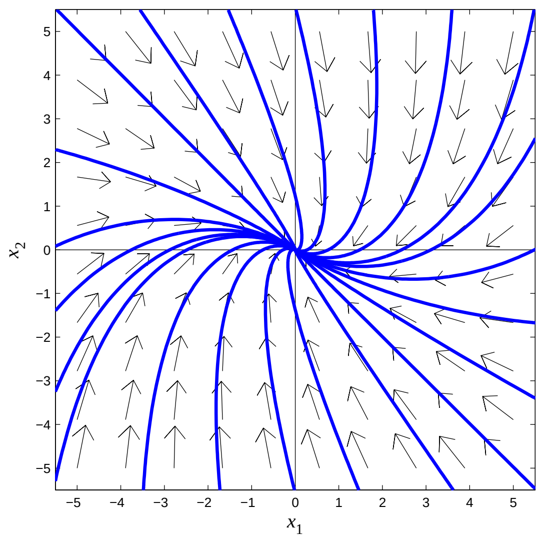
of the given system $\mathbf{x}' = \mathbf{A}\mathbf{x}$.

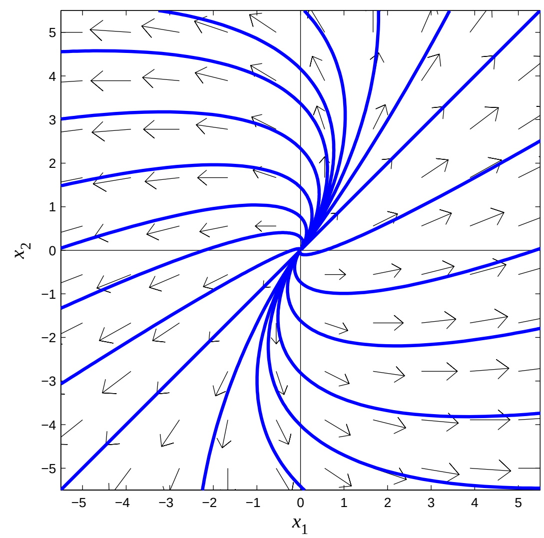
1. Characteristic equation $\qquad \lambda^2 + 6\lambda + 9 = 0$
Repeated eigenvalue $\qquad \lambda = -3$
Generalized eigenvector $\qquad \mathbf{w} = [1 \quad 0]^T$

$$\mathbf{v} = (\mathbf{A} - \lambda\mathbf{I})\mathbf{w} = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$x_1(t) = (c_1 + c_2 + c_2 t)e^{-3t}$$
$$x_2(t) = (-c_1 - c_2 t)e^{-3t}.$$

The left-hand figure at the top of the next page shows a direction field and typical solution curves.





2. Characteristic equation $\lambda^2 - 4\lambda + 4 = 0$
Repeated eigenvalue $\lambda = 2$
Generalized eigenvector $\mathbf{w} = [1 \quad 0]^T$

$$\mathbf{v} = (\mathbf{A} - \lambda\mathbf{I})\mathbf{w} = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$x_1(t) = (c_1 + c_2 + c_2 t)e^{2t}$$
$$x_2(t) = (c_1 + \qquad c_2 t)e^{2t}.$$

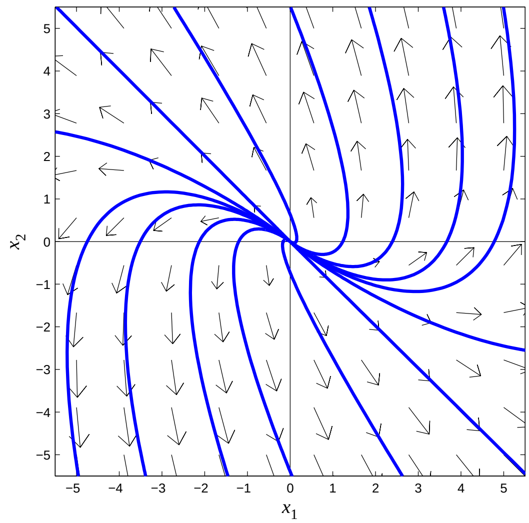
The right-hand figure above shows a direction field and typical solution curves.

3. Characteristic equation $\lambda^2 - 6\lambda + 9 = 0$
Repeated eigenvalue $\lambda = 3$
Generalized eigenvector $\mathbf{w} = [\,1 \quad 0]^T$

$$\mathbf{v} = (\mathbf{A} - \lambda\mathbf{I})\mathbf{w} = \begin{bmatrix} -2 & -2 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \end{bmatrix}$$

$$x_1(t) = (-2c_1 + c_2 - 2c_2 t)e^{3t}$$
$$x_2(t) = (\ 2c_1 + \qquad 2c_2 t)e^{3t}.$$

The figure at the top of the next page shows a direction field and typical solution curves.



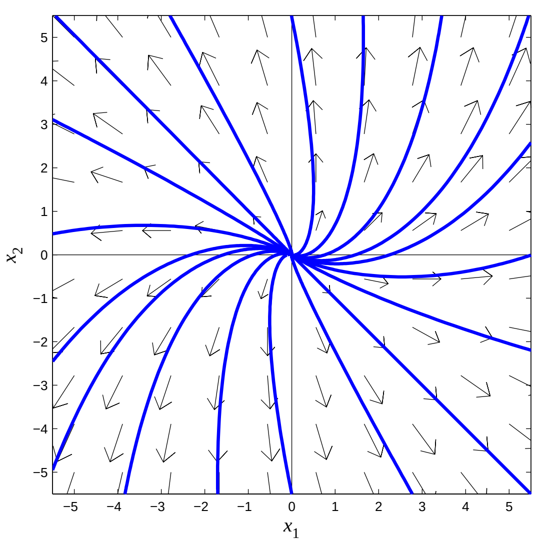
4. Characteristic equation $\lambda^2 - 8\lambda + 16 = 0$
Repeated eigenvalue $\lambda = 4$
Generalized eigenvector $\mathbf{w} = [\,1 \quad 0]^T$

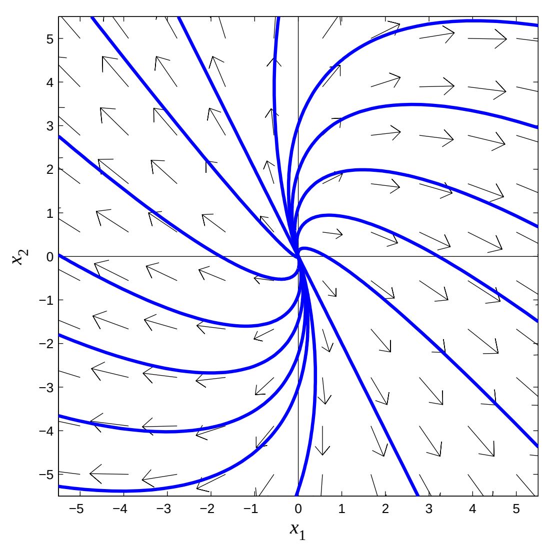
$$\mathbf{v} = (\mathbf{A} - \lambda\mathbf{I})\mathbf{w} = \begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$x_1(t) = (-c_1 + c_2 - c_2 t)e^{4t}$$

$$x_2(t) = (\,c_1 + \qquad c_2 t)e^{4t}.$$

The left-hand figure below shows a direction field and typical solution curves.





5. Characteristic equation $\lambda^2 - 10\lambda + 25 = 0$
Repeated eigenvalue $\lambda = 5$
Generalized eigenvector $\mathbf{w} = [1 \quad 0]^T$

$$\mathbf{v} = (\mathbf{A} - \lambda \mathbf{I})\mathbf{w} = \begin{bmatrix} 2 & 1 \\ -4 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ -4 \end{bmatrix}$$

$$x_1(t) = (2c_1 + c_2 + 2c_2 t)e^{5t}$$

$$x_2(t) = (-4c_1 \quad - 4c_2 t)e^{5t}.$$

The right-hand figure at the bottom of the preceding page shows a direction field and typical solution curves.

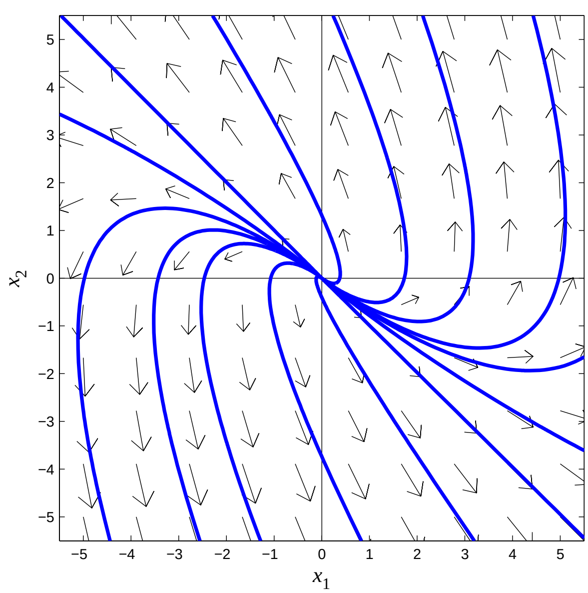
6. Characteristic equation $\lambda^2 - 10\lambda + 25 = 0$
Repeated eigenvalue $\lambda = 5$
Generalized eigenvector $\mathbf{w} = [1 \quad 0]^T$

$$\mathbf{v} = (\mathbf{A} - \lambda \mathbf{I})\mathbf{w} = \begin{bmatrix} -4 & -4 \\ 4 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -4 \\ 4 \end{bmatrix}$$

$$x_1(t) = (-4c_1 + c_2 - 4c_2 t)e^{5t}$$

$$x_2(t) = (4c_1 \quad + 4c_2 t)e^{5t}.$$

The figure below shows a direction field and typical solution curves.



In each of Problems 7–10 the characteristic polynomial is easily calculated by expansion along the row or column of **A** that contains two zeros. The matrix **A** has only two distinct eigenvalues, so we write $\lambda_1, \lambda_2, \lambda_3$ with either $\lambda_1 = \lambda_2$ or $\lambda_2 = \lambda_3$. Nevertheless, we find that it has 3 linearly independent eigenvectors $\mathbf{v}_1, \mathbf{v}_2$, and $\mathbf{v}_3$. We list also the scalar components $x_1(t), x_2(t), x_3(t)$ of the general solution $\mathbf{x}(t) = c_1\mathbf{v}_1e^{\lambda_1 t} + c_2\mathbf{v}_2e^{\lambda_2 t} + c_3\mathbf{v}_3e^{\lambda_3 t}$ of the system.

7. Characteristic equation $-\lambda^3 + 13\lambda^2 - 40\lambda + 36 = -(\lambda-2)^2(\lambda-9)$
Eigenvalues $\lambda = 2,\ 2,\ 9$
Eigenvectors $[1\quad 1\quad 0]^T,\ [1\quad 0\quad 1]^T,\ [0\quad 1\quad 0]^T$

$$\begin{aligned} x_1(t) &= c_1e^{2t} + c_2e^{2t} \\ x_2(t) &= c_1e^{2t} + c_3e^{9t} \\ x_3(t) &= c_2e^{2t} \end{aligned}$$

8. Characteristic equation $-\lambda^3 + 33\lambda^2 - 351\lambda + 1183 = -(\lambda-13)^2(\lambda-7)$
Eigenvalues $\lambda = 7,\ 13,\ 13$
Eigenvectors $[2\quad -3\quad 1]^T,\ [0\quad 0\quad 1]^T,\ [-1\quad 1\quad 0]^T,$

$$\begin{aligned} x_1(t) &= 2c_1e^{7t} - c_3e^{13t} \\ x_2(t) &= -3c_1e^{7t} + c_3e^{13t} \\ x_3(t) &= c_1e^{7t} + c_2e^{13t} \end{aligned}$$

9. Characteristic equation $-\lambda^3 + 19\lambda^2 - 115\lambda + 225 = -(\lambda-5)^2(\lambda-9)$
Eigenvalues $\lambda = 5,\ 5,\ 9$
Eigenvectors $[1\quad 2\quad 0]^T,\ [7\quad 0\quad 2]^T,\ [3\quad 0\quad 1]^T$

$$\begin{aligned} x_1(t) &= c_1e^{5t} + 7c_2e^{5t} + 3c_3e^{9t} \\ x_2(t) &= 2c_1e^{5t} \\ x_3(t) &= 2c_2e^{5t} + c_3e^{9t} \end{aligned}$$

10. Characteristic equation $-\lambda^3 + 13\lambda^2 - 51\lambda + 63 = -(\lambda-3)^2(\lambda-7)$
Eigenvalues $\lambda = 3,\ 3,\ 7$
Eigenvectors $[5\quad 2\quad 0]^T,\ [-3\quad 0\quad 1]^T,\ [2\quad 1\quad 0]^T$

$$\begin{aligned} x_1(t) &= 5c_1e^{3t} - 3c_2e^{3t} + 2c_3e^{7t} \\ x_2(t) &= 2c_1e^{3t} + c_3e^{7t} \\ x_3(t) &= c_2e^{3t} \end{aligned}$$

In each of Problems 11–14, the characteristic equation is $-\lambda^3 - 3\lambda^2 - 3\lambda - 1 = -(\lambda+1)^3$. Hence $\lambda = -1$ is a triple eigenvalue of defect 2, and we find that $(\mathbf{A}-\lambda\mathbf{I})^3 = \mathbf{0}$. In each problem we start with $\mathbf{v}_3 = [1 \;\; 0 \;\; 0]^T$ and then calculate $\mathbf{v}_2 = (\mathbf{A}-\lambda\mathbf{I})\mathbf{v}_3$ and $\mathbf{v}_1 = (\mathbf{A}-\lambda\mathbf{I})\mathbf{v}_2 \neq 0$. It follows that $(\mathbf{A}-\lambda\mathbf{I})\mathbf{v}_1 = (\mathbf{A}-\lambda\mathbf{I})^2\mathbf{v}_2 = (\mathbf{A}-\lambda\mathbf{I})^3\mathbf{v}_3 = \mathbf{0}$, so the vector $\mathbf{v}_1$ (if nonzero) is an ordinary eigenvector associated with the triple eigenvalue λ. Hence $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a length 3 chain of generalized eigenvectors, and the corresponding general solution is described by

$$\mathbf{x}(t) = e^{-t}[c_1\mathbf{v}_1 + c_2(\mathbf{v}_1 t + \mathbf{v}_2) + c_3(\mathbf{v}_1 t^2/2 + \mathbf{v}_2 t + \mathbf{v}_3)].$$

We give the scalar components $x_1(t)$, $x_2(t)$, $x_3(t)$ of $\mathbf{x}(t)$.

11. $\mathbf{v}_1 = [0 \;\; 1 \;\; 0]^T$, $\mathbf{v}_2 = [-2 \;\; -1 \;\; 1]^T$, $\mathbf{v}_3 = [1 \;\; 0 \;\; 0]^T$

$x_1(t) = e^{-t}(-2c_2 + c_3 - 2c_3 t)$

$x_2(t) = e^{-t}(c_1 - c_2 + c_2 t - c_3 t + c_3 t^2/2)$

$x_3(t) = e^{-t}(c_2 + c_3 t)$

12. $\mathbf{v}_1 = [1 \;\; 1 \;\; 0]^T$, $\mathbf{v}_2 = [0 \;\; 0 \;\; 1]^T$, $\mathbf{v}_3 = [1 \;\; 0 \;\; 0]^T$

$x_1(t) = e^{-t}(c_1 + c_3 + c_2 t + c_3 t^2/2)$

$x_2(t) = e^{-t}(c_1 + c_2 t + c_3 t^2/2)$

$x_3(t) = e^{-t}(c_2 + c_3 t)$

13. Here we are stymied initially, because if $\mathbf{v}_3 = [1 \;\; 0 \;\; 0]^T$ then $(\mathbf{A}-\lambda\mathbf{I})\mathbf{v}_3 = \mathbf{0}$ does not qualify as a (nonzero) generalized eigenvector. We there make a fresh start with $\mathbf{v}_3 = [0 \;\; 1 \;\; 0]^T$, and now we get the desired nonzero generalized eigenvectors upon successive multiplication by $\mathbf{A}-\lambda\mathbf{I}$.

$\mathbf{v}_1 = [1 \;\; 0 \;\; 0]^T$, $\mathbf{v}_2 = [0 \;\; 2 \;\; 1]^T$, $\mathbf{v}_3 = [0 \;\; 1 \;\; 0]^T$

$x_1(t) = e^{-t}(c_1 + c_2 t + c_3 t^2/2)$

$x_2(t) = e^{-t}(2c_2 + c_3 + 2c_3 t)$

$x_3(t) = e^{-t}(c_2 + c_3 t)$

14. $\mathbf{v}_1 = [5 \;\; -25 \;\; -5]^T$, $\mathbf{v}_2 = [1 \;\; -5 \;\; 4]^T$, $\mathbf{v}_3 = [1 \;\; 0 \;\; 0]^T$

$$x_1(t) = e^{-t}(5c_1 + c_2 + c_3 + 5c_2 t + c_3 t + 5c_3 t^2/2)$$

$$x_2(t) = e^{-t}(-25c_1 - 5c_2 - 25c_2 t - 5c_3 t - 25c_3 t^2/2)$$

$$x_3(t) = e^{-t}(-5c_1 + 4c_2 - 5c_2 t + 4c_3 t - 5c_3 t^2/2)$$

In each of Problems 15–18, the characteristic equation is $-\lambda^3 + 3\lambda^2 - 3\lambda + 1 = -(\lambda - 1)^3$. Hence $\lambda = 1$ is a triple eigenvalue of defect 1, and we find that $(\mathbf{A} - \lambda\mathbf{I})^2 = \mathbf{0}$. First we find the two linearly independent (ordinary) eigenvectors $\mathbf{u}_1$ and $\mathbf{u}_2$ associated with λ. Then we start with $\mathbf{v}_2 = [1 \quad 0 \quad 0]^T$ and calculate $\mathbf{v}_1 = (\mathbf{A} - \lambda\mathbf{I})\mathbf{v}_2 \neq 0$. It follows that $(\mathbf{A} - \lambda\mathbf{I})\mathbf{v}_1 = (\mathbf{A} - \lambda\mathbf{I})^2\mathbf{v}_2 = \mathbf{0}$, so $\mathbf{v}_1$ is an ordinary eigenvector associated with λ. However, $\mathbf{v}_1$ is a linear combination of $\mathbf{u}_1$ and $\mathbf{u}_2$, so $\mathbf{v}_1 e^t$ is a linear combination of the independent solutions $\mathbf{u}_1 e^t$ and $\mathbf{u}_2 e^t$. But $\{\mathbf{v}_1, \mathbf{v}_2\}$ is a length 2 chain of generalized eigenvectors associated with λ, so $(\mathbf{v}_1 t + \mathbf{v}_2)e^t$ is the desired third independent solution. The corresponding general solution is described by

$$\mathbf{x}(t) = e^t [c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + c_3(\mathbf{v}_1 t + \mathbf{v}_2)]$$

We give the scalar components $x_1(t), x_2(t), x_3(t)$ of $\mathbf{x}(t)$.

15. $\mathbf{u}_1 = [3 \quad -1 \quad 0]^T \qquad \mathbf{u}_2 = [0 \quad 0 \quad 1]^T$

$\mathbf{v}_1 = [-3 \quad 1 \quad 1]^T \qquad \mathbf{v}_2 = [1 \quad 0 \quad 0]^T$

$$x_1(t) = e^t(3c_1 + c_3 - 3c_3 t)$$

$$x_2(t) = e^t(-c_1 + c_3 t)$$

$$x_3(t) = e^t(c_2 + c_3 t)$$

16. $\mathbf{u}_1 = [3 \quad -2 \quad 0]^T \qquad \mathbf{u}_2 = [3 \quad 0 \quad -2]^T$

$\mathbf{v}_1 = [0 \quad -2 \quad 2]^T \qquad \mathbf{v}_2 = [1 \quad 0 \quad 0]^T$

$$x_1(t) = e^t(3c_1 + 3c_2 + c_3)$$

$$x_2(t) = e^t(-2c_1 - 2c_3 t)$$

$$x_3(t) = e^t(-2c_2 + 2c_3 t)$$

17. $\mathbf{u}_1 = [2 \quad 0 \quad -9]^T \qquad \mathbf{u}_2 = [1 \quad -3 \quad 0]^T$

$\mathbf{v}_1 = [0 \quad 6 \quad -9]^T \qquad \mathbf{v}_2 = [0 \quad 1 \quad 0]^T$

(Either $\mathbf{v}_2 = [1 \quad 0 \quad 0]^T$ or $\mathbf{v}_2 = [0 \quad 0 \quad 1]^T$ can be used also, but they yield different forms of the solution than given in the book's answer section.)

$x_1(t) = e^t(2c_1 + c_2)$

$x_2(t) = e^t(-3c_2 + c_3 + 6c_3 t)$

$x_3(t) = e^t(-9c_1 - 9c_3 t)$

18. $\mathbf{u}_1 = [-1 \quad 0 \quad 1]^T \quad \mathbf{u}_2 = [-2 \quad 1 \quad 0]^T$

$\mathbf{v}_1 = [0 \quad 1 \quad -2]^T \quad \mathbf{v}_2 = [1 \quad 0 \quad 0]^T$

$x_1(t) = e^t(-c_1 - 2c_2 + c_3)$

$x_2(t) = e^t(c_2 + c_3 t)$

$x_3(t) = e^t(c_1 - 2c_3 t)$

19. Characteristic equation $\lambda^4 - 2\lambda^2 + 1 = 0$

Double eigenvalue $\lambda = -1$ with eigenvectors

$$\mathbf{v}_1 = [1 \quad 0 \quad 0 \quad 1]^T \text{ and } \mathbf{v}_2 = [0 \quad 0 \quad 1 \quad 0]^T.$$

Double eigenvalue $\lambda = +1$ with eigenvectors

$$\mathbf{v}_3 = [0 \quad 1 \quad 0 \quad -2]^T \text{ and } \mathbf{v}_4 = [1 \quad 0 \quad 3 \quad 0]^T.$$

General solution

$$\mathbf{x}(t) = e^{-t}(c_1\mathbf{v}_1 + c_2\mathbf{v}_2) + e^t(c_3\mathbf{v}_3 + c_4\mathbf{v}_4)$$

Scalar components

$$x_1(t) = c_1e^{-t} + c_4e^t$$

$$x_2(t) = c_3e^t$$

$$x_3(t) = c_2e^{-t} + 3c_4e^t$$

$$x_4(t) = c_1e^{-t} - 2c_3e^t$$

20. Characteristic equation $\lambda^4 - 8\lambda^3 + 24\lambda^2 - 32\lambda + 16 = (\lambda - 2)^4 = 0$

Eigenvalue $\lambda = 2$ with multiplicity 4 and defect 3.

We find that $(\mathbf{A} - \lambda\mathbf{I})^3 \neq 0$ but $(\mathbf{A} - \lambda\mathbf{I})^4 = 0$. We therefore start with $\mathbf{v}_4 = [0 \quad 0 \quad 0 \quad 1]^T$ and define $\mathbf{v}_3 = (\mathbf{A} - \lambda\mathbf{I})\mathbf{v}_4$, $\mathbf{v}_2 = (\mathbf{A} - \lambda\mathbf{I})\mathbf{v}_3$, $\mathbf{v}_2 = (\mathbf{A} - \lambda\mathbf{I})\mathbf{v}_3$, and $\mathbf{v}_1 = (\mathbf{A} - \lambda\mathbf{I})\mathbf{v}_2 \neq 0$. This gives the length 4 chain $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ with

$$\mathbf{v}_1 = [1 \quad 0 \quad 0 \quad 0]^T \qquad \mathbf{v}_2 = [0 \quad 1 \quad 0 \quad 0]^T$$
$$\mathbf{v}_3 = [1 \quad 0 \quad 1 \quad 0]^T \qquad \mathbf{v}_4 = [0 \quad 0 \quad 0 \quad 1]^T.$$

The corresponding general solution is given by

$$\mathbf{x}(t) = e^{-t}\,[c_1\mathbf{v}_1 + c_2(\mathbf{v}_1 t + \mathbf{v}_2) + c_3(\mathbf{v}_1 t^2/2 + \mathbf{v}_2 t + \mathbf{v}_3) + c_4(\mathbf{v}_1 t^3/6 + \mathbf{v}_2 t^2/2 + \mathbf{v}_3 t + \mathbf{v}_4)]$$

with scalar components

$$\begin{aligned}
x_1(t) &= e^{2t}(c_1 + c_3 + c_2 t + c_4 t + c_3 t^2/2 + c_4 t^3/6)\\
x_2(t) &= e^{2t}(c_2 + c_3 t + c_4 t^2/2)\\
x_3(t) &= e^{2t}(c_3 + c_4 t)\\
x_4(t) &= e^{2t}(c_4).
\end{aligned}$$

21. Characteristic equation $\lambda^4 - 4\lambda^3 + 6\lambda^2 - 4\lambda + 1 = (\lambda - 1)^4 = 0$
Eigenvalue $\lambda = 1$ with multiplicity 4 and defect 2.

We find that $(\mathbf{A} - \lambda\mathbf{I})^2 \neq 0$ but $(\mathbf{A} - \lambda\mathbf{I})^3 = 0$. We therefore start with $\mathbf{v}_3 = [1\ \ 0\ \ 0\ \ 0]^T$ and define $\mathbf{v}_2 = (\mathbf{A} - \lambda\mathbf{I})\mathbf{v}_3$ and $\mathbf{v}_1 = (\mathbf{A} - \lambda\mathbf{I})\mathbf{v}_2 \neq 0$, thereby obtaining the length 3 chain $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ with

$$\mathbf{v}_1 = [0\ \ 0\ \ 0\ \ 1]^T, \quad \mathbf{v}_2 = [-2\ \ 1\ \ 1\ \ 0]^T, \quad \mathbf{v}3 = [1\ \ 0\ \ 0\ \ 0]^T.$$

Then we find the second ordinary eigenvector $\mathbf{v}_4 = [0\ \ 0\ \ 1\ \ 0]^T$. The corresponding general solution

$$\mathbf{x}(t) = e^{t}\,[c_1\mathbf{v}_1 + c_2(\mathbf{v}_1 t + \mathbf{v}_2) + c_3(\mathbf{v}_1 t^2/2 + \mathbf{v}_2 t + \mathbf{v}_3) + c_4\mathbf{v}_4]$$

has scalar components

$$\begin{aligned}
x_1(t) &= e^{t}(-2c_2 + c_3 - 2c_3 t)\\
x_2(t) &= e^{t}(c_2 + c_3 t)\\
x_3(t) &= e^{t}(c_2 + c_4 + c_3 t).\\
x_4(t) &= e^{t}(c_1 + c_2 t + c_3 t^2/2.)
\end{aligned}$$

22. Same eigenvalue and chain structure as in Problem 21, but with generalized eigenvectors

$$\begin{aligned}
\mathbf{v}_1 &= [1\ \ 0\ \ 0\ \ -2]^T & \mathbf{v}_2 &= [3\ \ -2\ \ 1\ \ -6]^T\\
\mathbf{v}_3 &= [0\ \ 1\ \ 0\ \ 0]^T & \mathbf{v}_4 &= [1\ \ 0\ \ 0\ \ 0]^T
\end{aligned}$$

where $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a length 3 chain and $\mathbf{v}_4$ is an ordinary eigenvector. The general solution $\mathbf{x}(t)$ defined as in Problem 21 has scalar components

$$
\begin{aligned}
x_1(t) &= e^t(c_1 + 3c_2 + c_4 + c_2 t + 3c_3 t + c_3 t^2/2) \\
x_2(t) &= e^t(-2c_2 + c_3 - 2c_3 t) \\
x_3(t) &= e^t(c_2 + c_3 t) \\
x_4(t) &= e^t(-2c_1 - 6c_2 - 2c_2 t - 6c_3 t - c_3 t^2)
\end{aligned}
$$

In Problems 23 and 24 there are only two distinct eigenvalues λ_1 and λ_2. However, the eigenvector equation $(\mathbf{A} - \lambda\mathbf{I})\mathbf{v} = 0$ yields the three linearly independent eigenvectors $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ that are given. We list the scalar components of the corresponding general solution $\mathbf{x}(t) = c_1\mathbf{v}_1 e^{\lambda_1 t} + c_2\mathbf{v}_2 e^{\lambda_2 t} + c_3\mathbf{v}_3 e^{\lambda_2 t}$.

23. $\lambda_1 = -1$: $\{\mathbf{v}_1\}$ with $\mathbf{v}_1 = [1 \quad -1 \quad 2]^T$

$\lambda_2 = 3$: $\{\mathbf{v}_2\}$ with $\mathbf{v}_2 = [4 \quad 0 \quad 9]^T$ and

$\{\mathbf{v}_3\}$ with $\mathbf{v}_3 = [0 \quad 2 \quad 1]^T$

Scalar components

$$
\begin{aligned}
x_1(t) &= c_1 e^{-t} + 4c_2 e^{3t} \\
x_2(t) &= -c_1 e^{-t} + 2c_3 e^{3t} \\
x_3(t) &= 2c_1 e^{-t} + 9c_2 e^{3t} + c_3 e^{3t}
\end{aligned}
$$

24. $\lambda_1 = -2$: $\{\mathbf{v}_1\}$ with $\mathbf{v}_1 = [5 \quad 3 \quad -3]^T$

$\lambda_2 = 3$: $\{\mathbf{v}_2\}$ with $\mathbf{v}_2 = [4 \quad 0 \quad -1]^T$ and

$\{\mathbf{v}_3\}$ with $\mathbf{v}_3 = [2 \quad -1 \quad 0]^T$

Scalar components

$$
\begin{aligned}
x_1(t) &= 5c_1 e^{-2t} + 4c_2 e^{3t} + 2c_3 e^{3t} \\
x_2(t) &= 3c_1 e^{-2t} - c_3 e^{3t} \\
x_3(t) &= -3c_1 e^{-2t} - c_2 e^{3t}
\end{aligned}
$$

In Problems 25, 26, and 28 there is given a single eigenvalue λ of multiplicity 3. We find that $(\mathbf{A} - \lambda\mathbf{I})^2 \neq 0$ but $(\mathbf{A} - \lambda\mathbf{I})^3 = 0$. We therefore start with $\mathbf{v}_3 = [1 \quad 0 \quad 0]^T$ and define $\mathbf{v}_2 = (\mathbf{A} - \lambda\mathbf{I})\mathbf{v}_3$ and $\mathbf{v}_1 = (\mathbf{A} - \lambda\mathbf{I})\mathbf{v}_2 \neq 0$, thereby obtaining the length 3 chain $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ of generalized eigenvectors based on the ordinary eigenvector $\mathbf{v}_1$. We list the scalar components of the corresponding general solution

$$
\mathbf{x}(t) = c_1\mathbf{v}_1 e^{\lambda t} + c_2(\mathbf{v}_1 t + \mathbf{v}_2)e^{\lambda t} + c_3(\mathbf{v}_1 t^2/2 + \mathbf{v}_2 t + \mathbf{v}_3)e^{\lambda t}.
$$

25. $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ with

$$\mathbf{v}_1 = [-1 \quad 0 \quad -1]^T, \quad \mathbf{v}_2 = [-4 \quad -1 \quad 0]^T, \quad \mathbf{v}_3 = [1 \quad 0 \quad 0]^T$$

Scalar components

$$\begin{aligned} x_1(t) &= e^{2t}(-c_1 - 4c_2 + c_3 - c_2 t - 4c_3 t - c_3 t^2/2) \\ x_2(t) &= e^{2t}(-c_2 - c_3 t) \\ x_3(t) &= e^{2t}(-c_1 - c_2 t - c_3 t^2/2) \end{aligned}$$

26. $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ with

$$\mathbf{v}_1 = [0 \quad 2 \quad 2]^T, \quad \mathbf{v}_2 = [2 \quad 1 \quad -3]^T, \quad \mathbf{v}_3 = [1 \quad 0 \quad 0]^T$$

General solution

$$\begin{aligned} x_1(t) &= e^{3t}(2c_2 + c_3 + 2c_3 t) \\ x_2(t) &= e^{3t}(2c_1 + c_2 + 2c_2 t + c_3 t + c_3 t^2) \\ x_3(t) &= e^{3t}(2c_1 - 3c_2 + 2c_2 t - 3c_3 t + c_3 t^2) \end{aligned}$$

27. We find that the triple eigenvalue $\lambda = 2$ has the two linearly independent eigenvectors $[1 \quad 1 \quad 0]^T$ and $[-1 \quad 0 \quad 1]^T$. Next we find that $(\mathbf{A} - \lambda\mathbf{I}) \neq 0$ but $(\mathbf{A} - \lambda\mathbf{I})^2 = 0$. We therefore start with $\mathbf{v}_2 = [1 \quad 0 \quad 0]^T$ and define

$$\mathbf{v}_1 = (\mathbf{A} - \lambda\mathbf{I})\mathbf{v}_2 = [-5 \quad 3 \quad 8]^T \neq \mathbf{0},$$

thereby obtaining the length 2 chain $\{\mathbf{v}_1, \mathbf{v}_2\}$ of generalized eigenvectors based on the ordinary eigenvector $\mathbf{v}_1$. If we take $\mathbf{v}_3 = [1 \quad 1 \quad 0]^T$, then the general solution $\mathbf{x}(t) = e^{2t}[c_1\mathbf{v}_1 + c_2(\mathbf{v}_1 t + \mathbf{v}_2) + c_3\mathbf{v}_3]$ has scalar components

$$\begin{aligned} x_1(t) &= e^{2t}(-5c_1 + c_2 + c_3 - 5c_2 t) \\ x_2(t) &= e^{2t}(3c_1 + 3c_2 t) \\ x_3(t) &= e^{2t}(8c_1 + 8c_2 t). \end{aligned}$$

28. $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ with

$$\mathbf{v}_1 = [119 \quad -289 \quad 0]^T, \quad \mathbf{v}_2 = [-17 \quad 34 \quad 17]^T, \quad \mathbf{v}_3 = [1 \quad 0 \quad 0]^T$$

General solution

$$\begin{aligned} x_1(t) &= e^{2t}(119c_1 - 17c_2 + c_3 + 119c_2 t - 17c_3 t + 119c_3 t^2/2) \\ x_2(t) &= e^{2t}(-289c_1 + 34c_2 - 289c_2 t + 34c_3 t - 289c_3 t^2/2) \\ x_3(t) &= e^{2t}(17c_2 + 17c_3 t) \end{aligned}$$

In Problems 29 and 30 the matrix $\mathbf{A}$ has two distinct eigenvalues λ_1 and λ_2 each having multiplicity 2 and defect 1. First, we select $\mathbf{v}_2$ so that $\mathbf{v}_1 = (\mathbf{A} - \lambda_1\mathbf{I})\mathbf{v}_2 \neq \mathbf{0}$ but $(\mathbf{A} - \lambda_1\mathbf{I})\mathbf{v}_1 = \mathbf{0}$, so $\{\mathbf{v}_1, \mathbf{v}_2\}$ is a length 2 chain based on $\mathbf{v}_1$. Next, we select $\mathbf{u}_2$ so that $\mathbf{u}_1 = (\mathbf{A} - \lambda_1\mathbf{I})\mathbf{u}_2 \neq \mathbf{0}$ but $(\mathbf{A} - \lambda_1\mathbf{I})\mathbf{u}_1 = \mathbf{0}$, so $\{\mathbf{u}_1, \mathbf{u}_2\}$ is a length 2 chain based on $\mathbf{u}_1$. We give the scalar components of the corresponding general solution

$$\mathbf{x}(t) = e^{\lambda_1 t}[c_1\mathbf{v}_1 + c_2(\mathbf{v}_1 t + \mathbf{v}_2)] + e^{\lambda_2 t}[c_3\mathbf{u}_1 + c_4(\mathbf{u}_1 t + \mathbf{u}_2)].$$

29. $\lambda = -1$: $\{\mathbf{v}_1, \mathbf{v}_2\}$ with $\mathbf{v}_1 = [1 \ \ -3 \ \ -1 \ \ -2]^T$ and $\mathbf{v}_2 = [0 \ \ 1 \ \ 0 \ \ 0]^T$,
$\lambda = 2$: $\{\mathbf{u}_1, \mathbf{u}_2\}$ with $\mathbf{u}_1 = [0 \ \ -1 \ \ 1 \ \ 0]^T$ and $\mathbf{u}_2 = [0 \ \ 0 \ \ 2 \ \ 1]^T$

Scalar components

$$\begin{aligned}
x_1(t) &= e^{-t}(c_1 + c_2 t) \\
x_2(t) &= e^{-t}(-3c_1 + c_2 - 3c_2 t) + e^{2t}(-c_3 - c_4 t) \\
x_3(t) &= e^{-t}(-c_1 - c_2 t) + e^{2t}(c_3 + 2c_4 + c_4 t) \\
x_4(t) &= e^{-t}(-2c_1 - 2c_2 t) + e^{2t}(c_4)
\end{aligned}$$

30. $\lambda = -1$: $\{\mathbf{v}_1, \mathbf{v}_2\}$ with $\mathbf{v}_1 = [0 \ \ 1 \ \ -1 \ \ -3]^T$ and $\mathbf{v}_2 = [0 \ \ 0 \ \ 1 \ \ 2]^T$,

$\lambda = 2$: $\{\mathbf{u}_1, \mathbf{u}_2\}$ with $\mathbf{u}_1 = [-1 \ \ 0 \ \ 0 \ \ 0]^T$ and $\mathbf{u}_2 = [0 \ \ 0 \ \ 3 \ \ 5]^T$

Scalar components

$$\begin{aligned}
x_1(t) &= e^{2t}(-c_3 - c_4 t) \\
x_2(t) &= e^{-t}(c_1 + c_2 t) \\
x_3(t) &= e^{-t}(-c_1 + c_2 - c_2 t) + e^{2t}(3c_4) \\
x_4(t) &= e^{-t}(-3c_1 + 2c_2 - 3c_2 t) + e^{2t}(5c_4)
\end{aligned}$$

31. We have the single eigenvalue $\lambda = 1$ of multiplicity 4. Starting with $\mathbf{v}_3 = [1 \ \ 0 \ \ 0 \ \ 0]^T$, we calculate $\mathbf{v}_2 = (\mathbf{A} - \lambda\mathbf{I})\mathbf{v}_3$ and $\mathbf{v}_1 = (\mathbf{A} - \lambda\mathbf{I})\mathbf{v}_2 \neq \mathbf{0}$, and find that $(\mathbf{A} - \lambda\mathbf{I})\mathbf{v}_1 = \mathbf{0}$. Therefore $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a length 3 chain based on the ordinary eigenvector $\mathbf{v}_1$. Next, the eigenvector equation $(\mathbf{A} - \lambda\mathbf{I})\mathbf{v} = \mathbf{0}$ yields the second linearly independent eigenvector $\mathbf{v}_4 = [0 \ \ 1 \ \ 3 \ \ 0]^T$. With

$$\mathbf{v}_1 = [42 \ \ 7 \ \ -21 \ \ -42]^T, \qquad \mathbf{v}_2 = [34 \ \ 22 \ \ -10 \ \ -27]^T,$$
$$\mathbf{v}_3 = [1 \ \ 0 \ \ 0 \ \ 0]^T \quad \text{and} \quad \mathbf{v}_4 = [0 \ \ 1 \ \ 3 \ \ 0]$$

the general solution

$$\mathbf{x}(t) = e^t[c_1\mathbf{v}_1 + c_2(\mathbf{v}_1 t + \mathbf{v}_2) + c_3(\mathbf{v}_1 t^2/2 + \mathbf{v}_2 t + \mathbf{v}_3) + c_4\mathbf{v}_4]$$

has scalar components

$$
\begin{aligned}
x_1(t) &= e^t(42c_1 + 34c_2 + c_3 + 42c_2t + 34c_3t + 21c_3t^2)\\
x_2(t) &= e^t(7c_1 + 22c_2 + c_4 + 7c_2t + 22c_3t + 7c_3t^2/2)\\
x_3(t) &= e^t(-21c_1 - 10c_2 + 3c_4 - 21c_2t - 10c_3t - 21c_3t^2/2)\\
x_4(t) &= e^t(-42c_1 - 27c_2 - 42c_2t - 27c_3t - 21c_3t^2).
\end{aligned}
$$

32. Here we find that the matrix **A** has five linearly independent eigenvectors:

$\lambda = 2$: eigenvectors $\mathbf{v}_1 = [8 \quad 0 \ {-3} \quad 1 \quad 0]^T$ and $\mathbf{v}_2 = [1 \quad 0 \quad 0 \quad 0 \quad 3]^T$

$\lambda = 3$: eigenvectors $\mathbf{v}_3 = [3 \ {-2} \ {-1} \quad 0 \quad 0]^T$, $\mathbf{v}_4 = [2 \ {-2} \quad 0 \ {-3} \quad 0]^T$,

$\mathbf{v}_5 = [1 \ {-1} \quad 0 \quad 0 \quad 3]^T$

The general solution

$$\mathbf{x}(t) = e^{2t}(c_1\mathbf{v}_1 + c_2\mathbf{v}_2) + e^{3t}(c_3\mathbf{v}_3 + c_4\mathbf{v}_4 + c_5\mathbf{v}_5)$$

has scalar components

$$
\begin{aligned}
x_1(t) &= e^{2t}(8c_1 + c_2) + e^{3t}(3c_3 + 2c_4 + c_5)\\
x_2(t) &= e^{3t}(-2c_3 - 2c_4 - c_5)\\
x_3(t) &= e^{2t}(-3c_1) + e^{3t}(-c_3)\\
x_4(t) &= e^{2t}(c_1) + e^{3t}(-3c_4)\\
x_3(t) &= e^{2t}(3c_2) + e^{3t}(3c_5)
\end{aligned}
$$

33. The chain $\{\mathbf{v}_1, \mathbf{v}_2\}$ was found using the matrices

$$\mathbf{A} - \lambda\mathbf{I} = \begin{bmatrix} 4i & -4 & 1 & 0 \\ 4 & 4i & 0 & 1 \\ 0 & 0 & 4i & -4 \\ 0 & 0 & 4 & 4i \end{bmatrix} \rightarrow \begin{bmatrix} 1 & i & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

and

$$(\mathbf{A} - \lambda\mathbf{I})^2 = \begin{bmatrix} -32 & -32i & 8i & -8 \\ 32i & -32 & 8 & 8i \\ 0 & 0 & -32 & -32i \\ 0 & 0 & 32i & -32 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & i & 0 & 0 \\ 0 & 0 & 1 & i \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

where $\to$ signifies reduction to row-echelon form. The resulting real-valued solution vectors are

$$\mathbf{x}_1(t) = e^{3t}\,[\ \cos 4t \quad \sin 4t \quad 0 \quad 0\ ]^T$$
$$\mathbf{x}_2(t) = e^{3t}\,[-\sin 4t \quad \cos 4t \quad 0 \quad 0\ ]^T$$
$$\mathbf{x}_3(t) = e^{3t}\,[\ t\cos 4t \quad t\sin 4t \quad \cos 4t \quad \sin 4t\ ]^T$$
$$\mathbf{x}_4(t) = e^{3t}\,[-t\sin 4t \quad t\cos 4t \quad -\sin 4t \quad \cos 4t\ ]^T.$$

34. The chain $\{\mathbf{v}_1, \mathbf{v}_2\}$ was found using the matrices

$$\mathbf{A}-\lambda\mathbf{I} = \begin{bmatrix} 3i & 0 & -8 & -3 \\ -18 & -3-3i & 0 & 0 \\ -9 & -3 & -27-3i & -9 \\ 33 & 10 & 90 & 30-3i \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 3+3i \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

and

$$(\mathbf{A}-\lambda\mathbf{I})^2 = \begin{bmatrix} -36 & -6 & -54+48i & -18+18i \\ 54+108i & 18i & 144 & 54 \\ 54i & 18i & -18+162i & 54i \\ -198i & -60i & 6-540i & -18-180i \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -3i & -i \\ 0 & 1 & 9+10i & 3+3i \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

where $\to$ signifies reduction to row-echelon form. The resulting real-valued solution vectors are

$$\mathbf{x}_1(t) = e^{2t}\,[\ \sin 3t \quad 3\cos 3t - 3\sin 3t \quad 0 \quad \sin 3t\ ]^T$$
$$\mathbf{x}_2(t) = e^{2t}\,[\ -\cos 3t \quad 3\sin 3t + 3\cos 3t \quad 0 \quad -\cos 3t\ ]^T$$
$$\mathbf{x}_3(t) = e^{2t}\,[3\cos 3t + t\sin 3t \quad (3t-10)\cos 3t - (3t+9)\sin 3t \quad \sin 3t \quad t\sin 3t\ ]^T$$
$$\mathbf{x}_4(t) = e^{2t}\,[-t\cos 3t + 3\sin 3t \quad (3t+9)\cos 3t + (3t-10)\sin 3t \quad -\cos 3t \quad -t\cos 3t]^T.$$

35. The coefficient matrix

$$\mathbf{A} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 1 & -2 & 1 \\ 1 & -1 & 1 & -2 \end{bmatrix}$$

has eigenvalues

$\lambda = 0$ with eigenvector $\mathbf{v}_1 = [1 \quad 1 \quad 0 \quad 0]^T$
$\lambda = -1$ with eigenvectors $\mathbf{v}_2 = [1 \quad 0 \quad -1 \quad 0]^T$ and $\mathbf{v}_3 = [0 \quad 1 \quad 0 \quad -1]^T$,
$\lambda = -2$ with eigenvector $\mathbf{v}_4 = [1 \quad -1 \quad -2 \quad 2]^T$.

When we impose the given initial conditions on the general solution

$$\mathbf{x}(t) = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 e^{-t} + c_3\mathbf{v}_3 e^{-t} + c_4\mathbf{v}_4 e^{-2t}$$

we find that $c_1 = v_0$, $c_2 = c_3 = -v_0$, $c_4 = 0$. Hence the position functions of the two masses are given by

$$x_1(t) = x_2(t) = v_0(1 - e^{-t}).$$

Each mass travels a distance v_0 before stopping.

36. The coefficient matrix is the same as in Problem 35 except that $a_{44} = -1$. Now the matrix $\mathbf{A}$ has the eigenvalue $\lambda = 0$ with eigenvector $\mathbf{v}_0 = [1 \quad 1 \quad 0 \quad 0]^T$, and the triple eigenvalue $\lambda = -1$ with associated length 2 chain $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ consisting of the generalized eigenvectors

$$\begin{aligned} \mathbf{v}_1 &= [0 \quad 1 \quad 0 \quad -1]^T \\ \mathbf{v}_2 &= [1 \quad 0 \quad -1 \quad 1]^T \\ \mathbf{v}_3 &= [1 \quad 0 \quad 0 \quad 0]^T. \end{aligned}$$

When we impose the given initial conditions on the general solution

$$\mathbf{x}(t) = c_0\mathbf{v}_0 + e^{-t}\,[c_1\mathbf{v}_1 + c_2(\mathbf{v}_1 t + \mathbf{v}_2) + c_3(\mathbf{v}_1 t^2/2 + \mathbf{v}_2 t + \mathbf{v}_3)]$$

we find that $c_0 = 2v_0$, $c_1 = -2v_0$, $c_2 = c_3 = -v_0$. Hence the position functions of the two masses are given by

$$\begin{aligned} x_1(t) &= v_0(2 - 2e^{-t} - te^{-t}), \\ x_2(t) &= v_0(2 - 2e^{-t} - te^{-t} - t^2e^{-t}/2). \end{aligned}$$

Each travels a distance $2v_0$ before stopping.

In Problems 37–46 we use the eigenvectors and generalized eigenvectors found in Problems 23–32 to construct a matrix $\mathbf{Q}$ such that $\mathbf{J} = \mathbf{Q}^{-1}\mathbf{AQ}$ is a Jordan normal form of the given matrix $\mathbf{A}$.

37. $\mathbf{v}_1 = [1 \quad -1 \quad 2]^T$, $\mathbf{v}_2 = [4 \quad 0 \quad 9]^T$, $\mathbf{v}_3 = [0 \quad 2 \quad 1]^T$

$$\mathbf{Q} = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 \end{bmatrix} = \begin{bmatrix} 1 & 4 & 0 \\ -1 & 0 & 2 \\ 2 & 9 & 1 \end{bmatrix}$$

$$\mathbf{J} = \mathbf{Q}^{-1}\mathbf{AQ} = \frac{1}{2}\begin{bmatrix} -18 & -4 & 8 \\ 5 & 1 & -2 \\ -9 & -1 & 4 \end{bmatrix}\begin{bmatrix} 39 & 8 & -16 \\ -36 & -5 & 16 \\ 72 & 16 & -29 \end{bmatrix}\begin{bmatrix} 1 & 4 & 0 \\ -1 & 0 & 2 \\ 2 & 9 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

38. $\mathbf{v}_1 = [5 \quad 3 \ -3]^T, \quad \mathbf{v}_2 = [4 \quad 0 \ -1]^T, \quad \mathbf{v}_3 = [2 \ -1 \quad 0]^T$

$$\mathbf{Q} = [\mathbf{v}_1 \quad \mathbf{v}_2 \quad \mathbf{v}_3] = \begin{bmatrix} 5 & 4 & 2 \\ 3 & 0 & -1 \\ -3 & -1 & 0 \end{bmatrix}$$

$$\mathbf{J} = \mathbf{Q}^{-1}\mathbf{AQ} = \begin{bmatrix} -1 & -2 & -4 \\ 3 & 6 & 11 \\ -3 & -7 & -12 \end{bmatrix}\begin{bmatrix} 28 & 50 & 100 \\ 15 & 33 & 60 \\ -15 & -30 & -57 \end{bmatrix}\begin{bmatrix} 5 & 4 & 2 \\ 3 & 0 & -1 \\ -3 & -1 & 0 \end{bmatrix} = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

39. $\mathbf{v}_1 = [-1 \quad 0 \ -1]^T, \quad \mathbf{v}_2 = [-4 \ -1 \quad 0]^T, \quad \mathbf{v}_3 = [1 \quad 0 \quad 0]^T$

$$\mathbf{Q} = [\mathbf{v}_1 \quad \mathbf{v}_2 \quad \mathbf{v}_3] = \begin{bmatrix} -1 & -4 & 1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{bmatrix}$$

$$\mathbf{J} = \mathbf{Q}^{-1}\mathbf{AQ} = \begin{bmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ 1 & -4 & -1 \end{bmatrix}\begin{bmatrix} -2 & 17 & 4 \\ -1 & 6 & 1 \\ 0 & 1 & 2 \end{bmatrix}\begin{bmatrix} -1 & -4 & 1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$$

40. $\mathbf{v}_1 = [0 \quad 2 \quad 2]^T, \quad \mathbf{v}_2 = [2 \quad 1 \ -3]^T, \quad \mathbf{v}_3 = [1 \quad 0 \quad 0]^T$

$$\mathbf{Q} = [\mathbf{v}_1 \quad \mathbf{v}_2 \quad \mathbf{v}_3] = \begin{bmatrix} 0 & 2 & 1 \\ 2 & 1 & 0 \\ 2 & -3 & 0 \end{bmatrix}$$

$$\mathbf{J} = \mathbf{Q}^{-1}\mathbf{AQ} = \frac{1}{8}\begin{bmatrix} 0 & 3 & 1 \\ 0 & 2 & -2 \\ 8 & -4 & 4 \end{bmatrix}\begin{bmatrix} 5 & -1 & 1 \\ 1 & 3 & 0 \\ -3 & 2 & 1 \end{bmatrix}\begin{bmatrix} 0 & 2 & 1 \\ 2 & 1 & 0 \\ 2 & -3 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 1 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{bmatrix}$$

41. $\mathbf{v}_1 = [-5 \quad 3 \quad 8]^T, \quad \mathbf{v}_2 = [1 \quad 0 \quad 0]^T, \quad \mathbf{v}_3 = [1 \quad 1 \quad 0]^T$

$$\mathbf{Q} = [\mathbf{v}_1 \quad \mathbf{v}_2 \quad \mathbf{v}_3] = \begin{bmatrix} -5 & 1 & 1 \\ 3 & 0 & 1 \\ 8 & 0 & 0 \end{bmatrix}$$

$$\mathbf{J} = \mathbf{Q}^{-1}\mathbf{A}\mathbf{Q} = \frac{1}{8}\begin{bmatrix} 0 & 0 & 1 \\ 8 & -8 & 8 \\ 0 & 8 & -3 \end{bmatrix}\begin{bmatrix} -3 & 5 & -5 \\ 3 & -1 & 3 \\ 8 & -8 & 10 \end{bmatrix}\begin{bmatrix} -5 & 1 & 1 \\ 3 & 0 & 1 \\ 8 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

42. $\mathbf{v}_1 = [119 \quad -289 \quad 0]^T, \quad \mathbf{v}_2 = [-17 \quad 34 \quad 17]^T, \quad \mathbf{v}_3 = [1 \quad 0 \quad 0]^T$

$$\mathbf{Q} = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 \end{bmatrix} = \begin{bmatrix} 119 & -17 & 1 \\ -289 & 34 & 0 \\ 0 & 17 & 0 \end{bmatrix}$$

$$\mathbf{J} = \mathbf{Q}^{-1}\mathbf{A}\mathbf{Q} = \frac{1}{289}\begin{bmatrix} 0 & -1 & 2 \\ 0 & 0 & 17 \\ 289 & 119 & 51 \end{bmatrix}\begin{bmatrix} -15 & -7 & 4 \\ 34 & 16 & -11 \\ 17 & 7 & 5 \end{bmatrix}\begin{bmatrix} 119 & -17 & 1 \\ -289 & 34 & 0 \\ 0 & 17 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$$

43. $\mathbf{v}_1 = [1 \quad -3 \quad -1 \quad -2]^T, \quad \mathbf{v}_2 = [0 \quad 1 \quad 0 \quad 0]^T,$
$\mathbf{u}_1 = [0 \quad -1 \quad 1 \quad 0]T, \quad \mathbf{u}_2 = [0 \quad 0 \quad 2 \quad 1]^T$

$$\mathbf{Q} = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{u}_1 & \mathbf{u}_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -3 & 1 & -1 & 0 \\ -1 & 0 & 1 & 2 \\ -2 & 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{J} = \mathbf{Q}^{-1}\mathbf{A}\mathbf{Q}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & -2 \\ -3 & 0 & 1 & -2 \\ 2 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} -1 & 1 & 1 & -2 \\ 7 & -4 & -6 & 11 \\ 5 & -1 & 1 & 3 \\ 6 & -2 & -2 & 6 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 & 0 \\ -3 & 1 & -1 & 0 \\ -1 & 0 & 1 & 2 \\ -2 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

44. $\mathbf{v}_1 = [0 \quad 1 \quad -1 \quad -3]^T, \quad \mathbf{v}_2 = [0 \quad 0 \quad 1 \quad 2]^T,$
$\mathbf{u}_1 = [-1 \quad 0 \quad 0 \quad 0]^T \quad \mathbf{u}_2 = [0 \quad 0 \quad 3 \quad 5]^T$

$$\mathbf{Q} = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{u}_1 & \mathbf{u}_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 3 \\ -3 & 2 & 0 & 5 \end{bmatrix}$$

$$\mathbf{J} = \mathbf{Q}^{-1}\mathbf{AQ}$$

$$= \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 4 & -5 & 3 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 \end{bmatrix} \begin{bmatrix} 2 & 1 & -2 & 1 \\ 0 & 3 & -5 & 3 \\ 0 & -13 & 22 & -12 \\ 0 & -27 & 45 & -25 \end{bmatrix} \begin{bmatrix} 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 3 \\ -3 & 2 & 0 & 5 \end{bmatrix} = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

45. $\mathbf{v}_1 = [42 \quad 7 \quad -21 \quad -42]^T$, $\quad \mathbf{v}_2 = [34 \quad 22 \quad -10 \quad -27]^T$,

$\mathbf{v}_3 = [1 \quad 0 \quad 0 \quad 0]^T$, $\quad \mathbf{v}_4 = [0 \quad 1 \quad 3 \quad 0]^T$

$$\mathbf{Q} = [\mathbf{v}_1 \quad \mathbf{v}_2 \quad \mathbf{v}_3 \quad \mathbf{v}_4] = \begin{bmatrix} 42 & 34 & 1 & 0 \\ 7 & 22 & 0 & 1 \\ -21 & -10 & 0 & 3 \\ -42 & -27 & 0 & 0 \end{bmatrix}$$

$$\mathbf{J} = \mathbf{Q}^{-1}\mathbf{AQ}$$

$$= \frac{1}{2058} \begin{bmatrix} 0 & -81 & 27 & -76 \\ 0 & 126 & -42 & 42 \\ 2058 & -882 & 294 & 1764 \\ 0 & -147 & 735 & -392 \end{bmatrix} \begin{bmatrix} 2 & 1 & -2 & 1 \\ 0 & 3 & -5 & 3 \\ 0 & -13 & 22 & -12 \\ 0 & -27 & 45 & -25 \end{bmatrix} \begin{bmatrix} 42 & 34 & 1 & 0 \\ 7 & 22 & 0 & 1 \\ -21 & -10 & 0 & 3 \\ -42 & -27 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

46. $\mathbf{v}_1 = [8 \quad 0 \quad -3 \quad 1 \quad 0]^T$, $\quad \mathbf{v}_2 = [1 \quad 0 \quad 0 \quad 0 \quad 3]^T$

$\mathbf{v}_3 = [3 \quad -2 \quad -1 \quad 0 \quad 0]^T$, $\quad \mathbf{v}_4 = [2 \quad -2 \quad 0 \quad -3 \quad 0]^T$, $\quad \mathbf{v}_5 = [1 \quad -1 \quad 0 \quad 0 \quad 3]^T$

$$\mathbf{Q} = [\mathbf{v}_1 \quad \mathbf{v}_2 \quad \mathbf{v}_3 \quad \mathbf{v}_4 \quad \mathbf{v}_5] = \begin{bmatrix} 8 & 1 & 3 & 2 & 1 \\ 0 & 0 & -2 & -2 & -1 \\ -3 & 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & -3 & 0 \\ 0 & 3 & 0 & 0 & 3 \end{bmatrix}$$

$$\mathbf{J} = \mathbf{Q}^{-1}\mathbf{A}\mathbf{Q}$$

$$= \frac{1}{3}\begin{bmatrix} -9 & 0 & -27 & -6 & 3 \\ 48 & 3 & 138 & 30 & -15 \\ 27 & 0 & 78 & 18 & -9 \\ -3 & 0 & -9 & -3 & 1 \\ -48 & -3 & -138 & -30 & 16 \end{bmatrix}\begin{bmatrix} 11 & -1 & 26 & 6 & -3 \\ 0 & 3 & 0 & 0 & 0 \\ -9 & 0 & -24 & -6 & 3 \\ 3 & 0 & 9 & 5 & -1 \\ -48 & -3 & -138 & -30 & 18 \end{bmatrix}\begin{bmatrix} 8 & 1 & 3 & 2 & 1 \\ 0 & 0 & -2 & -2 & -1 \\ -3 & 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & -3 & 0 \\ 0 & 3 & 0 & 0 & 3 \end{bmatrix}$$

$$\mathbf{J} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

SECTION 7.6

NUMERICAL METHODS FOR SYSTEMS

In Problems 1-8 we first write the given system in the form $x' = f(t,x,y), \quad y' = g(t,x,y)$. Then we use the template

$$\begin{aligned} h &= 0.1; \quad t_1 = t_0 + h \\ x_1 &= x_0 + h\,f(t_0,x_0,y_0); \quad y_1 = y_0 + h\,g(t_0,x_0,y_0) \\ x_2 &= x_1 + h\,f(t_1,x_1,y_1); \quad y_2 = y_1 + h\,g(t_1,x_1,y_1) \end{aligned}$$

(with the given values of t_0, x_0, and y_0) to calculate the Euler approximations $x_1 \approx x(0.1)$, $y_1 \approx y(0.1)$ and $x_2 \approx x(0.2)$, $y_2 \approx y(0.2)$ in part (a). We give these approximations and the actual values $x_{\text{act}} = x(0.2)$, $y_{\text{act}} = y(0.2)$ in tabular form. We use the template

$$\begin{aligned} h &= 0.2; \quad t_1 = t_0 + h \\ u_1 &= x_0 + h\,f(t_0,x_0,y_0); \quad v_1 = y_0 + h\,g(t_0,x_0,y_0) \\ x_1 &= x_0 + \tfrac{1}{2}h\left[f(t_0,x_0,y_0) + f(t_1,u_1,v_1)\right] \\ y_1 &= y_0 + \tfrac{1}{2}h\left[g(t_0,x_0,y_0) + g(t_1,u_1,v_1)\right] \end{aligned}$$

to calculate the improved Euler approximations $u_1 \approx x(0.2)$, $u_1 \approx y(0.2)$ and $x_1 \approx x(0.2)$, $y_1 \approx y(0.2)$ in part (b). We give these approximations and the actual values $x_{\text{act}} = x(0.2)$, $y_{\text{act}} = y(0.2)$ in tabular form. We use the template

$$h = 0.2;$$
$$F_1 = f(t_0, x_0, y_0); \quad G_1 = g(t_0, x_0, y_0)$$
$$F_2 = f(t_0 + \tfrac{1}{2}h, x_0 + \tfrac{1}{2}hF_1, y_0 + \tfrac{1}{2}hG_1); \quad G_2 = g(t_0 + \tfrac{1}{2}h, x_0 + \tfrac{1}{2}hF_1, y_0 + \tfrac{1}{2}hG_1)$$
$$F_3 = f(t_0 + \tfrac{1}{2}h, x_0 + \tfrac{1}{2}hF_2, y_0 + \tfrac{1}{2}hG_2); \quad G_3 = g(t_0 + \tfrac{1}{2}h, x_0 + \tfrac{1}{2}hF_2, y_0 + \tfrac{1}{2}hG_2)$$
$$F_4 = f(t_0 + h, x_0 + hF_3, y_0 + hG_3); \quad G_4 = g(t_0 + h, x_0 + hF_3, y_0 + hG_3)$$
$$x_1 = x_0 + \frac{h}{6}\left(F_1 + 2F_2 + 2F_3 + F_4\right); \quad y_1 = y_0 + \frac{h}{6}\left(G_1 + 2G_2 + 2G_3 + G_4\right)$$

to calculate the intermediate slopes and Runge-Kutta approximations $x_1 \approx x(0.2)$, $y_1 \approx y(0.2)$ for part (c). Again, we give the results in tabular form.

1. **(a)**

x_1	y_1	x_2	y_2	x_{act}	y_{act}
0.4	2.2	0.88	2.5	1.0034	2.6408

(b)

u_1	v_1	x_1	y_1	x_{act}	y_{act}
0.8	2.4	0.96	2.6	1.0034	2.6408

(c)

F_1	G_1	F_2	G_2	F_3	G_3	F_4	G_4
4	2	4.8	3	5.08	3.26	6.32	4.684
x_1	y_1	x_{act}	y_{act}				
1.0027	2.6401	1.0034	2.6408				

2. **(a)**

x_1	y_1	x_2	y_2	x_{act}	y_{act}
0.9	–0.9	0.81	–0.81	0.8187	–0.8187

(b)

u_1	v_1	x_1	y_1	x_{act}	y_{act}
0.8	–0.8	0.82	–0.82	0.8187	–0.8187

(c)

F_1	G_1	F_2	G_2	F_3	G_3	F_4	G_4
–1	1	–0.9	0.9	–0.91	0.91	–0.818	0.818
x_1	y_1	x_{act}	y_{act}				
0.8187	–0.8187	0.8187	–0.8187				

3. **(a)**

x_1	y_1	x_2	y_2	x_{act}	y_{act}
1.7	1.5	2.81	2.31	3.6775	2.9628

(b)

u_1	v_1	x_1	y_1	x_{act}	y_{act}
2.4	2	3.22	2.62	3.6775	2.9628

(c)

F_1	G_1	F_2	G_2	F_3	G_3	F_4	G_4
7	5	11.1	8.1	13.57	9.95	23.102	17.122
x_1	y_1	x_{act}	y_{act}				
3.6481	2.9407	3.6775	2.9628				

4. **(a)**

x_1	y_1	x_2	y_2	x_{act}	y_{act}
1.9	–0.6	3.31	–1.62	4.2427	-2.4205

(b)

u_1	v_1	x_1	y_1	x_{act}	y_{act}
2.8	–1.2	3.82	–2.04	4.2427	-2.4205

(c)

F_1	G_1	F_2	G_2	F_3	G_3	F_4	G_4
9	–6	14.1	–10.2	16.59	–12.42	26.442	–20.94
x_1	y_1	x_{act}	y_{act}				
4.2274	–2.4060	4.2427	-2.4205				

5. **(a)**

x_1	y_1	x_2	y_2	x_{act}	y_{act}
0.9	3.2	–0.52	2.92	-0.5793	2.4488

(b)

u_1	v_1	x_1	y_1	x_{act}	y_{act}
–0.2	3.4	–0.84	2.44	-0.5793	2.4488

(c)

F_1	G_1	F_2	G_2	F_3	G_3	F_4	G_4
–11	2	–14.2	–2.8	–12.44	–3.12	–12.856	–6.704
x_1	y_1	x_{act}	y_{act}				
–0.5712	2.4485	-0.5793	2.4488				

6. **(a)**

x_1	y_1	x_2	y_2	x_{act}	y_{act}
–0.8	4.4	–1.76	4.68	-1.9025	4.4999

(b)

u_1	v_1	x_1	y_1	x_{act}	y_{act}
–1.6	4.8	–1.92	4.56	-1.9025	4.4999

(c)

F_1	G_1	F_2	G_2	F_3	G_3	F_4	G_4
–8	4	–9.6	2.8	–9.52	2.36	–10.848	0.664
x_1	y_1	x_{act}	y_{act}				
–1.9029	4.4995	-1.9025	4.4999				

7. **(a)**

x_1	y_1	x_2	y_2	x_{act}	y_{act}
2.5	1.3	3.12	1.68	3.2820	1.7902

(b)

u_1	v_1	x_1	y_1	x_{act}	y_{act}
3	1.6	3.24	1.76	3.2820	1.7902

(c)

F_1	G_1	F_2	G_2	F_3	G_3	F_4	G_4
5	3	6.2	3.8	6.48	4	8.088	5.096
x_1	y_1	x_{act}	y_{act}				
3.2816	1.7899	3.2820	1.7902				

8. **(a)**

x_1	y_1	x_2	y_2	x_{act}	y_{act}
0.9	–0.9	2.16	–0.63	2.5270	-0.3889

(b)

u_1	v_1	x_1	y_1	x_{act}	y_{act}
1.8	–0.8	2.52	–0.46	2.5270	-0.3889

(c)

F_1	G_1	F_2	G_2	F_3	G_3	F_4	G_4
9	1	12.6	2.7	12.87	3.25	16.02	5.498
x_1	y_1	x_{act}	y_{act}				
2.5320	–0.3867	2.5270	-0.3889				

In Problems 9-11 we use the same Runge-Kutta template as in part (c) of Problems 1–8 above, and give both the Runge-Kutta approximate values with step sizes $h = 0.1$ and $h = 0.05$, and also the actual values.

9. With $h = 0.1$: $x(1) \approx 3.99261$, $y(1) \approx 6.21770$
With $h = 0.05$: $x(1) \approx 3.99234$, $y(1) \approx 6.21768$
Actual values: $x(1) \approx 3.99232$, $y(1) \approx 6.21768$

10. With $h = 0.1$: $x(1) \approx 1.31498$, $y(1) \approx 1.02537$
With $h = 0.05$: $x(1) \approx 1.31501$, $y(1) \approx 1.02538$
Actual values: $x(1) \approx 1.31501$, $y(1) \approx 1.02538$

11. With $h = 0.1$: $x(1) \approx -0.05832$, $y(1) \approx 0.56664$
With $h = 0.05$: $x(1) \approx -0.05832$, $y(1) \approx 0.56665$
Actual values: $x(1) \approx -0.05832$, $y(1) \approx 0.56665$

12. We first convert the given initial value problem to the two-dimensional problem

$$x' = y, \qquad x(0) = 0,$$
$$y' = -x + \sin t, \qquad y(0) = 0.$$

Then with both step sizes $h = 0.1$ and $h = 0.05$ we get the actual value $x(1) \approx 0.15058$ accurate to 5 decimal places.

13. With $y = x'$ we want to solve numerically the initial value problem

$$x' = y, \qquad x(0) = 0$$
$$y' = -32 - 0.04y, \qquad y(0) = 288.$$

When we run Program RK2DIM with step size $h = 0.1$ we find that the change of sign in the velocity v occurs as follows:

t	x	v
7.6	1050.2	+2.8
7.7	1050.3	−0.4

Thus the bolt attains a maximum height of about 1050 feet in about 7.7 seconds.

14. Now we want to solve numerically the initial value problem

$$x' = y, \qquad x(0) = 0,$$
$$y' = -32 - 0.0002y^2, \qquad y(0) = 288.$$

Running Program RK2DIM with step size $h = 0.1$, we find that the bolt attains a maximum height of about 1044 ft in about 7.8 sec. Note that these values are comparable to those found in Problem 13.

15. With $y = x'$, and with x in miles and t in seconds, we want to solve numerically the initial value problem

$$x' = y$$
$$y' = -95485.5/(x^2 + 7920x + 15681600)$$
$$x(0) = 0, \qquad y(0) = 1.$$

We find (running RK2DIM with $h = 1$) that the projectile reaches a maximum height of about 83.83 miles in about 168 sec = 2 min 48 sec.

16. We first defined the MATLAB function

```
function  xp  =  fnball(t,x)
%  Defines the baseball system
%       x1'  =  x'  =  x3,   x3'  =  -cvx'
%       x2'  =  y'  =  x4,   x4'  =  -cvy'- g
%  with air resistance coefficient c.

g    =  32;
c    =  0.0025;
xp   =  x;
v   =  sqrt(x(3).^2) + x(4).^2);
xp(1)   =   x(3);
xp(2)   =   x(4);
xp(3)   =  -c*v*x(3);
xp(4)   =  -c*v*x(4) - g;
```

Then, using the n-dimensional program **rkn** with step size 0.1 and initial data corresponding to the indicated initial inclination angles, we got the following results:

Angle	Time	Range
40	5.0	352.9
45	5.4	347.2
50	5.8	334.2

We have listed the time to the nearest tenth of a second, but have interpolated to find the range in feet.

17. The data in Problem 16 indicate that the range increases when the initial angle is decreased below 45°. The further data

Angle	Range
41.0	352.1
40.5	352.6

40.0	352.9
39.5	352.8
39.0	352.7
35.0	350.8

indicate that a maximum range of about 353 ft is attained with $\alpha \approx 40^{\circ}$.

18. We "shoot" for the proper inclination angle by running program **rkn** (with $h = 0.1$) as follows:

Angle	Range
60	287.1
58	298.5
57.5	301.1

Thus we get a range of 300 ft with an initial angle just under 57.5°.

19. First we run program **rkn** (with $h = 0.1$) with $v_0 = 250$ ft/sec and obtain the following results:

t	x	y
5.0	457.43	103.90
6.0	503.73	36.36

Interpolation gives $x = 494.4$ when $y = 50$. Then a run with $v_0 = 255$ ft/sec gives the following results:

t	x	y
5.5	486.75	77.46
6.0	508.86	41.62

Finally a run with $v_0 = 253$ ft/sec gives these results:

t	x	y
5.5	484.77	75.44
6.0	506.82	39.53

Now $x \approx 500$ ft when $y = 50$ ft. Thus Babe Ruth's home run ball had an initial velocity of 253 ft/sec.

20. A run of program **rkn** with $h = 0.1$ and with the given data yields the following results:

t	x	y	v	α
5.5	989	539	162	+0.95
5.6	1005	539	161	−0.18
.	.	.	.	.

.	.	.	.	.
.	.	.	.	.
.	.	.	.	.
11.5	1868	16	214	-52
11.6	1881	-1	216	-53

The first two lines of data above indicate that the crossbow bolt attains a maximum height of about 1005 ft in about 5.6 sec. About 6 sec later (total time 11.6 sec) it hits the ground, having traveled about 1880 ft horizontally.

21. A run with $h = 0.1$ indicates that the projectile has a range of about 21,400 ft $\approx$ 4.05 mi and a flight time of about 46 sec. It attains a maximum height of about 8970 ft in about 17.5 sec. At time $t \approx 23$ sec it has its minimum velocity of about 368 ft/sec. It hits the ground ($t \approx 46$ sec) at an angle of about $77°$ with a velocity of about 518 ft/sec.

CHAPTER 8

MATRIX EXPONENTIAL METHODS

SECTION 8.1

In Problems 1–8 we first use the eigenvalues and eigenvectors of the coefficient matrix **A** to find first a fundamental matrix $\mathbf{\Phi}(t)$ for the homogeneous system $\mathbf{x}' = \mathbf{Ax}$. Then we apply the formula

$$\mathbf{x}(t) = \mathbf{\Phi}(t)\mathbf{\Phi}(0)^{-1}\mathbf{x}_0,$$

to find the solution vector $\mathbf{x}(t)$ that satisfies the initial condition $\mathbf{x}(0) = \mathbf{x}_0$. Formulas (11) and (12) in the text provide inverses of 2-by-2 and 3-by-3 matrices.

1. Eigensystem: $\lambda_1 = 1, \quad \mathbf{v}_1 = [1 \quad -1]^T; \qquad \lambda_2 = 3, \quad \mathbf{v}_2 = [1 \quad 1]^T$

$$\Phi(t) = \begin{bmatrix} e^{\lambda_1 t}\mathbf{v}_1 & e^{\lambda_2 t}\mathbf{v}_2 \end{bmatrix} = \begin{bmatrix} e^t & e^{3t} \\ -e^t & e^{3t} \end{bmatrix}$$

$$\mathbf{x}(t) = \begin{bmatrix} e^t & e^{3t} \\ -e^t & e^{3t} \end{bmatrix} \cdot \frac{1}{2}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ -2 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 5e^t + e^{3t} \\ -5e^t + e^{3t} \end{bmatrix}$$

2. Eigensystem: $\lambda_1 = 0, \quad \mathbf{v}_1 = [1 \quad 2]^T; \qquad \lambda_2 = 4, \quad \mathbf{v}_2 = [1 \quad -2]^T$

$$\Phi(t) = \begin{bmatrix} e^{\lambda_1 t}\mathbf{v}_1 & e^{\lambda_2 t}\mathbf{v}_2 \end{bmatrix} = \begin{bmatrix} 1 & e^{4t} \\ 2 & -2e^{4t} \end{bmatrix}$$

$$\mathbf{x}(t) = \begin{bmatrix} 1 & e^{4t} \\ 2 & -2e^{4t} \end{bmatrix} \cdot \frac{1}{4}\begin{bmatrix} 2 & 1 \\ 2 & -1 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 3 + 5e^{4t} \\ 6 - 10e^{4t} \end{bmatrix}$$

3. Eigensystem: $\lambda = 4i, \quad \mathbf{v} = [1 + 2i \quad 2]^T$

$$\Phi(t) = \begin{bmatrix} \mathrm{Re}(\mathbf{v}e^{\lambda t}) & \mathrm{Im}(\mathbf{v}e^{\lambda t}) \end{bmatrix} = \begin{bmatrix} \cos 4t - 2\sin 4t & 2\cos 4t + \sin 4t \\ 2\cos 4t & 2\sin 4t \end{bmatrix}$$

$$\mathbf{x}(t) = \begin{bmatrix} \cos 4t - 2\sin 4t & 2\cos 4t + \sin 4t \\ 2\cos 4t & 2\sin 4t \end{bmatrix} \cdot \frac{1}{4}\begin{bmatrix} 0 & 2 \\ 2 & -1 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} -5\sin 4t \\ 4\cos 4t - 2\sin 4t \end{bmatrix}$$

4. Eigensystem: $\lambda = 2, 2; \quad \{\mathbf{v}_1, \mathbf{v}_2\}$ with $\mathbf{v}_1 = [1 \quad 1]^{\mathrm{T}},\ \mathbf{v}_2 = [1 \quad 0]^{\mathrm{T}}$

$$\Phi(t) = \begin{bmatrix} \mathbf{v}_1 e^{\lambda t} & (\mathbf{v}_1 t + \mathbf{v}_2) e^{\lambda t} \end{bmatrix} = e^{2t}\begin{bmatrix} 1 & 1+t \\ 1 & t \end{bmatrix}$$

$$\mathbf{x}(t) = e^{2t}\begin{bmatrix} 1 & 1+t \\ 1 & t \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} = e^{2t}\begin{bmatrix} 1+t \\ t \end{bmatrix}$$

5. Eigensystem: $\lambda = 3i, \quad \mathbf{v} = [-1+i \quad 3]^{\mathrm{T}}$

$$\Phi(t) = \begin{bmatrix} \mathrm{Re}(\mathbf{v} e^{\lambda t}) & \mathrm{Im}(\mathbf{v} e^{\lambda t}) \end{bmatrix} = \begin{bmatrix} -\cos 3t - \sin 3t & \cos 3t - \sin 3t \\ 3\cos 3t & 3\sin 3t \end{bmatrix}$$

$$\mathbf{x}(t) = \begin{bmatrix} -\cos 3t - \sin 3t & \cos 3t - \sin 3t \\ 3\cos 3t & 3\sin 3t \end{bmatrix} \cdot \frac{1}{3}\begin{bmatrix} 0 & 1 \\ 3 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 3\cos 3t - \sin 3t \\ -3\cos 3t + 6\sin 3t \end{bmatrix}$$

6. Eigensystem: $\lambda = 5+4i, \quad \mathbf{v} = [1+2i \quad 2]^{\mathrm{T}}$

$$\Phi(t) = \begin{bmatrix} \mathrm{Re}(\mathbf{v} e^{\lambda t}) & \mathrm{Im}(\mathbf{v} e^{\lambda t}) \end{bmatrix} = e^{5t}\begin{bmatrix} \cos 4t - 2\sin 4t & 2\cos 4t + 2\sin 4t \\ 2\cos 4t & 2\sin 4t \end{bmatrix}$$

$$\mathbf{x}(t) = e^{5t}\begin{bmatrix} \cos 4t - 2\sin 4t & 2\cos 4t + 2\sin 4t \\ 2\cos 4t & 2\sin 4t \end{bmatrix} \cdot \frac{1}{4}\begin{bmatrix} 0 & 2 \\ 2 & -4 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 0 \end{bmatrix} = 2e^{5t}\begin{bmatrix} \cos 4t + \sin 4t \\ \sin 4t \end{bmatrix}$$

7. Eigensystem:
$\lambda_1 = 0, \quad \mathbf{v}_1 = [6 \ \ 2 \ \ 5]^{\mathrm{T}}; \quad \lambda_2 = 1, \quad \mathbf{v}_2 = [3 \ \ 1 \ \ 2]^{\mathrm{T}}; \quad \lambda_3 = -1, \quad \mathbf{v}_3 = [2 \ \ 1 \ \ 2]^{\mathrm{T}}$

$$\Phi(t) = \begin{bmatrix} e^{\lambda_1 t}\mathbf{v}_1 & e^{\lambda_2 t}\mathbf{v}_2 & e^{\lambda_3 t}\mathbf{v}_3 \end{bmatrix} = \begin{bmatrix} 6 & 3e^t & 2e^{-t} \\ 2 & e^t & e^{-t} \\ 5 & 2e^t & 2e^{-t} \end{bmatrix}$$

$$\mathbf{x}(t) = \begin{bmatrix} 6 & 3e^t & 2e^{-t} \\ 2 & e^t & e^{-t} \\ 5 & 2e^t & 2e^{-t} \end{bmatrix} \cdot \begin{bmatrix} 0 & -2 & 1 \\ 1 & 2 & -2 \\ -1 & 3 & 0 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -12 + 12e^t + 2e^{-t} \\ -4 + 4e^t + e^{-t} \\ -10 + 8e^t + 2e^{-t} \end{bmatrix}$$

8. Eigensystem:
$\lambda_1 = -2, \quad \mathbf{v}_1 = [0 \ \ 1 \ \ -1]^{\mathrm{T}}; \quad \lambda_2 = 1, \quad \mathbf{v}_2 = [1 \ \ -1 \ \ 0]^{\mathrm{T}}; \quad \lambda_3 = 3, \quad \mathbf{v}_3 = [1 \ \ -1 \ \ 1]^{\mathrm{T}}$

$$\Phi(t) = \begin{bmatrix} e^{\lambda_1 t}\mathbf{v}_1 & e^{\lambda_2 t}\mathbf{v}_2 & e^{\lambda_3 t}\mathbf{v}_3 \end{bmatrix} = \begin{bmatrix} 0 & e^t & e^{3t} \\ e^{-2t} & -e^t & -e^{3t} \\ -e^{-2t} & 0 & e^{3t} \end{bmatrix}$$

$$\mathbf{x}(t) = \begin{bmatrix} 0 & e^t & e^{3t} \\ e^{-2t} & -e^t & -e^{3t} \\ -e^{-2t} & 0 & e^{3t} \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & -1 \\ 1 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} e^t \\ -e^t + e^{-2t} \\ -e^{-2t} \end{bmatrix}$$

In each of Problems 9-20 we first solve the given linear system to find two linearly independent solutions $\mathbf{x}_1$ and $\mathbf{x}_2$, then set up the fundamental matrix $\mathbf{\Phi}(t) = [\mathbf{x}_1(t) \quad \mathbf{x}_2(t)]$, and finally calculate the matrix exponential $e^{\mathbf{A}t} = \Phi(t)\,\Phi(0)^{-1}$.

9. Eigensystem: $\lambda_1 = 1, \quad \mathbf{v}_1 = [1 \quad 1]^T; \qquad \lambda_2 = 3, \quad \mathbf{v}_2 = [2 \quad 1]^T$

$$\Phi(t) = \begin{bmatrix} e^{\lambda_1 t}\mathbf{v}_1 & e^{\lambda_2 t}\mathbf{v}_2 \end{bmatrix} = \begin{bmatrix} e^t & 2e^{3t} \\ e^t & e^{3t} \end{bmatrix}$$

$$e^{\mathbf{A}t} = \begin{bmatrix} e^t & 2e^{3t} \\ e^t & e^{3t} \end{bmatrix} \begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} -e^t + 2e^{3t} & 2e^t - 2e^{3t} \\ -e^t + e^{3t} & 2e^t - e^{3t} \end{bmatrix}$$

10. Eigensystem: $\lambda_1 = 0, \quad \mathbf{v}_1 = [1 \quad 1]^T; \qquad \lambda_2 = 2, \quad \mathbf{v}_2 = [3 \quad 2]^T$

$$\Phi(t) = \begin{bmatrix} e^{\lambda_1 t}\mathbf{v}_1 & e^{\lambda_2 t}\mathbf{v}_2 \end{bmatrix} = \begin{bmatrix} 1 & 3e^{2t} \\ 1 & 2e^{2t} \end{bmatrix}$$

$$e^{\mathbf{A}t} = \begin{bmatrix} 1 & 3e^{2t} \\ 1 & 2e^{2t} \end{bmatrix} \begin{bmatrix} -2 & 3 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} -2 + 3e^{2t} & 3 - 3e^{2t} \\ -2 + 2e^{2t} & 3 - 2e^{2t} \end{bmatrix}$$

11. Eigensystem: $\lambda_1 = 2, \quad \mathbf{v}_1 = [1 \quad 1]^T; \qquad \lambda_2 = 3, \quad \mathbf{v}_2 = [3 \quad 2]^T$

$$\Phi(t) = \begin{bmatrix} e^{\lambda_1 t}\mathbf{v}_1 & e^{\lambda_2 t}\mathbf{v}_2 \end{bmatrix} = \begin{bmatrix} e^{2t} & 3e^{3t} \\ e^{2t} & 2e^{3t} \end{bmatrix}$$

$$e^{\mathbf{A}t} = \begin{bmatrix} e^{2t} & 3e^{3t} \\ e^{2t} & 2e^{3t} \end{bmatrix} \begin{bmatrix} -2 & 3 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} -2e^{2t} + 3e^{3t} & 3e^{2t} - 3e^{3t} \\ -2e^{2t} + 2e^{3t} & 3e^{2t} - 2e^{3t} \end{bmatrix}$$

12. Eigensystem: $\lambda_1 = 1, \quad \mathbf{v}_1 = [1 \quad 1]^{\mathrm{T}}; \qquad \lambda_2 = 2, \quad \mathbf{v}_2 = [4 \quad 3]^{\mathrm{T}}$

$$\Phi(t) = \begin{bmatrix} e^{\lambda_1 t}\mathbf{v}_1 & e^{\lambda_2 t}\mathbf{v}_2 \end{bmatrix} = \begin{bmatrix} e^t & 4e^{2t} \\ e^t & 3e^{2t} \end{bmatrix}$$

$$e^{\mathbf{A}t} = \begin{bmatrix} e^t & 4e^{2t} \\ e^t & 3e^{2t} \end{bmatrix} \begin{bmatrix} -3 & 4 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} -3e^t + 4e^{2t} & 4e^t - 4e^{2t} \\ -3e^t + 3e^{2t} & 4e^t - 3e^{2t} \end{bmatrix}$$

13. Eigensystem: $\lambda_1 = 1, \quad \mathbf{v}_1 = [1 \quad 1]^{\mathrm{T}}; \qquad \lambda_2 = 3, \quad \mathbf{v}_2 = [4 \quad 3]^{\mathrm{T}}$

$$\Phi(t) = \begin{bmatrix} e^{\lambda_1 t}\mathbf{v}_1 & e^{\lambda_2 t}\mathbf{v}_2 \end{bmatrix} = \begin{bmatrix} e^t & 4e^{3t} \\ e^t & 3e^{3t} \end{bmatrix}$$

$$e^{\mathbf{A}t} = \begin{bmatrix} e^t & 4e^{3t} \\ e^t & 3e^{3t} \end{bmatrix} \begin{bmatrix} -3 & 4 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} -3e^t + 4e^{3t} & 4e^t - 4e^{3t} \\ -3e^t + 3e^{3t} & 4e^t - 3e^{3t} \end{bmatrix}$$

14. Eigensystem: $\lambda_1 = 1, \quad \mathbf{v}_1 = [2 \quad 3]^{\mathrm{T}}; \qquad \lambda_2 = 3, \quad \mathbf{v}_2 = [3 \quad 4]^{\mathrm{T}}$

$$\Phi(t) = \begin{bmatrix} e^{\lambda_1 t}\mathbf{v}_1 & e^{\lambda_2 t}\mathbf{v}_2 \end{bmatrix} = \begin{bmatrix} 2e^t & 3e^{2t} \\ 3e^t & 4e^{2t} \end{bmatrix}$$

$$e^{\mathbf{A}t} = \begin{bmatrix} 2e^t & 3e^{2t} \\ 3e^t & 4e^{2t} \end{bmatrix} \begin{bmatrix} -4 & 3 \\ 3 & -2 \end{bmatrix} = \begin{bmatrix} -8e^t + 9e^{2t} & 6e^t - 6e^{2t} \\ -12e^t + 12e^{2t} & 9e^t - 8e^{2t} \end{bmatrix}$$

15. Eigensystem: $\lambda_1 = 1, \quad \mathbf{v}_1 = [2 \quad 1]^{\mathrm{T}}; \qquad \lambda_2 = 2, \quad \mathbf{v}_2 = [5 \quad 2]^{\mathrm{T}}$

$$\Phi(t) = \begin{bmatrix} e^{\lambda_1 t}\mathbf{v}_1 & e^{\lambda_2 t}\mathbf{v}_2 \end{bmatrix} = \begin{bmatrix} 2e^t & 5e^{2t} \\ e^t & 2e^{2t} \end{bmatrix}$$

$$e^{\mathbf{A}t} = \begin{bmatrix} 2e^t & 5e^{2t} \\ e^t & 2e^{2t} \end{bmatrix} \begin{bmatrix} -2 & 5 \\ 1 & -2 \end{bmatrix} = \begin{bmatrix} -4e^t + 5e^{2t} & 10e^t - 10e^{2t} \\ -2e^t + 2e^{2t} & 5e^t - 4e^{2t} \end{bmatrix}$$

16. Eigensystem: $\lambda_1 = 1, \quad \mathbf{v}_1 = [3 \quad 2]^{\mathrm{T}}; \qquad \lambda_2 = 2, \quad \mathbf{v}_2 = [5 \quad 3]^{\mathrm{T}}$

$$\Phi(t) = \begin{bmatrix} e^{\lambda_1 t}\mathbf{v}_1 & e^{\lambda_2 t}\mathbf{v}_2 \end{bmatrix} = \begin{bmatrix} 3e^t & 5e^{2t} \\ 2e^t & 3e^{2t} \end{bmatrix}$$

$$e^{\mathbf{A}t} = \begin{bmatrix} 3e^t & 5e^{2t} \\ 2e^t & 3e^{2t} \end{bmatrix} \begin{bmatrix} -3 & 5 \\ 2 & -3 \end{bmatrix} = \begin{bmatrix} -9e^t + 10e^{2t} & 15e^t - 15e^{2t} \\ -6e^t + 6e^{2t} & 10e^t - 9e^{2t} \end{bmatrix}$$

17. Eigensystem: $\lambda_1 = 2, \quad \mathbf{v}_1 = [1 \quad -1]^{\mathrm{T}}; \qquad \lambda_2 = 4, \quad \mathbf{v}_2 = [1 \quad 1]^{\mathrm{T}}$

$$\Phi(t) = \begin{bmatrix} e^{\lambda_1 t}\mathbf{v}_1 & e^{\lambda_2 t}\mathbf{v}_2 \end{bmatrix} = \begin{bmatrix} e^{2t} & e^{4t} \\ -e^{2t} & e^{4t} \end{bmatrix}$$

$$e^{\mathbf{A}t} = \begin{bmatrix} e^{2t} & e^{4t} \\ -e^{2t} & e^{4t} \end{bmatrix} \cdot \frac{1}{2}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} e^{2t}+e^{4t} & -e^{2t}+e^{4t} \\ -e^{2t}+e^{4t} & e^{2t}+e^{4t} \end{bmatrix}$$

18. Eigensystem: $\lambda_1 = 2, \quad \mathbf{v}_1 = [1 \quad -1]^{\mathrm{T}}; \qquad \lambda_2 = 6, \quad \mathbf{v}_2 = [1 \quad 1]^{\mathrm{T}}$

$$\Phi(t) = \begin{bmatrix} e^{\lambda_1 t}\mathbf{v}_1 & e^{\lambda_2 t}\mathbf{v}_2 \end{bmatrix} = \begin{bmatrix} e^{2t} & e^{6t} \\ -e^{2t} & e^{6t} \end{bmatrix}$$

$$e^{\mathbf{A}t} = \begin{bmatrix} e^{2t} & e^{6t} \\ -e^{2t} & e^{6t} \end{bmatrix} \cdot \frac{1}{2}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} e^{2t}+e^{6t} & -e^{2t}+e^{6t} \\ -e^{2t}+e^{6t} & e^{2t}+e^{6t} \end{bmatrix}$$

19. Eigensystem: $\lambda_1 = 5, \quad \mathbf{v}_1 = [1 \quad -2]^{\mathrm{T}}; \qquad \lambda_2 = 10, \quad \mathbf{v}_2 = [2 \quad 1]^{\mathrm{T}}$

$$\Phi(t) = \begin{bmatrix} e^{\lambda_1 t}\mathbf{v}_1 & e^{\lambda_2 t}\mathbf{v}_2 \end{bmatrix} = \begin{bmatrix} e^{5t} & 2e^{10t} \\ -2e^{5t} & e^{10t} \end{bmatrix}$$

$$e^{\mathbf{A}t} = \begin{bmatrix} e^{5t} & 2e^{10t} \\ -2e^{5t} & e^{10t} \end{bmatrix} \cdot \frac{1}{5}\begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} e^{5t}+4e^{10t} & -2e^{5t}+2e^{10t} \\ -2e^{5t}+2e^{10t} & 4e^{5t}+e^{10t} \end{bmatrix}$$

20. Eigensystem: $\lambda_1 = 5, \quad \mathbf{v}_1 = [1 \quad -2]^{\mathrm{T}}; \qquad \lambda_2 = 15, \quad \mathbf{v}_2 = [2 \quad 1]^{\mathrm{T}}$

$$\Phi(t) = \begin{bmatrix} e^{\lambda_1 t}\mathbf{v}_1 & e^{\lambda_2 t}\mathbf{v}_2 \end{bmatrix} = \begin{bmatrix} e^{5t} & 2e^{15t} \\ -2e^{5t} & e^{15t} \end{bmatrix}$$

$$e^{\mathbf{A}t} = \begin{bmatrix} e^{5t} & 2e^{15t} \\ -2e^{5t} & e^{15t} \end{bmatrix} \cdot \frac{1}{5}\begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} e^{5t}+4e^{15t} & -2e^{5t}+2e^{15t} \\ -2e^{5t}+2e^{15t} & 4e^{5t}+e^{15t} \end{bmatrix}$$

21. $\mathbf{A}^2 = \mathbf{0}$ so $e^{\mathbf{A}t} = \mathbf{I} + \mathbf{A}t = \begin{bmatrix} 1+t & -t \\ t & 1-t \end{bmatrix}$

22. $\mathbf{A}^2 = \mathbf{0}$ so $e^{\mathbf{A}t} = \mathbf{I} + \mathbf{A}t = \begin{bmatrix} 1+6t & 4t \\ -9t & 1-6t \end{bmatrix}$

23. $\mathbf{A}^3 = \mathbf{0}$ so $e^{\mathbf{A}t} = \mathbf{I} + \mathbf{A}t + \frac{1}{2}\mathbf{A}^2t^2 = \begin{bmatrix} 1+t & -t & -t-t^2 \\ t & 1-t & t-t^2 \\ 0 & 0 & 1 \end{bmatrix}$

24. $\mathbf{A}^3 = \mathbf{0}$ so $e^{\mathbf{A}t} = \mathbf{I} + \mathbf{A}t + \frac{1}{2}\mathbf{A}^2t^2 = \begin{bmatrix} 1+3t & 0 & -3t \\ 5t+18t^2 & 1 & 7t-18t^2 \\ 3t & 0 & 1-3t \end{bmatrix}$

25. $\mathbf{A} = 2\mathbf{I} + \mathbf{B}$ where $\mathbf{B}^2 = \mathbf{0}$, so $e^{\mathbf{A}t} = e^{2\mathbf{I}t}e^{\mathbf{B}t} = (e^{2t}\mathbf{I})(\mathbf{I} + \mathbf{B}t)$. Hence

$$e^{\mathbf{A}t} = \begin{bmatrix} e^{2t} & 5te^{2t} \\ 0 & e^{2t} \end{bmatrix}, \qquad \mathbf{x}(t) = e^{\mathbf{A}t}\begin{bmatrix} 4 \\ 7 \end{bmatrix} = e^{2t}\begin{bmatrix} 4+35t \\ 7 \end{bmatrix}$$

26. $\mathbf{A} = 7\mathbf{I} + \mathbf{B}$ where $\mathbf{B}^2 = \mathbf{0}$, so $e^{\mathbf{A}t} = e^{7\mathbf{I}t}e^{\mathbf{B}t} = (e^{7t}\mathbf{I})(\mathbf{I} + \mathbf{B}t)$. Hence

$$e^{\mathbf{A}t} = \begin{bmatrix} e^{7t} & 0 \\ 11te^{7t} & e^{7t} \end{bmatrix}, \qquad \mathbf{x}(t) = e^{\mathbf{A}t}\begin{bmatrix} 5 \\ -10 \end{bmatrix} = e^{7t}\begin{bmatrix} 5 \\ -10+55t \end{bmatrix}$$

27. $\mathbf{A} = \mathbf{I} + \mathbf{B}$ where $\mathbf{B}^3 = \mathbf{0}$, so $e^{\mathbf{A}t} = e^{\mathbf{I}t}e^{\mathbf{B}t} = (e^t\,\mathbf{I})(\mathbf{I} + t + \tfrac{1}{2}\mathbf{B}^2t^2)$. Hence

$$e^{\mathbf{A}t} = \begin{bmatrix} e^t & 2te^t & (3t+2t^2)e^t \\ 0 & e^t & 2te^t \\ 0 & 0 & e^t \end{bmatrix}, \quad \mathbf{x}(t) = e^{\mathbf{A}t}\begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} = e^t\begin{bmatrix} 4+28t+12t^2 \\ 5+12t \\ 6 \end{bmatrix}$$

28. $\mathbf{A} = 5\mathbf{I} + \mathbf{B}$ where $\mathbf{B}^3 = \mathbf{0}$, so $e^{\mathbf{A}t} = e^{5\mathbf{I}t}e^{\mathbf{B}t} = (e^{5t}\,\mathbf{I})(\mathbf{I} + \mathbf{B}t + \tfrac{1}{2}\mathbf{B}^2t^2)$. Hence

$$e^{\mathbf{A}t} = \begin{bmatrix} e^{5t} & 0 & 0 \\ 10te^{5t} & e^{5t} & 0 \\ (20t+150t^2)e^{5t} & 30te^{5t} & e^{5t} \end{bmatrix}, \quad \mathbf{x}(t) = e^{\mathbf{A}t}\begin{bmatrix} 40 \\ 50 \\ 60 \end{bmatrix} = e^{5t}\begin{bmatrix} 40 \\ 50+400t \\ 60+2300t+6000t^2 \end{bmatrix}$$

29. $\mathbf{A} = \mathbf{I} + \mathbf{B}$ where $\mathbf{B}^4 = \mathbf{0}$, so $e^{\mathbf{A}t} = e^{\mathbf{I}t}e^{\mathbf{B}t} = (e^t\,\mathbf{I})(\mathbf{I} + \mathbf{B}t + \tfrac{1}{2}\mathbf{B}^2t^2 + \tfrac{1}{6}\mathbf{B}^3t^3)$. Hence

$$e^{\mathbf{A}t} = e^t \begin{bmatrix} 1 & 2t & 3t+6t^2 & 4t+6t^2+4t^3 \\ 0 & 1 & 6t & 3t+6t^2 \\ 0 & 0 & 1 & 2t \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{x}(t) = e^{\mathbf{A}t}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = e^t \begin{bmatrix} 1+9t+12t^2+4t^3 \\ 1+9t+6t^2 \\ 1+2t \\ 1 \end{bmatrix}$$

30. $\mathbf{A} = 3\mathbf{I}+\mathbf{B}$ where $\mathbf{B}^4 = \mathbf{0}$, so $e^{\mathbf{A}t} = e^{3\mathbf{I}t}e^{\mathbf{B}t} = (e^{3t}\,\mathbf{I})(\mathbf{I}+\mathbf{B}t+\frac{1}{2}\mathbf{B}^2t^2+\frac{1}{6}\mathbf{B}^3t^3)$. Hence

$$e^{\mathbf{A}t} = e^{3t} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 6t & 1 & 0 & 0 \\ 9t+18t^2 & 6t & 1 & 0 \\ 12t+54t^2+36t^3 & 9t+18t^2 & 6t & 1 \end{bmatrix}, \quad \mathbf{x}(t) = e^{\mathbf{A}t}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = e^{3t} \begin{bmatrix} 1 \\ 1+6t \\ 1+15t+18t^2 \\ 1+27t+72t^2+36t^3 \end{bmatrix}$$

33. $e^{\mathbf{A}t} = \mathbf{I}\cosh t+\mathbf{A}\sinh t = \begin{bmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{bmatrix}$, so the general solution of $\mathbf{x}' = \mathbf{A}\mathbf{x}$ is

$$\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{c} = \begin{bmatrix} c_1\cosh t+c_2\sinh t \\ c_1\sinh t+c_2\cosh t \end{bmatrix}.$$

34. Direct calculation gives $\mathbf{A}^2 = -4\mathbf{I}$, and it follows that $\mathbf{A}^3 = -4\mathbf{A}$ and $\mathbf{A}^4 = 16\mathbf{I}$. Therefore

$$\begin{aligned} e^{\mathbf{A}t} &= \mathbf{I}+\mathbf{A}t-\frac{4\mathbf{I}t^2}{2!}-\frac{4\mathbf{A}t^3}{3!}+\frac{16\mathbf{I}t^4}{4!}+\frac{16\mathbf{A}t^5}{5!}+\cdots \\ &= \mathbf{I}\left[1-\frac{(2t)^2}{2!}+\frac{(2t)^4}{4!}+\cdots\right]+\frac{1}{2}\mathbf{A}\left[(2t)-\frac{(2t)^3}{3!}+\frac{(2t)^5}{5!}+\cdots\right] \\ e^{\mathbf{A}t} &= \mathbf{I}\cos 2t+\tfrac{1}{2}\mathbf{A}\sin 2t \end{aligned}$$

In Problems 35–40 we give first the linearly independent generalized eigenvectors $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$ of the matrix $\mathbf{A}$ and the corresponding solution vectors $\mathbf{x}_1(t), \mathbf{x}_2(t), \cdots, \mathbf{x}_n(t)$ defined by Eq. (34) in the text, then the fundamental matrix $\Phi(t) = \left[\mathbf{x}_1(t) \quad \mathbf{x}_2(t) \quad \cdots \quad \mathbf{x}_n(t)\right]$. Finally we calculate the exponential matrix $e^{\mathbf{A}t} = \Phi(t)\Phi(0)^{-1}$.

35. $\lambda = 3$: $\quad \mathbf{u}_1 = [4 \quad 0]^T, \quad \mathbf{u}_2 = [0 \quad 1]^T$

$\{\mathbf{u}_1, \mathbf{u}_2\}$ is a length 2 chain based on the ordinary (rank 1) eigenvector $\mathbf{u}_1$, so $\mathbf{u}_2$ is a generalized eigenvector of rank 2.

$$\mathbf{x}_1(t) = e^{\lambda t}\mathbf{u}_1, \quad \mathbf{x}_2(t) = e^{\lambda t}\left(\mathbf{u}_2 + (\mathbf{A}-\lambda\mathbf{I})\mathbf{u}_2 t\right)$$

$$\Phi(t) = [\mathbf{x}_1(t) \quad \mathbf{x}_2(t)] = e^{3t}\begin{bmatrix} 4 & 4t \\ 0 & 1 \end{bmatrix}$$

$$e^{\mathbf{A}t} = e^{3t}\begin{bmatrix} 4 & 4t \\ 0 & 1 \end{bmatrix}\cdot\frac{1}{4}\begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix} = e^{3t}\begin{bmatrix} 1 & 4t \\ 0 & 1 \end{bmatrix}$$

36. $\lambda = 1: \quad \mathbf{u}_1 = [8 \quad 0 \quad 0]^{\mathrm{T}}, \quad \mathbf{u}_2 = [5 \quad 4 \quad 0]^{\mathrm{T}}, \quad \mathbf{u}_3 = [0 \quad 1 \quad 1]^{\mathrm{T}}$

$\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is a length 3 chain based on the ordinary (rank 1) eigenvector $\mathbf{u}_1$, so $\mathbf{u}_2$ and $\mathbf{u}_3$ are generalized eigenvectors of ranks 2 and 3 (respectively).

$$\mathbf{x}_1(t) = e^{\lambda t}\mathbf{u}_1, \quad \mathbf{x}_2(t) = e^{\lambda t}\left(\mathbf{u}_2 + (\mathbf{A}-\lambda\mathbf{I})\mathbf{u}_2 t\right),$$

$$\mathbf{x}_3(t) = e^{\lambda t}\left(\mathbf{u}_3 + (\mathbf{A}-\lambda\mathbf{I})\mathbf{u}_3 t + (\mathbf{A}-\lambda\mathbf{I})^2\mathbf{u}_3 t^2/2\right)$$

$$\Phi(t) = [\mathbf{x}_1(t) \quad \mathbf{x}_2(t) \quad \mathbf{x}_3(t)] = e^{t}\begin{bmatrix} 8 & 5+8t & 5t+4t^2 \\ 0 & 4 & 1+4t \\ 0 & 0 & 1 \end{bmatrix}$$

$$e^{\mathbf{A}t} = e^{t}\begin{bmatrix} 8 & 5+8t & 5t+4t^2 \\ 0 & 4 & 1+4t \\ 0 & 0 & 1 \end{bmatrix}\cdot\frac{1}{32}\begin{bmatrix} 4 & -5 & 5 \\ 0 & 8 & -8 \\ 0 & 0 & 32 \end{bmatrix} = e^{t}\begin{bmatrix} 1 & 2t & 3t+4t^2 \\ 0 & 1 & 4t \\ 0 & 0 & 1 \end{bmatrix}$$

37. $\lambda_1 = 2: \quad \mathbf{u}_1 = [1 \quad 0 \quad 0]^{\mathrm{T}}, \quad \mathbf{x}_1(t) = e^{\lambda_1 t}\mathbf{u}_1$

$\lambda_2 = 1: \quad \mathbf{u}_2 = [9 \quad -3 \quad 0]^{\mathrm{T}}, \quad \mathbf{u}_3 = [10 \quad 1 \quad -1]^{\mathrm{T}}$

$\{\mathbf{u}_2, \mathbf{u}_3\}$ is a length 2 chain based on the ordinary (rank 1) eigenvector $\mathbf{u}_2$, so $\mathbf{u}_3$ is a generalized eigenvector of rank 2.

$$\mathbf{x}_2(t) = e^{\lambda_2 t}\mathbf{u}_2, \quad \mathbf{x}_3(t) = e^{\lambda_2 t}\left(\mathbf{u}_3 + (\mathbf{A}-\lambda_2\mathbf{I})\mathbf{u}_3 t\right)$$

$$\Phi(t) = [\mathbf{x}_1(t) \quad \mathbf{x}_2(t) \quad \mathbf{x}_3(t)] = \begin{bmatrix} e^{2t} & 9e^{t} & (10+9t)e^{t} \\ 0 & -3e^{t} & (1-3t)e^{t} \\ 0 & 0 & -e^{t} \end{bmatrix}$$

$$e^{\mathbf{A}t} = \begin{bmatrix} e^{2t} & 9e^{t} & (10+9t)e^{t} \\ 0 & -3e^{t} & (1-3t)e^{t} \\ 0 & 0 & -e^{t} \end{bmatrix}\cdot\frac{1}{3}\begin{bmatrix} 3 & 9 & 13 \\ 0 & -1 & -1 \\ 0 & 0 & -3 \end{bmatrix}$$

$$= \begin{bmatrix} e^{2t} & -3e^{t}+3e^{2t} & (-13-9t)e^{t}+13e^{2t} \\ 0 & e^{t} & 3t\,e^{t} \\ 0 & 0 & e^{t} \end{bmatrix}$$

38. $\lambda_1 = 10: \quad \mathbf{u}_1 = [4 \quad 1 \quad 0]^{\mathrm{T}}, \quad \mathbf{x}_1(t) = e^{\lambda_1 t}\mathbf{u}_1$

$\lambda_2 = 5: \quad \mathbf{u}_2 = [50 \quad 0 \quad 0]^{\mathrm{T}}, \quad \mathbf{u}_3 = [0 \quad 4 \quad -1]^{\mathrm{T}}$

$\{\mathbf{u}_2, \mathbf{u}_3\}$ is a length 2 chain based on the ordinary (rank 1) eigenvector $\mathbf{u}_2$, so $\mathbf{u}_3$ is a generalized eigenvector of rank 2.

$\mathbf{x}_2(t) = e^{\lambda_2 t}\mathbf{u}_2, \quad \mathbf{x}_3(t) = e^{\lambda_2 t}\left(\mathbf{u}_3 + (\mathbf{A}-\lambda_2\mathbf{I})\mathbf{u}_3 t\right)$

$$\Phi(t) = [\mathbf{x}_1(t) \quad \mathbf{x}_2(t) \quad \mathbf{x}_3(t)] = \begin{bmatrix} 4e^{10t} & 50e^{5t} & 50t\,e^{5t} \\ e^{10t} & 0 & 4e^{5t} \\ 0 & 0 & -e^{5t} \end{bmatrix}$$

$$e^{\mathbf{A}t} = \begin{bmatrix} 4e^{10t} & 50e^{5t} & 50t\,e^{5t} \\ e^{10t} & 0 & 4e^{5t} \\ 0 & 0 & -e^{5t} \end{bmatrix} \cdot \frac{1}{50} \begin{bmatrix} 0 & 50 & 200 \\ 1 & -4 & -16 \\ 0 & 0 & -50 \end{bmatrix}$$

$$= \begin{bmatrix} e^{5t} & 4e^{10t}-4e^{5t} & 16e^{10t}-(16+50t)e^{5t} \\ 0 & e^{10t} & 4e^{10t}-4e^{5t} \\ 0 & 0 & e^{5t} \end{bmatrix}$$

39. $\lambda_2 = 1: \quad \mathbf{u}_1 = [3 \quad 0 \quad 0 \quad 0]^{\mathrm{T}}, \quad \mathbf{u}_2 = [0 \quad 1 \quad 0 \quad 0]^{\mathrm{T}}$

$\{\mathbf{u}_1, \mathbf{u}_2\}$ is a length 2 chain based on the ordinary (rank 1) eigenvector $\mathbf{u}_1$, so $\mathbf{u}_2$ is a generalized eigenvector of rank 2.

$\mathbf{x}_1(t) = e^{\lambda_1 t}\mathbf{u}_1, \quad \mathbf{x}_2(t) = e^{\lambda_1 t}\left(\mathbf{u}_2 + (\mathbf{A}-\lambda_1\mathbf{I})\mathbf{u}_2 t\right)$

$\lambda_2 = 2: \quad \mathbf{u}_3 = [144 \quad 36 \quad 12 \quad 0]^{\mathrm{T}}, \quad \mathbf{u}_4 = [0 \quad 27 \quad 17 \quad 4]^{\mathrm{T}}$

$\{\mathbf{u}_3, \mathbf{u}_4\}$ is a length 2 chain based on the ordinary (rank 1) eigenvector $\mathbf{u}_3$, so $\mathbf{u}_4$ is a generalized eigenvector of rank 2.

$\mathbf{x}_3(t) = e^{\lambda_2 t}\mathbf{u}_3, \quad \mathbf{x}_4(t) = e^{\lambda_2 t}\left(\mathbf{u}_4 + (\mathbf{A}-\lambda_2\mathbf{I})\mathbf{u}_4 t\right)$

$$\Phi(t) = [\mathbf{x}_1(t) \quad \mathbf{x}_2(t) \quad \mathbf{x}_3(t) \quad \mathbf{x}_4(t)] = \begin{bmatrix} 3e^{t} & 3t\,e^{t} & 144e^{2t} & 144t\,e^{2t} \\ 0 & e^{t} & 36e^{2t} & (27+36t)\,e^{2t} \\ 0 & 0 & 12e^{2t} & (17+12t)\,e^{2t} \\ 0 & 0 & 0 & 4e^{2t} \end{bmatrix}$$

$$e^{\mathbf{A}t} = \begin{bmatrix} 3e^t & 3te^t & 144e^{2t} & 144te^{2t} \\ 0 & e^t & 36e^{2t} & (27+36t)e^{2t} \\ 0 & 0 & 12e^{2t} & (17+12t)e^{2t} \\ 0 & 0 & 0 & 4e^{2t} \end{bmatrix} \cdot \frac{1}{48} \begin{bmatrix} 16 & 0 & -192 & 816 \\ 0 & 48 & -144 & 288 \\ 0 & 0 & 4 & -17 \\ 0 & 0 & 0 & 12 \end{bmatrix}$$

$$= \begin{bmatrix} e^t & 3te^t & (-12-9t)e^t + 12te^{2t} & (51+18t)e^t + (-51+36t)e^{2t} \\ 0 & e^t & -3e^t + 3e^{2t} & 6e^t + (-6+9t)e^{2t} \\ 0 & 0 & e^{2t} & 3te^{2t} \\ 0 & 0 & 0 & e^{2t} \end{bmatrix}$$

40. $\lambda_1 = 3: \quad \mathbf{u}_1 = [100 \quad 20 \quad 4 \quad 1]^{\mathrm{T}}, \quad \mathbf{x}_1(t) = e^{\lambda_1 t}\mathbf{u}_1$

$\lambda = 2: \quad \mathbf{u}_2 = [16 \quad 0 \quad 0 \quad 0]^{\mathrm{T}}, \quad \mathbf{u}_3 = [0 \quad 4 \quad 0 \quad 0]^{\mathrm{T}}, \quad \mathbf{u}_4 = [0 \quad -1 \quad 1 \quad 0]^{\mathrm{T}}$

$\{\mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4\}$ is a length 3 chain based on the ordinary (rank 1) eigenvector $\mathbf{u}_2$, so $\mathbf{u}_3$ and $\mathbf{u}_4$ are generalized eigenvectors of ranks 2 and 3 (respectively).

$$\mathbf{x}_2(t) = e^{\lambda_2 t}\mathbf{u}_2, \quad \mathbf{x}_3(t) = e^{\lambda_2 t}\left(\mathbf{u}_3 + (\mathbf{A} - \lambda_2\mathbf{I})\mathbf{u}_3 t\right),$$

$$\mathbf{x}_4(t) = e^{\lambda_2 t}\left(\mathbf{u}_4 + (\mathbf{A} - \lambda_2\mathbf{I})\mathbf{u}_4 t + (\mathbf{A} - \lambda_2\mathbf{I})^2\mathbf{u}_4 t^2/2\right)$$

$$\mathbf{\Phi}(t) = [\mathbf{x}_1(t) \quad \mathbf{x}_2(t) \quad \mathbf{x}_3(t) \quad \mathbf{x}_4(t)] = \begin{bmatrix} 100e^{3t} & 16e^{2t} & 16te^{2t} & 8t^2e^{2t} \\ 20e^{3t} & 0 & 4e^{2t} & (-1+4t)e^{2t} \\ 4e^{3t} & 0 & 0 & e^{2t} \\ e^{3t} & 0 & 0 & 0 \end{bmatrix}$$

$$e^{\mathbf{A}t} = \begin{bmatrix} 100e^{3t} & 16e^{2t} & 16te^{2t} & 8t^2e^{2t} \\ 20e^{3t} & 0 & 4e^{2t} & (-1+4t)e^{2t} \\ 4e^{3t} & 0 & 0 & e^{2t} \\ e^{3t} & 0 & 0 & 0 \end{bmatrix} \cdot \frac{1}{16} \begin{bmatrix} 0 & 0 & 0 & 16 \\ 1 & 0 & 0 & -100 \\ 0 & 4 & 4 & -96 \\ 0 & 0 & 16 & -64 \end{bmatrix}$$

$$= \begin{bmatrix} e^{2t} & 4te^{2t} & (4t+8t^2)e^{2t} & 100e^{3t} - (100+96t+32t^2)e^{2t} \\ 0 & e^{2t} & 4te^{2t} & 20e^{3t} - (20+16t)e^{2t} \\ 0 & 0 & e^{2t} & 4e^{3t} - 4e^{2t} \\ 0 & 0 & 0 & e^{3t} \end{bmatrix}$$

SECTION 8.2

NONHOMOGENEOUS LINEAR SYSTEMS

1. Substitution of the trial solution $x_p(t) = a,\; y_p(t) = b$ yields the equations $a + 2b + 3 = 0,\; 2a + b - 2 = 0$ with solution $a = 7/3,\; b = -8/3$. Thus we obtain the particular solution $x(t) = 7/3,\; y(t) = -8/3$.

2. When we substitute the trial solution $x_p(t) = a_1 + b_1 t,\; y_p(t) = a_2 + b_2 t$ and collect coefficients, we get the equations

$$\begin{aligned} 2a_1 + 3a_2 + 5 &= b_1 \qquad & 2b_1 + 3b_2 &= 0 \\ 2a_1 + a_2 &= b_2 \qquad & 2b_1 + b_2 &= 2. \end{aligned}$$

We first solve the second pair for $b_1 = 3/2,\; b_2 = -1$. Then we can solve the first pair for $a_1 = 1/8,\; a_2 = -5/4$. This gives the particular solution

$$x(t) = \frac{1}{8}(1 + 12t), \qquad y(t) = -\frac{1}{4}(5 + 4t).$$

3. When we substitute the trial solution

$$x_p = a_1 + b_1 t + c_1 t^2, \qquad y_p = a_2 + b_2 t + c_2 t^2$$

and collect coefficients, we get the equations

$$\begin{aligned} 3a_1 + 4a_2 &= b_1 \qquad & 3b_1 + 4b_2 &= 2c_1 \qquad & 3c_1 + 4c_2 &= 0 \\ 3a_1 + 2a_2 &= b_2 \qquad & 3b_1 + 2b_2 &= 2c_2 \qquad & 3c_1 + 2c_2 + 1 &= 0. \end{aligned}$$

Working backwards, we solve first for $c_1 = -2/3,\; c_2 = 1/2$, then for $b_1 = 10/9,\; b_2 = -7/6$, and finally for $a_1 = -31/27,\; a_2 = 41/36$. This determines the particular solution $x_p(t),\; y_p(t)$. Next, the coefficient matrix of the associated homogeneous system has eigenvalues $\lambda_1 = -1$ and $\lambda_2 = 6$ with eigenvectors $\mathbf{v}_1 = [1 \quad -1]^{\mathrm{T}}$ and $\mathbf{v}_2 = [4 \quad 3]^{\mathrm{T}}$, respectively, so the complementary solution is given by

$$x_c(t) = c_1 e^{-t} + 4c_2 e^{6t}, \qquad x_c(t) = -c_1 e^{-t} + 3c_2 e^{6t}.$$

When we impose the initial conditions $x(0) = 0,\; y(0) = 0$ on the general solution $x(t) = x_c(t) + x_p(t),\quad y(t) = y_c(t) + y_p(t)$ we find that $c_1 = 8/7,\; c_2 = 1/756$. This finally gives the desired particular solution

$$x(t) = \frac{1}{756}(\ 864\,e^{-t} + 4\,e^{6t} - 868 + 840\,t - 504\,t^2)$$

$$y(t) = \frac{1}{756}(-864\,e^{-t} + 3\,e^{6t} + 861 - 882\,t + 378\,t^2).$$

4. The coefficient matrix of the associated homogeneous system has eigenvalues $\lambda_1 = -5$ and $\lambda_2 = -2$ with eigenvectors $\mathbf{v}_1 = [1 \quad 1]^T$ and $\mathbf{v}_2 = [1 \quad -6]^T$, respectively, so the complementary solution is given by

$$x_c(t) = c_1 e^{5t} + c_2 e^{-2t}, \quad y_c(t) = c_1 e^{5t} - 6c_2 e^{-2t}.$$

Then we try $x_p(t) = a\,e^t$, $y_p(t) = b\,e^t$ and find readily the particular solution $x_p(t) = -\frac{1}{12}e^t$, $y_p(t) = -\frac{3}{4}e^t$. Thus the general solution is

$$x(t) = c_1 e^{5t} + c_2 e^{-2t} - \tfrac{1}{12}e^t, \qquad y(t) = c_1 e^{5t} - 6c_2 e^{-2t} - \tfrac{3}{4}e^t.$$

Finally we apply the initial conditions $x(0) = y(0) = 1$ to determine $c_1 = 33/28$ and $c_2 = -2/21$. The resulting particular solution is given by

$$x(t) = \frac{1}{84}(99\,e^{5t} - 8e^{-2t} - 7e^t), \qquad y(t) = \frac{1}{84}(99\,e^{5t} + 48e^{-2t} - 63e^t).$$

5. The coefficient matrix of the associated homogeneous system has eigenvalues $\lambda_1 = -1$ and $\lambda_2 = 5$, so the nonhomogeneous term e^{-t} duplicates part of the complementary solution. We therefore try the particular solution

$$x_p(t) = a_1 + b_1 e^{-t} + c_1 t\,e^{-t}, \quad y_p(t) = a_2 + b_2 e^{-t} + c_2 t\,e^{-t}.$$

Upon solving the six linear equations we get by collecting coefficients after substitution of this trial solution into the given nonhomogeneous system, we obtain the particular solution

$$x(t) = \frac{1}{3}(-12 - e^{-t} - 7t\,e^{-t}), \quad y(t) = \frac{1}{3}(-6 - 7t\,e^{-t}).$$

6. The coefficient matrix of the associated homogeneous system has eigenvalues $\lambda = \frac{1}{2}\left(7 \pm \sqrt{89}\right)$ so there is no duplication. We therefore try the particular solution

$$x_p(t) = b_1 e^t + c_1 t\,e^t, \quad y_p(t) = b_2 e^t + c_2 t\,e^t.$$

Upon solving the four linear equations we get by collecting coefficients after substitution of this trial solution into the given nonhomogeneous system, we obtain the particular solution

$$x(t) = -\frac{1}{256}(91+16t)e^t, \quad y(t) = \frac{1}{32}(25+16t)e^t.$$

7. First we try the particular solution

$$x_p(t) = a_1 \sin t + b_1 \cos t, \qquad y_p(t) = a_2 \sin t + b_2 \cos t.$$

Upon solving the four linear equations we get by collecting coefficients after substitution of this trial solution into the given nonhomogeneous system, we find that $a_1 = -21/82$, $b_1 = -25/82$, $a_2 = -15/41$, $b_2 = -12/41$. The coefficient matrix of the associated homogeneous system has eigenvalues $\lambda_1 = 1$ and $\lambda_2 = -9$ with eigenvectors $\mathbf{v}_1 = [1 \quad 1]^T$ and $\mathbf{v}_2 = [2 \quad -3]^T$, respectively, so the complementary solution is given by

$$x_c(t) = c_1 e^t + 2c_2 e^{-9t}, \qquad y_c(t) = c_1 e^t - 3c_2 e^{-9t}.$$

When we impose the initial conditions $x(0) = 1$, $y(0) = 0$, we find that $c_1 = 9/10$ and $c_2 = 83/410$. It follows that the desired particular solution $x = x_c + x_p$, $y = y_c + y_p$ is given by

$$x(t) = \frac{1}{410}(369e^t + 166e^{-9t} - 125\cos t - 105\sin t)$$
$$y(t) = \frac{1}{410}(369e^t - 249e^{-9t} - 120\cos t - 150\sin t).$$

8. The coefficient matrix of the associated homogeneous system has eigenvalues $\lambda = \pm 2i$, so the complementary function involves $\cos 2t$ and $\sin 2t$. There being therefore no duplication, we substitute the trial solution

$$x_p(t) = a_1 \sin t + b_1 \cos t, \qquad y_p(t) = a_2 \sin t + b_2 \cos t$$

into the given nonhomogeneous system. Upon solving the four linear equations that result upon collection of coefficients, we obtain the particular solution

$$x(t) = \frac{1}{3}(17\cos t + 2\sin t), \qquad y(t) = \frac{1}{3}(3\cos t + 5\sin t).$$

9. Here the associated homogeneous system is the same as in Problem 8, so the nonhomogeneous term $\cos 2t$ term duplicates the complementary function. We therefore substitute the trial solution

$$\begin{aligned} x_p(t) &= a_1 \sin 2t + b_1 \cos 2t + c_1 t \sin 2t + d_1 t \cos 2t \\ y_p(t) &= a_2 \sin 2t + b_2 \cos 2t + c_2 t \sin 2t + d_2 t \cos 2t \end{aligned}$$

and use a computer algebra system to solve the system of 8 linear equations that results when we collect coefficients in the usual way. This gives the particular solution

$$x(t) = \frac{1}{4}(\sin 2t + 2t\cos 2t + t\sin 2t), \qquad y(t) = \frac{1}{4}t\sin 2t.$$

10. The coefficient matrix of the associated homogeneous system has eigenvalues $\lambda = \pm i\sqrt{3}$, so there is no duplication. Substitution of the trial solution

$$x_p(t) = a_1 e^t \cos t + b_1 e^t \sin t, \qquad y_p(t) = a_2 e^t \cos t + b_2 e^t \sin t$$

yields the equations

$$\begin{aligned} 2a_2 + b_1 &= 0 & \qquad -2a_1 + 2a_2 + b_2 &= 0 \\ 2b_2 - a_1 &= 0 & \qquad -a_2 - 2b_1 + 2b_2 &= 1. \end{aligned}$$

The first two equations enable us to eliminate two of the variables immediately, and we readily solve for the values $a_1 = 4/13$, $a_2 = 3/13$, $b_1 = -6/13$, $b_2 = 2/13$ that give the particular solution

$$x(t) = \frac{1}{13}e^t(4\cos t - 6\sin t), \qquad y(t) = \frac{1}{13}e^t(3\cos t + 2\sin t).$$

11. The coefficient matrix of the associated homogeneous system has eigenvalues $\lambda_1 = 0$ and $\lambda_2 = 4$, so there is duplication of constant terms. We therefore substitute the particular solution

$$x_p(t) = a_1 + b_1 t, \qquad y_p(t) = a_2 + b_2 t$$

and solve the resulting equations for $a_1 = -2$, $a_2 = 0$, $b_1 = -2$, $b_2 = 1$. The eigenvectors of the coefficient matrix associated with the eigenvalues $\lambda_1 = 0$ and $\lambda_2 = 4$ are $\mathbf{v}_1 = [2 \quad -1]^{\mathrm{T}}$ and $\mathbf{v}_2 = [2 \quad 1]^{\mathrm{T}}$, respectively, so the general solution of the given nonhomogeneous system is given by

$$x(t) = 2c_1 + 2c_2 e^{4t} - 2 - 2t, \qquad y(t) = -c_1 + c_2 e^{4t} + t.$$

When we impose the initial conditions $x(0) = 1$, $y(0) = -1$ we find readily that $c_1 = 5/4$, $c_2 = 1/4$. This gives the desired particular solution

$$x(t) = \tfrac{1}{2}(1-4t+e^{4t}), \qquad y(t) = \tfrac{1}{4}(-5+4t+e^{4t}).$$

12. The coefficient matrix of the associated homogeneous system has eigenvalues $\lambda_1 = 0$ and $\lambda_2 = 2$, so there is duplication of constant terms in the first natural attempt. We must multiply the *t*-terms by t and include all lower degree terms in the trial solution. Thus we substitute the the trial solution

$$x_p(t) = a_1 + b_1 t + c_1 t^2, \qquad y_p(t) = a_2 + b_2 t + c_2 t^2.$$

The resulting six equations in the coefficients are satisfied by $a_1 = b_1 = a_2 = b_2 = 0$, $c_1 = 1,\ c_2 = -1$. This gives the particular solution $x(t) = t^2,\ \ y(t) = -t^2$.

13. The coefficient matrix of the associated homogeneous system has eigenvalues $\lambda_1 = 1$ and $\lambda_2 = 3$, so there is duplication of e^t terms. We therefore substitute the trial solution

$$x_p(t) = (a_1 + b_1 t)e^t, \quad y_p(t) = (a_2 + b_2 t)e^t.$$

This leads readily to the particular solution

$$x(t) = \frac{1}{2}(1+5t)e^t, \qquad y(t) = -\frac{5}{2}t\,e^t.$$

14. The coefficient matrix of the associated homogeneous system has eigenvalues $\lambda_1 = 0$ and $\lambda_2 = 4$, so there is duplication of both constant terms and e^{4t} terms. We therefore substitute the particular solution

$$x_p(t) = a_1 + b_1 t + c_1 e^{4t} + d_1 t\,e^{4t}, \quad y_p(t) = a_2 + b_2 t + c_2 e^{4t} + d_2 t\,e^{4t}.$$

When we use a computer algebra system to solve the resulting system of 8 equations in 8 unknowns, we find that a_2 and c_2 can be chosen arbitrarily. With both zero we get the particular solution

$$x(t) = \frac{1}{8}\left(-2+4t-e^{4t}+2t\,e^{4t}\right), \qquad y(t) = \frac{t}{2}\left(-2+e^{4t}\right).$$

In Problems 15 and 16 the amounts $x_1(t)$ and $x_2(t)$ in the two tanks satisfy the equations

$$x_1' = rc_0 - k_1 x_1, \qquad x_2' = k_1 x_1 - k_2 x_2$$

where $k_i = r/V_i$ in terms of the flow rate r, the inflowing concentration c_0, and the volumes V_1 and V_2 of the two tanks.

15. **(a)** We solve the initial value problem

$$\begin{aligned} x_1' &= 20 - x_1/10, & x_1(0) &= 0 \\ x_2' &= x_1/10 - x_2/20, & x_2(0) &= 0 \end{aligned}$$

for $x_1(t) = 200(1-e^{-t/10}), \quad x_2(t) = 400(1+e^{-t/10}-2e^{-t/20})$.

(b) Evidently $x_1(t) \to 200$ gal and $x_2(t) \to 400$ gal as $t \to \infty$.

(c) It takes about 6 min 56 sec for tank 1 to reach a salt concentration of 1 lb/gal, and about 24 min 34 sec for tank 2 to reach this concentration.

16. **(a)** We solve the initial value problem

$$\begin{aligned} x_1' &= 30 - x_1/20, & x_1(0) &= 0 \\ x_2' &= x_1/20 - x_2/10, & x_2(0) &= 0 \end{aligned}$$

for $x_1(t) = 600(1-e^{-t/20}), \quad x_2(t) = 300(1+e^{-t/10}-2e^{-t/20})$.

(b) Evidently $x_1(t) \to 600$ gal and $x_2(t) \to 300$ gal as $t \to \infty$.

(c) It takes about 8 min 7 sec for tank 1 to reach a salt concentration of 1 lb/gal, and about 17 min 13 sec for tank 2 to reach this concentration.

In Problems 17–34 we apply the variation of parameters formula in Eq. (28) of Section 8.2. The answers shown below were actually calculated using the Mathematica code listed in the application material for Section 8.2. For instance, for Problem 17 we first enter the coefficient matrix

```
A = {{6,  -7},
     {1,  -2}};
```

the initial vector

```
x0 = {{0},
      {0}};
```

and the vector

```
f[t_] := {{60},
          {90}};
```

of nonhomogeneous terms. It simplifies the notation to rename Mathematica's exponential matrix function by defining

```
exp[A_] := MatrixExp[A]
```

Then the integral in the variation of parameters formula is given by

```
integral =
Integrate[exp[-A*s] . f[s], {s, 0, t}] // Simplify
```

$$\begin{bmatrix} -102+7e^{-5t}+95e^{t} \\ -96+e^{-5t}+95e^{t} \end{bmatrix}.$$

Finally the desired particular solution is given by

```
solution =
exp[A*t] . (x0 + integral) // Simplify
```

$$\begin{bmatrix} 102-7e^{-5t}-95e^{t} \\ 96-e^{-5t}-95e^{t} \end{bmatrix}.$$

(Maple and MATLAB versions of this computation are provided in the applications manual that accompanies the textbook.)

In each succeeding problem, we need only substitute the given coefficient matrix **A**, initial vector **x0**, and the vector **f** of nonhomogeneous terms in the above commands, and then re-execute them in turn. We give below only the component functions of the final results.

17. $x_1(t) = 102-95e^{-t}-7e^{5t}, \quad x_2(t) = 96-95e^{-t}-e^{5t}$

18. $x_1(t) = 68-110t-75e^{-t}+7e^{5t}, \quad x_2(t) = 74-80t-75e^{-t}+e^{5t}$

19. $x_1(t) = -70-60t+16e^{-3t}+54e^{2t}, \quad x_2(t) = 5-60t-32e^{-3t}+27e^{2t}$

20. $x_1(t) = 3e^{2t}+60te^{2t}-3e^{-3t}, \quad x_2(t) = -6e^{2t}+30te^{2t}+6e^{-3t}$

21. $x_1(t) = -e^{-t}-14e^{2t}+15e^{3t}, \quad x_2(t) = -5e^{-t}-10e^{2t}+15e^{3t}$

22. $x_1(t) = -10e^{-t}-7te^{-t}+10e^{3t}-5te^{3t}, \quad x_2(t) = -15e^{-t}-35te^{-t}+15e^{3t}-5te^{3t}$

23. $x_1(t) = 3+11t+8t^2, \quad x_2(t) = 5+17t+24t^2$

24. $x_1(t) = 2 + t + \ln t, \quad x_2(t) = 5 + 3t - \dfrac{1}{t} + 3\ln t$

25. $x_1(t) = -1 + 8t + \cos t - 8\sin t, \quad x_2(t) = -2 + 4t + 2\cos t - 3\sin t$

26. $x_1(t) = 3\cos t - 32\sin t + 17t\cos t + 4t\sin t$
$x_2(t) = 5\cos t - 13\sin t + 6t\cos t + 5t\sin t$

27. $x_1(t) = 8t^3 + 6t^4, \quad x_2(t) = 3t^2 - 2t^3 + 3t^4$

28. $x_1(t) = -7 + 14t - 6t^2 + 4t^2\ln t, \quad x_2(t) = -7 + 9t - 3t^2 + \ln t - 2t\ln t + 2t^2\ln t$

29. $x_1(t) = t\cos t - \ln(\cos t)\sin t, \quad x_2(t) = t\sin t - \ln(\cos t)\cos t$

30. $x_1(t) = \frac{1}{2}t^2\cos 2t, \quad x_2(t) = \frac{1}{2}t^2\sin 2t$

31. $x_1(t) = (9t^2 + 4t^3)e^t, \quad x_2(t) = 6t^2e^t, \quad x_3(t) = 6te^t$

32. $x_1(t) = (44 + 18t)e^t + (-44 + 26t)e^{2t}, \quad x_2(t) = 6e^t + (-6 + 6t)e^{2t}, \quad x_3(t) = 2te^{2t}$

33. $x_1(t) = 15t^2 + 60t^3 + 95t^4 + 12t^5, \quad x_2(t) = 15t^2 + 55t^3 + 15t^4,$
$x_3(t) = 15t^2 + 20t^3, \quad x_4(t) = 15t^2$

34. $x_1(t) = 4t^3 + (4 + 16t + 8t^2)e^{2t}, \quad x_2(t) = 3t^2 + (2 + 4t)e^{2t},$
$x_3(t) = (2 + 4t + 2t^2)e^{2t}, \quad x_4(t) = (1 + t)e^{2t}$

SECTION 8.3

SPECTRAL DECOMPOSITION METHODS

In Problems 1–20 here we want to use projection matrices to find fundamental matrix solutions of the linear systems given in Problems 1–20 of Section 7.3. In each of Problems 1–16, the coefficient matrix $\mathbf{A}$ is 2×2 with distinct eigenvalues λ_1 and λ_2. We can therefore use the method of Example 1 in Section 8.3. That is, if we define the projection matrices

$$\mathbf{P}_1 = \frac{\mathbf{A} - \lambda_2\mathbf{I}}{\lambda_1 - \lambda_2} \quad \text{and} \quad \mathbf{P}_2 = \frac{\mathbf{A} - \lambda_1\mathbf{I}}{\lambda_2 - \lambda_1}, \tag{1}$$

then the desired fundamental matrix solution of the system $\mathbf{x}' = \mathbf{A}\mathbf{x}$ is the exponential matrix

$$e^{\mathbf{A}t} = e^{\lambda_1 t}\mathbf{P}_1 + e^{\lambda_2 t}\mathbf{P}_2. \tag{2}$$

We use the eigenvalues λ_1 and λ_2 given in the Section 7.3 solutions for these problems.

1. $\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}; \quad \lambda_1 = -1, \quad \lambda_2 = 3$

$$\mathbf{P}_1 = \frac{1}{-4}(\mathbf{A} - 3\mathbf{I}) = \frac{1}{2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, \quad \mathbf{P}_2 = \frac{1}{4}(\mathbf{A} + \mathbf{I}) = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

$$e^{\mathbf{A}t} = e^{-t}\mathbf{P}_1 + e^{3t}\mathbf{P}_2 = \frac{1}{2}\begin{bmatrix} e^{-t} + e^{3t} & -e^{-t} + e^{3t} \\ -e^{-t} + e^{3t} & e^{-t} + e^{3t} \end{bmatrix}$$

2. $\mathbf{A} = \begin{bmatrix} 2 & 3 \\ 2 & 1 \end{bmatrix}; \quad \lambda_1 = -1, \quad \lambda_2 = 4$

$$\mathbf{P}_1 = \frac{1}{-5}(\mathbf{A} - 4\mathbf{I}) = \frac{1}{5}\begin{bmatrix} 2 & -3 \\ -2 & 3 \end{bmatrix}, \quad \mathbf{P}_2 = \frac{1}{5}(\mathbf{A} + \mathbf{I}) = \frac{1}{5}\begin{bmatrix} 3 & 3 \\ 2 & 2 \end{bmatrix}$$

$$e^{\mathbf{A}t} = e^{-t}\mathbf{P}_1 + e^{4t}\mathbf{P}_2 = \frac{1}{5}\begin{bmatrix} 2e^{-t} + 3e^{4t} & -3e^{-t} + 3e^{4t} \\ -2e^{-t} + 2e^{4t} & 3e^{-t} + 2e^{4t} \end{bmatrix}$$

3. $\mathbf{A} = \begin{bmatrix} 3 & 4 \\ 3 & 2 \end{bmatrix}; \quad \lambda_1 = -1, \quad \lambda_2 = 6$

$$\mathbf{P}_1 = \frac{1}{-7}(\mathbf{A} - 6\mathbf{I}) = \frac{1}{7}\begin{bmatrix} 3 & -4 \\ -3 & 4 \end{bmatrix}, \quad \mathbf{P}_2 = \frac{1}{7}(\mathbf{A} + \mathbf{I}) = \frac{1}{7}\begin{bmatrix} 4 & 4 \\ 3 & 3 \end{bmatrix}$$

$$e^{\mathbf{A}t} = e^{-t}\mathbf{P}_1 + e^{6t}\mathbf{P}_2 = \frac{1}{7}\begin{bmatrix} 3e^{-t} + 4e^{6t} & -4e^{-t} + 4e^{6t} \\ -3e^{-t} + 3e^{6t} & 4e^{-t} + 3e^{6t} \end{bmatrix}$$

4. $\mathbf{A} = \begin{bmatrix} 4 & 1 \\ 6 & -1 \end{bmatrix}; \quad \lambda_1 = -2, \quad \lambda_2 = 5$

$$\mathbf{P}_1 = \frac{1}{-7}(\mathbf{A} - 5\mathbf{I}) = \frac{1}{7}\begin{bmatrix} 1 & -1 \\ -6 & 6 \end{bmatrix}, \quad \mathbf{P}_2 = \frac{1}{7}(\mathbf{A} + 2\mathbf{I}) = \frac{1}{7}\begin{bmatrix} 6 & 1 \\ 6 & 1 \end{bmatrix}$$

$$e^{\mathbf{A}t} = e^{-2t}\mathbf{P}_1 + e^{5t}\mathbf{P}_2 = \frac{1}{7}\begin{bmatrix} e^{-2t} + 6e^{5t} & -e^{-2t} + e^{5t} \\ -6e^{-2t} + 6e^{5t} & 6e^{-2t} + e^{5t} \end{bmatrix}$$

5. $\mathbf{A} = \begin{bmatrix} 6 & -7 \\ 1 & -2 \end{bmatrix}; \quad \lambda_1 = -1, \quad \lambda_2 = 5$

$$\mathbf{P}_1 = \frac{1}{-6}(\mathbf{A} - 5\mathbf{I}) = \frac{1}{6}\begin{bmatrix} -1 & 7 \\ -1 & 7 \end{bmatrix}, \quad \mathbf{P}_2 = \frac{1}{6}(\mathbf{A} + \mathbf{I}) = \frac{1}{6}\begin{bmatrix} 7 & -7 \\ 1 & -1 \end{bmatrix}$$

$$e^{\mathbf{A}t} = e^{-t}\mathbf{P}_1 + e^{5t}\mathbf{P}_2 = \frac{1}{6}\begin{bmatrix} -e^{-t} + 7e^{5t} & 7e^{-t} - 7e^{5t} \\ -e^{-t} + e^{5t} & 7e^{-t} - e^{5t} \end{bmatrix}$$

6. $\mathbf{A} = \begin{bmatrix} 9 & 5 \\ -6 & -2 \end{bmatrix}; \quad \lambda_1 = 3, \quad \lambda_2 = 4$

$$\mathbf{P}_1 = \frac{1}{-1}(\mathbf{A} - 4\mathbf{I}) = \begin{bmatrix} -5 & -5 \\ 6 & 6 \end{bmatrix}, \quad \mathbf{P}_2 = \frac{1}{1}(\mathbf{A} - 3\mathbf{I}) = \begin{bmatrix} 6 & 5 \\ -6 & -5 \end{bmatrix}$$

$$e^{\mathbf{A}t} = e^{3t}\mathbf{P}_1 + e^{4t}\mathbf{P}_2 = \begin{bmatrix} -5e^{3t} + 6e^{4t} & -5e^{3t} + 5e^{4t} \\ 6e^{3t} - 6e^{4t} & 6e^{3t} - 5e^{4t} \end{bmatrix}$$

7. $\mathbf{A} = \begin{bmatrix} -3 & 4 \\ 6 & -5 \end{bmatrix}; \quad \lambda_1 = -9, \quad \lambda_2 = 1$

$$\mathbf{P}_1 = \frac{1}{-10}(\mathbf{A} - \mathbf{I}) = \frac{1}{5}\begin{bmatrix} 2 & -2 \\ -3 & 3 \end{bmatrix}, \quad \mathbf{P}_2 = \frac{1}{10}(\mathbf{A} + 9\mathbf{I}) = \frac{1}{10}\begin{bmatrix} 3 & 2 \\ 3 & 2 \end{bmatrix}$$

$$e^{\mathbf{A}t} = e^{-9t}\mathbf{P}_1 + e^{t}\mathbf{P}_2 = \frac{1}{5}\begin{bmatrix} 2e^{-9t} + 3e^{t} & -2e^{-9t} + 2e^{t} \\ -3e^{-9t} + 3e^{t} & 3e^{-9t} + 2e^{t} \end{bmatrix}$$

8. $\mathbf{A} = \begin{bmatrix} 1 & -5 \\ 1 & -1 \end{bmatrix}; \quad \lambda_1 = -2i, \quad \lambda_2 = 2i$

$$\mathbf{P}_1 = \frac{1}{-4i}(\mathbf{A} - 2i\mathbf{I}) = \frac{1}{4}\begin{bmatrix} 2+i & -5i \\ i & 2-i \end{bmatrix}, \quad \mathbf{P}_2 = \frac{1}{4i}(\mathbf{A} + 2i\mathbf{I}) = \frac{1}{4}\begin{bmatrix} 2-i & 5i \\ -i & 2+i \end{bmatrix}$$

$$e^{\mathbf{A}t} = e^{-2it}\mathbf{P}_1 + e^{2it}\mathbf{P}_2 = (\cos 2t - i\sin 2t)\mathbf{P}_1 + (\cos 2t + i\sin 2t)\mathbf{P}_2$$

$$= \frac{1}{2}\begin{bmatrix} 2\cos 2t + \sin 2t & -5\sin 2t \\ \sin 2t & 2\cos 2t - \sin 2t \end{bmatrix}$$

9. $\mathbf{A} = \begin{bmatrix} 2 & -5 \\ 4 & 2 \end{bmatrix}; \quad \lambda_1 = -4i, \quad \lambda_2 = 4i$

$$\mathbf{P}_1 = \frac{1}{-8i}(\mathbf{A} - 4i\mathbf{I}) = \frac{1}{8}\begin{bmatrix} 4+2i & -5i \\ 4i & 4-2i \end{bmatrix}, \quad \mathbf{P}_2 = \frac{1}{8i}(\mathbf{A} + 4i\mathbf{I}) = \frac{1}{8}\begin{bmatrix} 4-2i & 5i \\ -4i & 4+2i \end{bmatrix}$$

$$e^{\mathbf{A}t} = e^{-4it}\mathbf{P}_1 + e^{4it}\mathbf{P}_2 = (\cos 4t - i\sin 4t)\mathbf{P}_1 + (\cos 4t + i\sin 4t)\mathbf{P}_2$$
$$= \frac{1}{4}\begin{bmatrix} 4\cos 4t + 2\sin 4t & -5\sin 4t \\ 4\sin 4t & 4\cos 4t - 2\sin 4t \end{bmatrix}$$

10. $$\mathbf{A} = \begin{bmatrix} -3 & -2 \\ 9 & 3 \end{bmatrix}; \quad \lambda_1 = -3i, \quad \lambda_2 = 3i$$
$$\mathbf{P}_1 = \frac{1}{-6i}(\mathbf{A} - 3i\mathbf{I}) = \frac{1}{6}\begin{bmatrix} 3-3i & -2i \\ 9i & 3+3i \end{bmatrix}, \quad \mathbf{P}_2 = \frac{1}{6i}(\mathbf{A} + 3i\mathbf{I}) = \frac{1}{6}\begin{bmatrix} 3+3i & 2i \\ -9i & 3-3i \end{bmatrix}$$
$$e^{\mathbf{A}t} = e^{-3it}\mathbf{P}_1 + e^{3it}\mathbf{P}_2 = (\cos 3t - i\sin 3t)\mathbf{P}_1 + (\cos 3t + i\sin 3t)\mathbf{P}_2$$
$$= \frac{1}{3}\begin{bmatrix} 3\cos 3t - 3\sin 3t & -2\sin 3t \\ 9\sin 3t & 3\cos 3t + 3\sin 3t \end{bmatrix}$$

11. $$\mathbf{A} = \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}; \quad \lambda_1 = 1-2i, \quad \lambda_2 = 1+2i$$
$$\mathbf{P}_1 = \frac{1}{-4i}(\mathbf{A} - (1+2i)\mathbf{I}) = \frac{1}{2}\begin{bmatrix} 1 & -i \\ i & 1 \end{bmatrix}, \quad \mathbf{P}_2 = \frac{1}{4i}(\mathbf{A} - (1-2i)\mathbf{I}) = \frac{1}{2}\begin{bmatrix} 1 & i \\ -i & 1 \end{bmatrix}$$
$$e^{\mathbf{A}t} = e^{(1-2i)t}\mathbf{P}_1 + e^{(1+2i)t}\mathbf{P}_2 = e^t(\cos 2t - i\sin 2t)\mathbf{P}_1 + e^t(\cos 2t + i\sin 2t)\mathbf{P}_2$$
$$= e^t\begin{bmatrix} \cos 2t & -\sin 2t \\ \sin 2t & \cos 2t \end{bmatrix}$$

12. $$\mathbf{A} = \begin{bmatrix} 1 & -5 \\ 1 & 3 \end{bmatrix}; \quad \lambda_1 = 2-2i, \quad \lambda_2 = 2+2i$$
$$\mathbf{P}_1 = \frac{1}{-4i}(\mathbf{A} - (2+2i)\mathbf{I}) = \frac{1}{4}\begin{bmatrix} 2-i & -5i \\ i & 2+i \end{bmatrix}$$
$$\mathbf{P}_2 = \frac{1}{4i}(\mathbf{A} - (2-2i)\mathbf{I}) = \frac{1}{4}\begin{bmatrix} 2+i & 5i \\ -i & 2-i \end{bmatrix}$$
$$e^{\mathbf{A}t} = e^{(2-2i)t}\mathbf{P}_1 + e^{(2+2i)t}\mathbf{P}_2 = e^{2t}(\cos 2t - i\sin 2t)\mathbf{P}_1 + e^{2t}(\cos 2t + i\sin 2t)\mathbf{P}_2$$
$$= \frac{e^{2t}}{2}\begin{bmatrix} 2\cos 2t - \sin 2t & -5\sin 2t \\ \sin 2t & 2\cos 2t + \sin 2t \end{bmatrix}$$

13. $$\mathbf{A} = \begin{bmatrix} 5 & -9 \\ 2 & -1 \end{bmatrix}; \quad \lambda_1 = 2-3i, \quad \lambda_2 = 2+3i$$

$$\mathbf{P}_1 = \frac{1}{-6i}\left(\mathbf{A}-(2+3i)\mathbf{I}\right) = \frac{1}{6}\begin{bmatrix} 3+3i & -9i \\ 2i & 3-3i \end{bmatrix}$$

$$\mathbf{P}_2 = \frac{1}{6i}\left(\mathbf{A}-(2-3i)\mathbf{I}\right) = \frac{1}{6}\begin{bmatrix} 3-3i & 9i \\ -2i & 3+3i \end{bmatrix}$$

$$e^{\mathbf{A}t} = e^{(2-3i)t}\mathbf{P}_1 + e^{(2+3i)t}\mathbf{P}_2 = e^{2t}(\cos 3t - i\sin 3t)\mathbf{P}_1 + e^{2t}(\cos 3t + i\sin 3t)\mathbf{P}_2$$

$$= \frac{e^{2t}}{3}\begin{bmatrix} 3\cos 3t + \sin 3t & -9\sin 3t \\ 2\sin 3t & 3\cos 3t - \sin 3t \end{bmatrix}$$

14. $\mathbf{A} = \begin{bmatrix} 3 & -4 \\ 4 & 3 \end{bmatrix}; \quad \lambda_1 = 3-4i, \quad \lambda_2 = 3+4i$

$$\mathbf{P}_1 = \frac{1}{-8i}\left(\mathbf{A}-(3+4i)\mathbf{I}\right) = \frac{1}{2}\begin{bmatrix} 1 & -i \\ i & 1 \end{bmatrix}, \quad \mathbf{P}_2 = \frac{1}{8i}\left(\mathbf{A}-(3-4i)\mathbf{I}\right) = \frac{1}{2}\begin{bmatrix} 1 & i \\ -i & 1 \end{bmatrix}$$

$$e^{\mathbf{A}t} = e^{(3-4i)t}\mathbf{P}_1 + e^{(3+4i)t}\mathbf{P}_2 = e^{3t}(\cos 4t - i\sin 4t)\mathbf{P}_1 + e^{3t}(\cos 4t + i\sin 4t)\mathbf{P}_2$$

$$= e^{3t}\begin{bmatrix} \cos 4t & -\sin 4t \\ \sin 4t & \cos 4t \end{bmatrix}$$

15. $\mathbf{A} = \begin{bmatrix} 7 & -5 \\ 4 & 3 \end{bmatrix}; \quad \lambda_1 = 5-4i, \quad \lambda_2 = 5+4i$

$$\mathbf{P}_1 = \frac{1}{-8i}\left(\mathbf{A}-(5+4i)\mathbf{I}\right) = \frac{1}{8}\begin{bmatrix} 4+2i & -5i \\ 4i & 4-2i \end{bmatrix}$$

$$\mathbf{P}_2 = \frac{1}{8i}\left(\mathbf{A}-(5-4i)\mathbf{I}\right) = \frac{1}{8}\begin{bmatrix} 4-2i & 5i \\ -4i & 4+2i \end{bmatrix}$$

$$e^{\mathbf{A}t} = e^{(5-4i)t}\mathbf{P}_1 + e^{(5+4i)t}\mathbf{P}_2 = e^{5t}(\cos 4t - i\sin 4t)\mathbf{P}_1 + e^{5t}(\cos 4t + i\sin 4t)\mathbf{P}_2$$

$$= \frac{e^{5t}}{4}\begin{bmatrix} 4\cos 4t + 2\sin 2t & -5\sin 4t \\ 4\sin 4t & 4\cos 4t - 2\sin 2t \end{bmatrix}$$

16. $\mathbf{A} = \begin{bmatrix} -50 & 20 \\ 100 & -60 \end{bmatrix}; \quad \lambda_1 = -100, \quad \lambda_2 = -10$

$$\mathbf{P}_1 = \frac{1}{-90}\left(\mathbf{A}+10\mathbf{I}\right) = \frac{1}{9}\begin{bmatrix} 4 & -2 \\ -10 & 5 \end{bmatrix}, \quad \mathbf{P}_2 = \frac{1}{90}\left(\mathbf{A}+100\mathbf{I}\right) = \frac{1}{9}\begin{bmatrix} 5 & 2 \\ 10 & 4 \end{bmatrix}$$

$$e^{\mathbf{A}t} = e^{-100t}\mathbf{P}_1 + e^{-10t}\mathbf{P}_2 = \frac{1}{9}\begin{bmatrix} 4e^{-100t} + 5e^{-10t} & -2e^{-100t} + 2e^{-10t} \\ -10e^{-100t} + 10e^{-10t} & 5e^{-100t} + 4e^{-10t} \end{bmatrix}$$

In Problems 17, 18, and 20 the coefficient matrix $\mathbf{A}$ is 3×3 with distinct eigenvalues $\lambda_1, \lambda_2, \lambda_3$. Looking at Equations (7) and (3) in Section 8.3, we see that

$$e^{\mathbf{A}t} = e^{\lambda_1 t}\mathbf{P}_1 + e^{\lambda_2 t}\mathbf{P}_2 + e^{\lambda_3 t}\mathbf{P}_3 \tag{3}$$

where the projection matrices are defined by

$$\mathbf{P}_1 = \frac{(\mathbf{A}-\lambda_2\mathbf{I})(\mathbf{A}-\lambda_3\mathbf{I})}{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)}, \quad \mathbf{P}_2 = \frac{(\mathbf{A}-\lambda_1\mathbf{I})(\mathbf{A}-\lambda_3\mathbf{I})}{(\lambda_2-\lambda_1)(\lambda_2-\lambda_3)}, \quad \mathbf{P}_3 = \frac{(\mathbf{A}-\lambda_1\mathbf{I})(\mathbf{A}-\lambda_2\mathbf{I})}{(\lambda_3-\lambda_1)(\lambda_3-\lambda_2)}. \tag{4}$$

17. $\mathbf{A} = \begin{bmatrix} 4 & 1 & 4 \\ 1 & 7 & 1 \\ 4 & 1 & 4 \end{bmatrix}; \quad \lambda_1 = 0, \quad \lambda_2 = 6, \quad \lambda_3 = 9$

$$\mathbf{P}_1 = \frac{1}{54}(\mathbf{A}-6\mathbf{I})(\mathbf{A}-9\mathbf{I}) = \frac{1}{2}\begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

$$\mathbf{P}_2 = \frac{1}{-18}(\mathbf{A}-0\mathbf{I})(\mathbf{A}-9\mathbf{I}) = \frac{1}{6}\begin{bmatrix} 1 & -2 & 1 \\ -2 & 4 & -2 \\ 1 & -2 & 1 \end{bmatrix}$$

$$\mathbf{P}_3 = \frac{1}{27}(\mathbf{A}-0\mathbf{I})(\mathbf{A}-6\mathbf{I}) = \frac{1}{3}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

$$e^{\mathbf{A}t} = \mathbf{P}_1 + e^{6t}\mathbf{P}_2 + e^{9t}\mathbf{P}_3 = \frac{1}{6}\begin{bmatrix} 3+e^{6t}+2e^{9t} & -2e^{6t}+2e^{9t} & -3+e^{6t}+2e^{9t} \\ -2e^{6t}+2e^{9t} & 4e^{6t}+2e^{9t} & -2e^{6t}+2e^{9t} \\ -3+e^{6t}+2e^{9t} & -2e^{6t}+2e^{9t} & 3+e^{6t}+2e^{9t} \end{bmatrix}$$

18. $\mathbf{A} = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 7 & 1 \\ 2 & 1 & 7 \end{bmatrix}; \quad \lambda_1 = 0, \quad \lambda_2 = 6, \quad \lambda_3 = 9$

$$\mathbf{P}_1 = \frac{1}{54}(\mathbf{A}-6\mathbf{I})(\mathbf{A}-9\mathbf{I}) = \frac{1}{18}\begin{bmatrix} 16 & -4 & -4 \\ -4 & 1 & 1 \\ -4 & 1 & 1 \end{bmatrix}$$

$$\mathbf{P}_2 = \frac{1}{-18}(\mathbf{A}-0\mathbf{I})(\mathbf{A}-9\mathbf{I}) = \frac{1}{2}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

$$\mathbf{P}_3 = \frac{1}{27}(\mathbf{A}-0\mathbf{I})(\mathbf{A}-6\mathbf{I}) = \frac{1}{9}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 4 & 4 \\ 2 & 4 & 4 \end{bmatrix}$$

$$e^{\mathbf{A}t} = \mathbf{P}_1 + e^{6t}\mathbf{P}_2 + e^{9t}\mathbf{P}_3 = \frac{1}{18}\begin{bmatrix} 16+2e^{9t} & -4+4e^{9t} & -4+4e^{9t} \\ -4e^{6t}+4e^{9t} & 1+9e^{6t}+8e^{9t} & 1-9e^{6t}+8e^{9t} \\ -4e^{6t}+4e^{9t} & 1-9e^{6t}+8e^{9t} & 1+9e^{6t}+8e^{9t} \end{bmatrix}$$

19. $\mathbf{A} = \begin{bmatrix} 4 & 1 & 1 \\ 1 & 4 & 1 \\ 1 & 1 & 4 \end{bmatrix}; \quad \lambda_1 = 6, \quad \lambda_2 = 3, \quad \lambda_3 = 3$

Here we have the eigenvalue $\lambda_1 = 6$ of multiplicity 1 and the eigenvalue $\lambda_2 = 3$ of multiplicity 2. By Example 2 in Section 8.3, the desired matrix exponential is given by

$$e^{\mathbf{A}t} = e^{\lambda_1 t}\mathbf{P}_1 + e^{\lambda_2 t}\mathbf{P}_2\left[\mathbf{I} + (\mathbf{A}-\lambda_2\mathbf{I})t\right] \qquad (5)$$

where $\mathbf{P}_1$ and $\mathbf{P}_2$ are the projection matrices of $\mathbf{A}$ corresponding to the eigenvalues λ_1 and λ_2. The reciprocal of the characteristic polynomial $p(\lambda) = (\lambda-6)(\lambda-3)^2$ has the partial fractions decomposition

$$\frac{1}{p(\lambda)} = \frac{1}{9(\lambda-6)} - \frac{\lambda}{9(\lambda-3)^2},$$

so $a_1(\lambda) = 1/9$ and $a_2(\lambda) = -\lambda/9$ in the notation of Equation (25) in the text. Therefore Equation (26) there gives

$$\mathbf{P}_1 = a_1(\mathbf{A})\cdot(\mathbf{A}-\lambda_2\mathbf{I})^2 = \frac{1}{9}(\mathbf{A}-3\mathbf{I})^2 = \frac{1}{3}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix},$$

$$\mathbf{P}_2 = a_2(\mathbf{A})\cdot(\mathbf{A}-\lambda_1\mathbf{I}) = -\frac{1}{9}\mathbf{A}(\mathbf{A}-6\mathbf{I}) = \frac{1}{3}\begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}.$$

Finally, Equation (5) above gives

$$e^{\mathbf{A}t} = e^{6t}\mathbf{P}_1 + e^{3t}\mathbf{P}_2\left(\mathbf{I} + (\mathbf{A}-3\mathbf{I})t\right) = \frac{1}{3}\begin{bmatrix} 3e^{3t}+e^{6t} & -e^{3t}+e^{6t} & -e^{3t}+e^{6t} \\ -e^{3t}+e^{6t} & 2e^{3t}+e^{6t} & -e^{3t}+e^{6t} \\ -e^{3t}+e^{6t} & -e^{3t}+e^{6t} & 2e^{3t}+e^{6t} \end{bmatrix}.$$

20. $\mathbf{A} = \begin{bmatrix} 5 & 1 & 3 \\ 1 & 7 & 1 \\ 3 & 1 & 5 \end{bmatrix}; \quad \lambda_1 = 2, \quad \lambda_2 = 6, \quad \lambda_3 = 9$

$$\mathbf{P}_1 = \frac{1}{28}(\mathbf{A}-6\mathbf{I})(\mathbf{A}-9\mathbf{I}) = \frac{1}{2}\begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

$$\mathbf{P}_2 = \frac{1}{-12}(\mathbf{A}-2\mathbf{I})(\mathbf{A}-9\mathbf{I}) = \frac{1}{6}\begin{bmatrix} 1 & -2 & 1 \\ -2 & 4 & -2 \\ 1 & -2 & 1 \end{bmatrix}$$

$$\mathbf{P}_3 = \frac{1}{21}(\mathbf{A}-2\mathbf{I})(\mathbf{A}-6\mathbf{I}) = \frac{1}{3}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

$$e^{\mathbf{A}t} = e^{2t}\mathbf{P}_1 + e^{6t}\mathbf{P}_2 + e^{9t}\mathbf{P}_3 = \frac{1}{6}\begin{bmatrix} 3e^{2t}+e^{6t}+2e^{9t} & -2e^{6t}+2e^{9t} & -3e^{2t}+e^{6t}+2e^{9t} \\ -2e^{6t}+2e^{9t} & 4e^{6t}+2e^{9t} & -2e^{6t}+2e^{9t} \\ -3e^{2t}+e^{6t}+2e^{9t} & -2e^{6t}+2e^{9t} & 3e^{2t}+e^{6t}+2e^{9t} \end{bmatrix}$$

In Problems 21–30 here we want to use projection matrices to find fundamental matrix solutions of the linear systems given in Problems 1–10 of Section 7.5. In each of Problems 21–26, the 2×2 coefficient matrix $\mathbf{A}$ has characteristic polynomial of the form $p(\lambda) = (\lambda-\lambda_1)^2$ and thus a single eigenvalue λ_1 of multiplicity 2. Consequently, Example 5 in Section 8.3 gives the desired fundamental matrix

$$e^{\mathbf{A}t} = e^{\lambda_1 t}\left[\mathbf{I} + (\mathbf{A}-\lambda_1\mathbf{I})t\right]. \tag{6}$$

21. $\mathbf{A} = \begin{bmatrix} -2 & 1 \\ -1 & -4 \end{bmatrix}; \quad \lambda_1 = -3$

$$e^{\mathbf{A}t} = e^{-3t}\left[\mathbf{I} + (\mathbf{A}+3\mathbf{I})t\right] = e^{-3t}\begin{bmatrix} 1+t & t \\ -t & 1-t \end{bmatrix}$$

22. $\mathbf{A} = \begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix}; \quad \lambda_1 = 2$

$$e^{\mathbf{A}t} = e^{2t}\left[\mathbf{I} + (\mathbf{A}-2\mathbf{I})t\right] = e^{2t}\begin{bmatrix} 1+t & -t \\ t & 1-t \end{bmatrix}$$

23. $\mathbf{A} = \begin{bmatrix} 1 & -2 \\ 2 & 5 \end{bmatrix}$; $\lambda_1 = 3$

$$e^{\mathbf{A}t} = e^{3t}\left[\mathbf{I} + (\mathbf{A} - 3\mathbf{I})t\right] = e^{3t}\begin{bmatrix} 1-2t & -2t \\ 2t & 1+2t \end{bmatrix}$$

24. $\mathbf{A} = \begin{bmatrix} 3 & -1 \\ 1 & 5 \end{bmatrix}$; $\lambda_1 = 4$

$$e^{\mathbf{A}t} = e^{4t}\left[\mathbf{I} + (\mathbf{A} - 4\mathbf{I})t\right] = e^{4t}\begin{bmatrix} 1-t & -t \\ t & 1+t \end{bmatrix}$$

25. $\mathbf{A} = \begin{bmatrix} 7 & 1 \\ -4 & 3 \end{bmatrix}$; $\lambda_1 = 5$

$$e^{\mathbf{A}t} = e^{5t}\left[\mathbf{I} + (\mathbf{A} - 5\mathbf{I})t\right] = e^{5t}\begin{bmatrix} 1-4t & -4t \\ 4t & 1+4t \end{bmatrix}$$

26. $\mathbf{A} = \begin{bmatrix} 1 & -4 \\ 4 & 9 \end{bmatrix}$; $\lambda_1 = 5$

$$e^{\mathbf{A}t} = e^{5t}\left[\mathbf{I} + (\mathbf{A} - 5\mathbf{I})t\right] = e^{5t}\begin{bmatrix} 1+2t & t \\ -4t & 1-2t \end{bmatrix}$$

Each of the 3×3 coefficient matrices in Problems 27–30 has a characteristic polynomial of the form $p(\lambda) = (\lambda - \lambda_1)((\lambda - \lambda_2)^2$ yielding an eigenvalue λ_1 of multiplicity 1 and an eigenvalue λ_2 of multiplicity 2. We therefore use the method explained in Problem 19 above, and list here the results of the principal steps in the calculation of the fundamental matrix $e^{\mathbf{A}t}$.

27. $\mathbf{A} = \begin{bmatrix} 2 & 0 & 0 \\ -7 & 9 & 7 \\ 0 & 0 & 2 \end{bmatrix}$; $p(\lambda) = (\lambda - 9)(\lambda - 2)^2$; $\lambda_1 = 9$, $\lambda_2 = 2$

$$\frac{1}{p(\lambda)} = \frac{1}{49(\lambda - 9)} - \frac{\lambda + 5}{49(\lambda - 3)^2}; \qquad a_1(\lambda) = \frac{1}{49}, \qquad a_2(\lambda) = -\frac{\lambda + 5}{49}$$

$$\mathbf{P}_1 = a_1(\mathbf{A})\cdot(\mathbf{A}-\lambda_2\mathbf{I})^2 = \frac{1}{49}\left(\mathbf{A}-2\mathbf{I}\right)^2 = \begin{bmatrix} 0 & 0 & 0 \\ -1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\mathbf{P}_2 = a_2(\mathbf{A})\cdot(\mathbf{A}-\lambda_1\mathbf{I}) = -\frac{1}{49}\left(\mathbf{A}+5\mathbf{I}\right)\left(\mathbf{A}-9\mathbf{I}\right) = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$e^{\mathbf{A}t} = e^{9t}\mathbf{P}_1 + e^{2t}\mathbf{P}_2\left(\mathbf{I}+(\mathbf{A}-2\mathbf{I})\,t\right) = \begin{bmatrix} e^{2t} & 0 & 0 \\ e^{2t}-e^{9t} & e^{9t} & -e^{2t}+e^{9t} \\ 0 & 0 & e^{2t} \end{bmatrix}$$

28. $$\mathbf{A} = \begin{bmatrix} 25 & 12 & 0 \\ -18 & -5 & 0 \\ 6 & 6 & 13 \end{bmatrix}; \qquad p(\lambda) = (\lambda-7)(\lambda-13)^2; \qquad \lambda_1 = 7, \quad \lambda_2 = 13$$

$$\frac{1}{p(\lambda)} = \frac{1}{36(\lambda-9)} + \frac{19-\lambda}{49(\lambda-3)^2}; \qquad a_1(\lambda) = \frac{1}{36}, \qquad a_2(\lambda) = \frac{19-\lambda}{36}$$

$$\mathbf{P}_1 = a_1(\mathbf{A})\cdot(\mathbf{A}-\lambda_2\mathbf{I})^2 = \frac{1}{36}\left(\mathbf{A}-13\mathbf{I}\right)^2 = \begin{bmatrix} -2 & -2 & 0 \\ 3 & 3 & 0 \\ -1 & -1 & 0 \end{bmatrix}$$

$$\mathbf{P}_2 = a_2(\mathbf{A})\cdot(\mathbf{A}-\lambda_1\mathbf{I}) = \frac{1}{36}\left(19\mathbf{I}-\mathbf{A}\right)\left(\mathbf{A}-7\mathbf{I}\right) = \begin{bmatrix} 3 & 2 & 0 \\ -3 & -2 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

$$e^{\mathbf{A}t} = e^{7t}\mathbf{P}_1 + e^{13t}\mathbf{P}_2\left(\mathbf{I}+(\mathbf{A}-13\mathbf{I})\,t\right) = \begin{bmatrix} -2e^{7t}+3e^{13t} & -2e^{7t}+2e^{13t} & 0 \\ 3e^{7t}-3e^{13t} & 3e^{7t}-2e^{13t} & 0 \\ -e^{7t}+e^{13t} & -e^{7t}+e^{13t} & e^{13t} \end{bmatrix}$$

29. $$\mathbf{A} = \begin{bmatrix} -19 & 12 & 84 \\ 0 & 5 & 0 \\ -8 & 4 & 33 \end{bmatrix}; \qquad p(\lambda) = (\lambda-9)(\lambda-5)^2; \qquad \lambda_1 = 9, \quad \lambda_2 = 5$$

$$\frac{1}{p(\lambda)} = \frac{1}{16(\lambda-9)} + \frac{1-\lambda}{16(\lambda-5)^2}; \qquad a_1(\lambda) = \frac{1}{16}, \qquad a_2(\lambda) = \frac{1-\lambda}{16}$$

$$\mathbf{P}_1 = a_1(\mathbf{A})\cdot(\mathbf{A}-\lambda_2\mathbf{I})^2 = \frac{1}{16}\left(\mathbf{A}-5\mathbf{I}\right)^2 = \begin{bmatrix} -6 & 3 & 21 \\ 0 & 0 & 0 \\ -2 & 1 & 7 \end{bmatrix}$$

$$\mathbf{P}_2 = a_2(\mathbf{A})\cdot(\mathbf{A}-\lambda_1\mathbf{I}) = \frac{1}{16}\left(\mathbf{I}-\mathbf{A}\right)\left(\mathbf{A}-9\mathbf{I}\right) = \begin{bmatrix} 7 & -3 & -21 \\ 0 & 1 & 0 \\ 2 & -1 & -6 \end{bmatrix}$$

$$e^{\mathbf{A}t} = e^{9t}\mathbf{P}_1 + e^{5t}\mathbf{P}_2\left(\mathbf{I}+(\mathbf{A}-5\mathbf{I})\,t\right) = \begin{bmatrix} 7e^{5t}-6e^{9t} & -3e^{5t}+3e^{9t} & -21e^{5t}+21e^{9t} \\ 0 & e^{5t} & 0 \\ 2e^{5t}-2e^{9t} & -e^{5t}+e^{9t} & -6e^{5t}+7e^{9t} \end{bmatrix}$$

30. $\mathbf{A} = \begin{bmatrix} -13 & 40 & -48 \\ -8 & 23 & -24 \\ 0 & 0 & 3 \end{bmatrix}$; $\quad p(\lambda) = (\lambda-7)(\lambda-3)^2$; $\quad \lambda_1 = 7, \quad \lambda_2 = 3$

$$\frac{1}{p(\lambda)} = \frac{1}{16(\lambda-7)} - \frac{1+\lambda}{16(\lambda-3)^2}; \qquad a_1(\lambda) = \frac{1}{16}, \qquad a_2(\lambda) = -\frac{1+\lambda}{16}$$

$$\mathbf{P}_1 = a_1(\mathbf{A})\cdot(\mathbf{A}-\lambda_2\mathbf{I})^2 = \frac{1}{16}\left(\mathbf{A}-3\mathbf{I}\right)^2 = \begin{bmatrix} -4 & 10 & -12 \\ -2 & 5 & -6 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\mathbf{P}_2 = a_2(\mathbf{A})\cdot(\mathbf{A}-\lambda_1\mathbf{I}) = -\frac{1}{16}\left(\mathbf{I}+\mathbf{A}\right)\left(\mathbf{A}-7\mathbf{I}\right) = \begin{bmatrix} 5 & -10 & 12 \\ 2 & -4 & 6 \\ 0 & 0 & 1 \end{bmatrix}$$

$$e^{\mathbf{A}t} = e^{7t}\mathbf{P}_1 + e^{3t}\mathbf{P}_2\left(\mathbf{I}+(\mathbf{A}-3\mathbf{I})\,t\right) = \begin{bmatrix} 5e^{3t}-4e^{7t} & -10e^{3t}+10e^{7t} & 12e^{3t}-12e^{7t} \\ 2e^{3t}-2e^{7t} & -4e^{3t}+5e^{7t} & 6e^{3t}-6e^{7t} \\ 0 & 0 & e^{3t} \end{bmatrix}$$

In Problems 31–40 we use the methods of this section to find the matrix exponentials that were *given* in in the statements of Problems 21–30 of Section 8.2. Once $e^{\mathbf{A}t}$ is known, the desired particular solution $\mathbf{x}(t)$ is provided by the variation of parameters formula

$$\mathbf{x}(t) = e^{\mathbf{A}t}\left(\mathbf{x}(0) + \int_0^t e^{-\mathbf{A}s}\,\mathbf{f}(s)\,ds\right) \tag{7}$$

of Section 8.2. We give first the calculation of the matrix exponential using projection matrices and then the final result, which we obtained in each case using a computer algebra system to evaluate the right-hand side in Equation (7) — as described in the remarks preceding Problems 21–30 of Section 8.2. We illustrate this highly formal process by giving intermediate results in Problems 31, 34, and 39.

31. $\mathbf{A} = \begin{bmatrix} 4 & -1 \\ 5 & 2 \end{bmatrix}, \quad \lambda_1 = -1, \quad \lambda_2 = 3$

$$\mathbf{P}_1 = \frac{1}{-4}(\mathbf{A} - 3\mathbf{I}) = \frac{1}{4}\begin{bmatrix} -1 & 1 \\ -5 & 5 \end{bmatrix}, \quad \mathbf{P}_2 = \frac{1}{4}(\mathbf{A} + \mathbf{I}) = \frac{1}{4}\begin{bmatrix} 5 & -1 \\ 5 & -1 \end{bmatrix}$$

$$e^{\mathbf{A}t} = e^{-t}\mathbf{P}_1 + e^{3t}\mathbf{P}_2 = \frac{1}{4}\begin{bmatrix} -e^{-t} + 5e^{3t} & e^{-t} - e^{3t} \\ -5e^{-t} + 5e^{3t} & 5e^{-t} - e^{3t} \end{bmatrix}$$

$$\mathbf{x}(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \qquad \mathbf{f}(t) = \begin{bmatrix} 18e^{2t} \\ 30e^{2t} \end{bmatrix}$$

$$e^{-\mathbf{A}s}\mathbf{f}(s) = \frac{1}{4}\begin{bmatrix} -e^{s} + 5e^{-3s} & e^{s} - e^{-3s} \\ -5e^{s} + 5e^{-3s} & 5e^{s} - e^{-3s} \end{bmatrix}\begin{bmatrix} 18e^{2s} \\ 30e^{2s} \end{bmatrix} = \begin{bmatrix} 15e^{-s} + 3e^{3s} \\ 15e^{-s} + 15e^{3s} \end{bmatrix}$$

$$\mathbf{x}(0) + \int_0^t e^{-\mathbf{A}s}\mathbf{f}(s)\,ds = \begin{bmatrix} 0 \\ 0 \end{bmatrix} + \int_0^t \begin{bmatrix} 15e^{-s} + 3e^{3s} \\ 15e^{-s} + 15e^{3s} \end{bmatrix} ds = \begin{bmatrix} 14 - 15e^{-t} + e^{3t} \\ 10 - 15e^{-t} + 5e^{3t} \end{bmatrix}$$

$$\mathbf{x}(t) = e^{\mathbf{A}t}\left(\mathbf{x}(0) + \int_0^t e^{-\mathbf{A}s}\mathbf{f}(s)\,ds\right) = \frac{1}{4}\begin{bmatrix} -e^{-t} + 5e^{3t} & e^{-t} - e^{3t} \\ -5e^{-t} + 5e^{3t} & 5e^{-t} - e^{3t} \end{bmatrix}\begin{bmatrix} 14 - 15e^{-t} + e^{3t} \\ 10 - 15e^{-t} + 5e^{3t} \end{bmatrix}$$

$$\mathbf{x}(t) = \begin{bmatrix} -e^{-t} - 14e^{2t} + 15e^{3t} \\ -5e^{-t} - 10e^{2t} + 15e^{3t} \end{bmatrix}$$

32. With the same matrix exponential as in Problem 31, but with $\mathbf{f}(t) = \begin{bmatrix} 28e^{-t} \\ 20e^{3t} \end{bmatrix}$.

$$\mathbf{x}(t) = \begin{bmatrix} (-10 - 7t)e^{-t} + (10 - 5t)e^{3t} \\ (-15 - 35t)e^{-t} + (15 - 5t)e^{3t} \end{bmatrix}$$

33. $\mathbf{A} = \begin{bmatrix} 3 & -1 \\ 9 & -3 \end{bmatrix}, \quad \lambda_1 = 0, \quad \lambda_2 = 0$

$$e^{\mathbf{A}t} = e^{-0t}\left[\mathbf{I} + (\mathbf{A} + 0\mathbf{I})t\right] = \mathbf{I} + \mathbf{A}t = \begin{bmatrix} 1 + 3t & -t \\ 9t & 1 - 3t \end{bmatrix} \qquad \text{(as in Problems 21-26)}$$

$$\mathbf{x}(0) = \begin{bmatrix} 3 \\ 5 \end{bmatrix}, \qquad \mathbf{f}(t) = \begin{bmatrix} 7 \\ 5 \end{bmatrix}$$

$$\mathbf{x}(t) = \begin{bmatrix} 3+11t+8t^2 \\ 5+17t+24t^2 \end{bmatrix}$$

34. With $e^{\mathbf{A}t}$ as in Problem 33, but with $\mathbf{x}(1) = \begin{bmatrix} 3 \\ 7 \end{bmatrix}$ and $\mathbf{f}(t) = \begin{bmatrix} 0 \\ 1/t^2 \end{bmatrix}$. Because $t_0 = 1$, we use the general variation of parameters formula in Eq. (25) of Section 8.2.

$$e^{-\mathbf{A}s}\mathbf{f}(s) = \begin{bmatrix} 1-3s & s \\ -9s & 1+3s \end{bmatrix}\begin{bmatrix} 0 \\ 1/s^2 \end{bmatrix} = \begin{bmatrix} 1/s \\ 3/s+1/s^2 \end{bmatrix}$$

$$e^{-\mathbf{A}}\mathbf{x}(1) + \int_1^t e^{-\mathbf{A}s}\mathbf{f}(s)\,ds = \begin{bmatrix} -2 & 1 \\ -9 & 4 \end{bmatrix}\begin{bmatrix} 3 \\ 7 \end{bmatrix} + \int_1^t \begin{bmatrix} 1/s \\ 3/s+1/s^2 \end{bmatrix} ds = \begin{bmatrix} 1+\ln t \\ 2+3\ln t - 1/t \end{bmatrix}$$

$$\mathbf{x}(t) = e^{\mathbf{A}t}\left(e^{-\mathbf{A}}\mathbf{x}(1) + \int_1^t e^{-\mathbf{A}s}\mathbf{f}(s)\,ds\right) = \begin{bmatrix} 1+3t & -t \\ 9t & 1-3t \end{bmatrix}\begin{bmatrix} 1+\ln t \\ 2+3\ln t - 1/t \end{bmatrix}$$

$$\mathbf{x}(t) = \begin{bmatrix} 2+t+\ln t \\ 5+3t-\dfrac{1}{t}+3\ln t \end{bmatrix}$$

35. $\mathbf{A} = \begin{bmatrix} 2 & -5 \\ 1 & -2 \end{bmatrix}, \quad \lambda_1 = -i, \quad \lambda_2 = i$

$$\mathbf{P}_1 = \frac{1}{-2i}(\mathbf{A} - i\mathbf{I}) = \frac{1}{2}\begin{bmatrix} 1+2i & -5i \\ i & 1-2i \end{bmatrix}, \quad \mathbf{P}_2 = \frac{1}{2i}(\mathbf{A} + i\mathbf{I}) = \frac{1}{2}\begin{bmatrix} 1-2i & 5i \\ -i & 1+2i \end{bmatrix}$$

$$e^{\mathbf{A}t} = e^{-it}\mathbf{P}_1 + e^{it}\mathbf{P}_2 = (\cos t - i\sin t)\mathbf{P}_1 + (\cos t + i\sin t)\mathbf{P}_2$$

$$= \begin{bmatrix} \cos t + 2\sin t & -5\sin t \\ \sin t & \cos t - 2\sin t \end{bmatrix}; \qquad \mathbf{x}(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \qquad \mathbf{f}(t) = \begin{bmatrix} 4t \\ 1 \end{bmatrix}.$$

The solution vector $\mathbf{x}(t) = \begin{bmatrix} -1+8t+\cos t - 8\sin t \\ -2+4t+2\cos t - 3\sin t \end{bmatrix}$ is derived by variation of parameters in the solution to Problem 25 of Section 8.2.

36. With $e^{\mathbf{A}t}$ as in Problem 35, but with $\mathbf{x}(0) = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$ and $\mathbf{f}(t) = \begin{bmatrix} 4\cos t \\ 6\sin t \end{bmatrix}$.

$$\mathbf{x}(t) = \begin{bmatrix} 3\cos t - 32\sin t + 17t\cos t + 4t\sin t \\ 5\cos t - 13\sin t + 6t\cos t + 5t\sin t \end{bmatrix}$$

37. $\mathbf{A} = \begin{bmatrix} 3 & -1 \\ 9 & -3 \end{bmatrix}, \quad \lambda_1 = 0, \quad \lambda_2 = 0$

$$e^{\mathbf{A}t} = e^{-0t}\left[\mathbf{I} + (\mathbf{A} + 0\mathbf{I})t\right] = \mathbf{I} + \mathbf{A}t = \begin{bmatrix} 1+2t & -4t \\ t & 1-2t \end{bmatrix} \qquad \text{(as in Problems 21–26)}$$

$$\mathbf{x}(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \qquad \mathbf{f}(t) = \begin{bmatrix} 36t^2 \\ 6t \end{bmatrix}$$

$$\mathbf{x}(t) = \begin{bmatrix} 8t^3 + 6t^4 \\ 3t^2 - 2t^3 + 3t^4 \end{bmatrix}$$

38. With $e^{\mathbf{A}t}$ as in Problem 37, but with $\mathbf{x}(1) = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $\mathbf{f}(t) = \begin{bmatrix} 4\ln t \\ 1/t \end{bmatrix}$. The details of the variation of parameters process are similar to those shown in Problem 34 above.

$$\mathbf{x}(t) = \begin{bmatrix} -7 + 14t - 6t^2 + 4t^2\ln t \\ -7 + 9t - 3t^2 + \ln t - 2t\ln t + 2t^2\ln t \end{bmatrix}$$

39. $\mathbf{A} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad \lambda_1 = -i, \quad \lambda_2 = i$

$$\mathbf{P}_1 = \frac{1}{-2i}(\mathbf{A} - i\mathbf{I}) = \frac{1}{2}\begin{bmatrix} 1 & -i \\ i & 1 \end{bmatrix}, \qquad \mathbf{P}_2 = \frac{1}{2i}(\mathbf{A} + i\mathbf{I}) = \frac{1}{2}\begin{bmatrix} 1 & i \\ -i & 1 \end{bmatrix}$$

$$\begin{aligned} e^{\mathbf{A}t} &= e^{-it}\mathbf{P}_1 + e^{it}\mathbf{P}_2 \\ &= (\cos t - i\sin t)\mathbf{P}_1 + (\cos t + i\sin t)\mathbf{P}_2 = \begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix} \end{aligned}$$

$$\mathbf{x}(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \qquad \mathbf{f}(t) = \begin{bmatrix} \sec t \\ 0 \end{bmatrix}$$

$$e^{-\mathbf{A}s}\mathbf{f}(s) = \begin{bmatrix} \cos s & \sin s \\ -\sin s & \cos s \end{bmatrix}\begin{bmatrix} \sec s \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ -\tan s \end{bmatrix}$$

$$\mathbf{x}(0) + \int_0^t e^{-\mathbf{A}s}\mathbf{f}(s)\,ds = \begin{bmatrix} 0 \\ 0 \end{bmatrix} + \int_0^t \begin{bmatrix} 1 \\ -\tan s \end{bmatrix} ds = \begin{bmatrix} t \\ \ln(\cos t) \end{bmatrix}$$

$$\mathbf{x}(t) = e^{\mathbf{A}t}\left(\mathbf{x}(0) + \int_0^t e^{-\mathbf{A}s}\mathbf{f}(s)\,ds\right) = \begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix}\begin{bmatrix} t \\ \ln(\cos t) \end{bmatrix}$$

$$\mathbf{x}(t) = \begin{bmatrix} t\cos t - \ln(\cos t)\sin t \\ t\sin t + \ln(\cos t)\cos t \end{bmatrix}$$

40. $\mathbf{A} = \begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix}, \quad \lambda_1 = -2i, \quad \lambda_2 = 2i$

$$\mathbf{P}_1 = \frac{1}{-4i}(\mathbf{A} - 2i\,\mathbf{I}) = \frac{1}{2}\begin{bmatrix} 1 & -i \\ i & 1 \end{bmatrix}, \quad \mathbf{P}_2 = \frac{1}{4i}(\mathbf{A} + 2i\,\mathbf{I}) = \frac{1}{2}\begin{bmatrix} 1 & i \\ -i & 1 \end{bmatrix}$$

$$e^{\mathbf{A}t} = e^{-2it}\mathbf{P}_1 + e^{2it}\mathbf{P}_2$$

$$= (\cos 2t - i\sin 2t)\mathbf{P}_1 + (\cos 2t + i\sin 2t)\mathbf{P}_2 = \begin{bmatrix} \cos 2t & -\sin 2t \\ \sin 2t & \cos 2t \end{bmatrix}$$

$$\mathbf{x}(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \qquad \mathbf{f}(t) = \begin{bmatrix} t\cos 2t \\ t\sin 2t \end{bmatrix}$$

$$\mathbf{x}(t) = \begin{bmatrix} \frac{1}{2}t^2\cos 2t \\ \frac{1}{2}t^2\sin 2t \end{bmatrix}$$

Each of the 3×3 coefficient matrices in Problems 41 and 42 has a characteristic polynomial of the form $p(\lambda) = (\lambda-\lambda_1)((\lambda-\lambda_2)^2$ yielding an eigenvalue λ_1 of multiplicity 1 and an eigenvalue λ_2 of multiplicity 2. We therefore use the method explained in Problem 19 above, and list here the results of the principal steps in the calculation of the fundamental matrix $e^{\mathbf{A}t}$.

41. $\mathbf{A} = \begin{bmatrix} 39 & 8 & -16 \\ -36 & -5 & 16 \\ 72 & 16 & -29 \end{bmatrix}; \qquad p(\lambda) = (\lambda+1)(\lambda-3)^2, \quad \lambda_1 = -1, \quad \lambda_3 = 3$

$$\frac{1}{p(\lambda)} = \frac{1}{16(\lambda+1)} + \frac{7-\lambda}{16(\lambda-3)^2}; \qquad a_1(\lambda) = \frac{1}{16}, \qquad a_2(\lambda) = \frac{7-\lambda}{16}$$

$$\mathbf{P}_1 = a_1(\mathbf{A})\cdot(\mathbf{A} - \lambda_2\mathbf{I})^2 = \frac{1}{16}(\mathbf{A} - 3\mathbf{I})^2 = \begin{bmatrix} -9 & -2 & 4 \\ 9 & 2 & -4 \\ -18 & -4 & 8 \end{bmatrix}$$

$$\mathbf{P}_2 = a_2(\mathbf{A})\cdot(\mathbf{A}-\lambda_1\mathbf{I}) = \frac{1}{16}(7\mathbf{I}-\mathbf{A})(\mathbf{A}+\mathbf{I}) = \begin{bmatrix} 10 & 2 & -4 \\ -9 & -1 & 4 \\ 18 & 4 & -7 \end{bmatrix}$$

$$e^{\mathbf{A}t} = e^{-t}\mathbf{P}_1 + e^{3t}\mathbf{P}_2\left(\mathbf{I}+(\mathbf{A}-3\mathbf{I})\,t\right) = \begin{bmatrix} -9e^{-t}+10e^{3t} & -2e^{-t}+2e^{3t} & 4e^{-t}-4e^{3t} \\ 9e^{-t}-9e^{3t} & 2e^{-t}-e^{3t} & -4e^{-t}+4e^{3t} \\ -18e^{-t}+18e^{3t} & -4e^{-t}+4e^{3t} & 8e^{-t}-7e^{3t} \end{bmatrix}$$

42. $\mathbf{A} = \begin{bmatrix} 28 & 50 & 100 \\ 15 & 33 & 60 \\ -15 & -30 & -57 \end{bmatrix}$; $\quad p(\lambda) = (\lambda+2)(\lambda-3)^2, \quad \lambda_1 = -2, \quad \lambda_3 = 3$

$$\frac{1}{p(\lambda)} = \frac{1}{25(\lambda+1)} + \frac{8-\lambda}{25(\lambda-3)^2}; \qquad a_1(\lambda) = \frac{1}{25}, \qquad a_2(\lambda) = \frac{7-\lambda}{25}$$

$$\mathbf{P}_1 = a_1(\mathbf{A})\cdot(\mathbf{A}-\lambda_2\mathbf{I})^2 = \frac{1}{25}(\mathbf{A}-3\mathbf{I})^2 = \begin{bmatrix} -5 & -10 & -20 \\ -3 & -6 & -12 \\ 3 & 6 & 12 \end{bmatrix}$$

$$\mathbf{P}_2 = a_2(\mathbf{A})\cdot(\mathbf{A}-\lambda_1\mathbf{I}) = \frac{1}{25}(8\mathbf{I}-\mathbf{A})(\mathbf{A}+2\mathbf{I}) = \begin{bmatrix} 6 & 10 & 20 \\ 3 & 7 & 12 \\ -3 & -6 & -11 \end{bmatrix}$$

$$e^{\mathbf{A}t} = e^{-2t}\mathbf{P}_1 + e^{3t}\mathbf{P}_2\left(\mathbf{I}+(\mathbf{A}-3\mathbf{I})\,t\right) = \begin{bmatrix} -5e^{-2t}+6e^{3t} & -10e^{-2t}+10e^{3t} & -20e^{-2t}+20e^{3t} \\ -3e^{-2t}+3e^{3t} & -6e^{-2t}+7e^{3t} & -12e^{-2t}+12e^{3t} \\ 3e^{-2t}-3e^{3t} & 6e^{-2t}-6e^{3t} & 12e^{-2t}-11e^{3t} \end{bmatrix}$$

In each of Problems 43 and 44, the given 3×3 coefficient matrix $\mathbf{A}$ has characteristic polynomial of the form $p(\lambda) = (\lambda-\lambda_1)^3$ and thus a single eigenvalue λ_1 of multiplicity 3. Consequently, Equations (25) and (26) in Section 8.3 of the text imply that $a_1(\lambda) = b_1(\lambda) = 1$ and hence that the associated projection matrix $\mathbf{P}_1 = \mathbf{I}$. Therefore Equation (35) in the text reduces (with $q = 1$ and $m_1 = 3$) to

$$e^{\mathbf{A}t} = e^{\lambda_1 t}\left[\mathbf{I}+(\mathbf{A}-\lambda_1\mathbf{I})t + \tfrac{1}{2}(\mathbf{A}-\lambda_1\mathbf{I})^2 t^2\right]. \tag{8}$$

43. $\mathbf{A} = \begin{bmatrix} -2 & 17 & 4 \\ -1 & 6 & 1 \\ 0 & 1 & 2 \end{bmatrix}$; $\quad p(\lambda) = (\lambda-2)^2, \qquad \lambda_1 = 2$

Substitution of $\mathbf{A}$ and $\lambda_1 = 2$ into the formula in (8) above yields

$$e^{\mathbf{A}t} = \frac{1}{2}e^{2t}\begin{bmatrix} -t^2-8t+2 & 4t^2+34t & t^2+8t \\ -2t & 8t+2 & 2t \\ -t^2 & 4t^2+2t & t^2+2 \end{bmatrix}.$$

44. $\mathbf{A} = \begin{bmatrix} 5 & -1 & 1 \\ 1 & 3 & 0 \\ -3 & 2 & 1 \end{bmatrix}; \qquad p(\lambda) = (\lambda-3)^2, \qquad \lambda_1 = 3$

Substitution of $\mathbf{A}$ and $\lambda_1 = 3$ into the formula in (8) above yields

$$e^{\mathbf{A}t} = \frac{1}{2}e^{3t}\begin{bmatrix} 4t+2 & -2t & 2t \\ 2t^2+2t & -t^2+2 & t^2 \\ 2t^2-6t & -t^2+4t & t^2-4t+2 \end{bmatrix}.$$

45. $\mathbf{A} = \begin{bmatrix} -1 & 1 & 1 & -2 \\ 7 & -4 & -6 & 11 \\ 5 & -1 & 1 & 3 \\ 6 & -2 & -2 & 6 \end{bmatrix}; \qquad p(\lambda) = (\lambda+1)^2(\lambda-2)^2, \qquad \lambda_1 = -1, \quad \lambda_2 = 2$

Since $\mathbf{A}$ has two eigenvalues $\lambda_1 = -1$ and $\lambda_2 = 2$ each of multiplicity 2, Equation (35) in the text reduces (with $q = m1 = m2 = 2$) to

$$e^{\mathbf{A}t} = e^{\lambda_1 t}\mathbf{P}_1\left[\mathbf{I} + (\mathbf{A}-\lambda_1\mathbf{I})t\right] + e^{\lambda_2 t}\mathbf{P}_2\left[\mathbf{I} + (\mathbf{A}-\lambda_2\mathbf{I})t\right]. \tag{9}$$

To calculate the projection matrices $\mathbf{P}_1$ and $\mathbf{P}_2$, we start with the partial fraction decomposition

$$\frac{1}{p(\lambda)} = \frac{2\lambda+5}{27(\lambda+1)^2} + \frac{7-2\lambda}{27(\lambda-2)^2}.$$

Then Equation (25) in the text implies that $a_1(\lambda) = (2\lambda+5)/27$ and $a_2(\lambda) = (7-2\lambda)/27$. Hence Equation (26) yields

$$\mathbf{P}_1 = \frac{1}{27}(2\mathbf{A}+5\mathbf{I})(\mathbf{A}+\mathbf{I})^2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -3 & 1 & 1 & -2 \\ -1 & 0 & 0 & 0 \\ -2 & 0 & 0 & 0 \end{bmatrix}$$

and

$$\mathbf{P}_2 = \frac{1}{27}(7\mathbf{I}-2\mathbf{A})(\mathbf{A}-2\mathbf{I})^2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 3 & 0 & -1 & 2 \\ 1 & 0 & 1 & 0 \\ 2 & 0 & 0 & 1 \end{bmatrix}.$$

When we substitute these eigenvalues and projection matrices in Eq. (9) above we get

$$e^{\mathbf{A}t} = \begin{bmatrix} e^{-t} & t\,e^{-t} & t\,e^{-t} & -2t\,e^{-t} \\ -3e^{-t}+(3-2t)e^{2t} & (1-3t)e^{-t} & (1-3t)e^{-t}-e^{2t} & 2(3t-1)e^{-t}+(2-t)e^{2t} \\ -e^{-t}+(1+2t)e^{2t} & -t\,e^{-t} & -t\,e^{-t}+e^{2t} & 2t\,e^{-t}+t\,e^{2t} \\ -2e^{-t}+2e^{2t} & -2t\,e^{-t} & -2t\,e^{-t} & 4t\,e^{-t}+e^{2t} \end{bmatrix}.$$

46. $\mathbf{A} = \begin{bmatrix} 35 & -12 & 4 & 30 \\ 22 & -8 & 3 & 19 \\ -10 & 3 & 0 & -9 \\ -27 & 9 & -3 & -23 \end{bmatrix}; \qquad p(\lambda) = (\lambda-1)^4, \qquad \lambda_1 = 1$

Thus the given 4×4 coefficient matrix $\mathbf{A}$ has characteristic polynomial of the form $p(\lambda)=(\lambda-\lambda_1)^4$ and thus a single eigenvalue $\lambda_1 = 1$ of multiplicity 4. Consequently, Equations (25) and (26) in Section 8.3 of the text imply that $a_1(\lambda)=b_1(\lambda)=1$ and hence that the associated projection matrix $\mathbf{P}_1 = \mathbf{I}$. Therefore Equation (35) in the text reduces (with $q = 1$ and $m_1 = 4$) to

$$e^{\mathbf{A}t} = e^{\lambda_1 t}\left[\mathbf{I}+(\mathbf{A}-\lambda_1\mathbf{I})t+\tfrac{1}{2}(\mathbf{A}-\lambda_1\mathbf{I})^2t^2+\tfrac{1}{6}(\mathbf{A}-\lambda_1\mathbf{I})^3t^3\right]. \qquad (10)$$

When we substitute $\mathbf{A}$ and $\lambda_1 = 1$ in this formula we get

$$e^{\mathbf{A}t} = \frac{1}{2}e^t\begin{bmatrix} 48t^2+68t+2 & -18t^2-24t & 6t^2+8t & 36t^2+60t \\ 7t^2+44t & -3t^2-18t+2 & t^2+6t & 6t^2+38t \\ -21t^2-20t & 9t^2+6t & -3t^2-2t+2 & -18t^2-18t \\ -42t^2-54t & 18t^2+18t & -6t^2-6t & -36t^2-48t+2 \end{bmatrix}.$$

In each of problems 47–50 we use the given values of k_1, k_2, and k_3 set up the second-order linear system $\mathbf{x}'' = \mathbf{A}\mathbf{x}$ with coefficient matrix

$$\mathbf{A} = \begin{bmatrix} -k_1-k_2 & k_2 \\ k_2 & -k_2-k_3 \end{bmatrix}.$$

We then calculate the particular solution

$$\mathbf{x}(t) = e^{\sqrt{\mathbf{A}}\,t}\mathbf{c}_1 + e^{-\sqrt{\mathbf{A}}\,t}\mathbf{c}_2 \tag{11}$$

given by Theorem 3 with $\mathbf{c}_1 = \begin{bmatrix}1 & 0\end{bmatrix}^T$ and $\mathbf{c}_2 = \begin{bmatrix}0 & 1\end{bmatrix}^T$. Given the distinct eigenvalues λ_1, λ_2 and the projection matrices $\mathbf{P}_1, \mathbf{P}_2$ of $\mathbf{A}$, the matrix exponential needed in (11) is given by

$$e^{\sqrt{\mathbf{A}}\,t} = e^{\sqrt{\lambda_1}\,t}\,\mathbf{P}_1 + e^{\sqrt{\lambda_2}\,t}\,\mathbf{P}_2. \tag{12}$$

47. $\mathbf{A} = \begin{bmatrix}-5 & 4\\ 4 & -5\end{bmatrix};\quad p(\lambda) = \lambda^2 - 10\lambda + 9,\quad \lambda_1 = -1,\quad \lambda_2 = -9$

$$\mathbf{P}_1 = \frac{1}{-10}(\mathbf{A}+9\mathbf{I}) = \frac{1}{2}\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix},\qquad \mathbf{P}_2 = \frac{1}{10}(\mathbf{A}+\mathbf{I}) = \frac{1}{2}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}$$

$$e^{\sqrt{\mathbf{A}}\,t} = e^{\sqrt{\lambda_1}\,t}\,\mathbf{P}_1 + e^{\sqrt{\lambda_2}\,t}\,\mathbf{P}_2 = e^{it}\,\mathbf{P}_1 + e^{3it}\,\mathbf{P}_2 = \frac{1}{2}\begin{bmatrix}e^{it}+e^{3it} & e^{it}-e^{3it}\\ e^{it}-e^{3it} & e^{it}+e^{3it}\end{bmatrix}$$

$$\mathbf{x}(t) = e^{\sqrt{\mathbf{A}}\,t}\mathbf{c}_1 + e^{-\sqrt{\mathbf{A}}\,t}\mathbf{c}_2 = \frac{1}{2}\begin{bmatrix}\left(e^{it}+e^{-it}\right)-\left(e^{3it}-e^{3it}\right)\\ \left(e^{it}+e^{-it}\right)+\left(e^{3it}-e^{3it}\right)\end{bmatrix} = \begin{bmatrix}\cos t\\ \cos t\end{bmatrix} + i\begin{bmatrix}-\sin 3t\\ \sin 3t\end{bmatrix}$$

Note that $\mathbf{x}(t)$ is a linear combination of two motions — one in which the two masses move in the same direction with frequency $\omega_1 = 1$ and with equal amplitudes, and one in which they move in opposite directions with frequency $\omega_2 = 3$ and with equal amplitudes.

48. $\mathbf{A} = \begin{bmatrix}-3 & 2\\ 2 & -3\end{bmatrix};\quad p(\lambda) = \lambda^2 - 6\lambda + 5,\quad \lambda_1 = -1,\quad \lambda_2 = -5$

$$\mathbf{P}_1 = \frac{1}{-6}(\mathbf{A}+5\mathbf{I}) = \frac{1}{2}\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix},\qquad \mathbf{P}_2 = \frac{1}{6}(\mathbf{A}+\mathbf{I}) = \frac{1}{2}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}$$

$$e^{\sqrt{\mathbf{A}}\,t} = e^{\sqrt{\lambda_1}\,t}\,\mathbf{P}_1 + e^{\sqrt{\lambda_2}\,t}\,\mathbf{P}_2 = e^{it}\,\mathbf{P}_1 + e^{\sqrt{5}\,it}\,\mathbf{P}_2 = \frac{1}{2}\begin{bmatrix}e^{it}+e^{\sqrt{5}\,it} & e^{it}-e^{\sqrt{5}\,it}\\ e^{it}-e^{\sqrt{5}\,it} & e^{it}+e^{\sqrt{5}\,it}\end{bmatrix}$$

$$\mathbf{x}(t) = e^{\sqrt{\mathbf{A}}\,t}\mathbf{c}_1 + e^{-\sqrt{\mathbf{A}}\,t}\mathbf{c}_2 = \frac{1}{2}\begin{bmatrix}\left(e^{it}+e^{-it}\right)-\left(e^{\sqrt{5}\,it}-e^{\sqrt{5}\,it}\right)\\ \left(e^{it}+e^{-it}\right)+\left(e^{\sqrt{5}\,it}-e^{\sqrt{5}\,it}\right)\end{bmatrix} = \begin{bmatrix}\cos t\\ \cos t\end{bmatrix} + i\begin{bmatrix}-\sin\sqrt{5}\,t\\ \sin\sqrt{5}\,t\end{bmatrix}$$

Note that $\mathbf{x}(t)$ is a linear combination of two motions — one in which the two masses move in the same direction with frequency $\omega_1 = 1$ and with equal amplitudes, and one in which they move in opposite directions with frequency $\omega_2 = \sqrt{5}$ and with equal amplitudes.

49. $\mathbf{A} = \begin{bmatrix} -3 & 1 \\ 1 & -3 \end{bmatrix}; \quad p(\lambda) = \lambda^2 - 6\lambda + 8, \quad \lambda_1 = -2, \quad \lambda_2 = -4$

$$\mathbf{P}_1 = \frac{1}{-6}(\mathbf{A} + 4\mathbf{I}) = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \quad \mathbf{P}_2 = \frac{1}{6}(\mathbf{A} + 2\mathbf{I}) = \frac{1}{2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$e^{\sqrt{\mathbf{A}}\,t} = e^{\sqrt{\lambda_1}\,t}\,\mathbf{P}_1 + e^{\sqrt{\lambda_2}\,t}\,\mathbf{P}_2 = e^{\sqrt{2}\,it}\,\mathbf{P}_1 + e^{2it}\,\mathbf{P}_2 = \frac{1}{2}\begin{bmatrix} e^{\sqrt{2}\,it} + e^{2it} & e^{\sqrt{2}\,it} - e^{2it} \\ e^{\sqrt{2}\,it} - e^{2it} & e^{\sqrt{2}\,it} + e^{2it} \end{bmatrix}$$

$$\mathbf{x}(t) = e^{\sqrt{\mathbf{A}}\,t}\mathbf{c}_1 + e^{-\sqrt{\mathbf{A}}\,t}\mathbf{c}_2 = \frac{1}{2}\begin{bmatrix} \left(e^{\sqrt{2}\,it} + e^{-\sqrt{2}\,it}\right) - \left(e^{2it} - e^{2it}\right) \\ \left(e^{\sqrt{2}\,it} + e^{-\sqrt{2}\,it}\right) + \left(e^{2it} - e^{2it}\right) \end{bmatrix} = \begin{bmatrix} \cos\sqrt{2}\,t \\ \cos\sqrt{2}\,t \end{bmatrix} + i\begin{bmatrix} -\sin 2t \\ \sin 2t \end{bmatrix}$$

Note that $\mathbf{x}(t)$ is a linear combination of two motions — one in which the two masses move in the same direction with frequency $\omega_1 = \sqrt{2}$ and with equal amplitudes, and one in which they move in opposite directions with frequency $\omega_2 = 2$ and with equal amplitudes.

50. $\mathbf{A} = \begin{bmatrix} -10 & 6 \\ 6 & -10 \end{bmatrix}; \quad p(\lambda) = \lambda^2 - 20\lambda + 64, \quad \lambda_1 = -4, \quad \lambda_2 = -16$

$$\mathbf{P}_1 = \frac{1}{-20}(\mathbf{A} + 16\mathbf{I}) = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \quad \mathbf{P}_2 = \frac{1}{20}(\mathbf{A} + 4\mathbf{I}) = \frac{1}{2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$e^{\sqrt{\mathbf{A}}\,t} = e^{\sqrt{\lambda_1}\,t}\,\mathbf{P}_1 + e^{\sqrt{\lambda_2}\,t}\,\mathbf{P}_2 = e^{2it}\,\mathbf{P}_1 + e^{4it}\,\mathbf{P}_2 = \frac{1}{2}\begin{bmatrix} e^{2it} + e^{4it} & e^{2it} - e^{4it} \\ e^{2it} - e^{4it} & e^{2it} + e^{4it} \end{bmatrix}$$

$$\mathbf{x}(t) = e^{\sqrt{\mathbf{A}}\,t}\mathbf{c}_1 + e^{-\sqrt{\mathbf{A}}\,t}\mathbf{c}_2 = \frac{1}{2}\begin{bmatrix} \left(e^{2it} + e^{-2it}\right) - \left(e^{4it} - e^{4it}\right) \\ \left(e^{2it} + e^{-2it}\right) + \left(e^{4it} - e^{4it}\right) \end{bmatrix} = \begin{bmatrix} \cos 2t \\ \cos 2t \end{bmatrix} + i\begin{bmatrix} -\sin 4t \\ \sin 4t \end{bmatrix}$$

Note that $\mathbf{x}(t)$ is a linear combination of two motions — one in which the two masses move in the same direction with frequency $\omega_1 = 2$ and with equal amplitudes, and one in which they move in opposite directions with frequency $\omega_2 = 4$ and with equal amplitudes.

CHAPTER 9

NONLINEAR SYSTEMS AND PHENOMENA

SECTION 9.1

STABILITY AND THE PHASE PLANE

1. The only solution of the homogeneous system $2x - y = 0,\ x - 3y = 0$ is the origin (0, 0). The only figure among Figs. 9.1.11 through 9.1.18 showing a single critical point at the origin is Fig. 9.1.13. Thus the only critical point of the given autonomous system is the saddle point (0, 0) shown in Figure 9.1.13 in the text.

2. The only solution of the system $x - y = 0,\ x + 3y + 4 = 0$ is the point (1, 1). The only figure among Figs. 9.1.11 through 9.1.18 showing a single critical point at (1, 1) is Fig. 9.1.15. Thus the only critical point of the given autonomous system is the node (1, 1) shown in Figure 9.1.15 in the text.

3. The only solution of the system $x - 2y + 3 = 0,\ x - y + 2 = 0$ is the point (–1, 1). The only figure among Figs. 9.1.11 through 9.1.18 showing a single critical point at (–1, 1) is Fig. 9.1.18. Thus the only critical point of the given autonomous system is the stable center (-1, 1) shown in Figure 9.1.18 in the text.

4. The only solution of the system $2x - 2y - 4 = 0,\ x + 4y + 3 = 0$ is the point (1, –1). The only figure among Figs. 9.1.11 through 9.1.18 showing a single critical point at (1, –1) is Fig. 9.1.12. Thus the only critical point of the given autonomous system is the spiral point (1,-1) shown in Figure 9.1.12 in the text.

5. The first equation $1 - y^2 = 0$ gives $y = 1$ or $y = -1$ at a critical point. Then the second equation $x + 2y = 0$ gives $x = -2$ or $x = 2$, respectively. The only figure among Figs. 9.1.11 through 9.1.18 showing two critical points at (–2, 1) and (2, –1) is Fig. 9.1.11. Thus the critical points of the given autonomous system are the spiral point (-2, 1) and the saddle point (2, 1) shown in Figure 9.1.11 in the text.

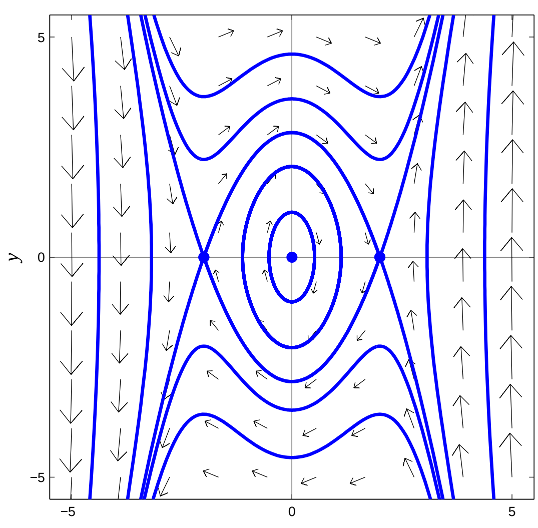
6. The second equation $4 - x^2 = 0$ gives $x = 2$ or $x = -2$ at a critical point. Then the first equation $2 - 4x - 15y = 0$ gives $y = -2/5$ or $x = 2/3$, respectively. The only figure among Figs. 9.1.11 through 9.1.18 showing two critical points at (–2, 2/3) and (2, –2/5) is Fig. 9.1.17. Thus the critical points of the given autonomous system are the spiral point (-2, 2/3) and the saddle point (2,-2/5) shown in Figure 9.1.17 in the text.

7. The first equation $4x - x^3 = 0$ gives $x = -2$, $x = 0$, or $x = 2$ at a critical point. Then the second equation $x - 2y = 0$ gives $y = -1$, $y = 0$, or $y = 1$, respectively. The only figure among Figs. 9.1.11 through 9.1.18 showing three critical points at $(-2, -1)$, $(0, 0)$, and $(2, 1)$ is Fig. 9.1.14. Thus the critical points of the given autonomous system are the spiral point (0, 0) and the saddle points (-2, 1) and (2, 1) shown in Figure 9.1.14 in the text.

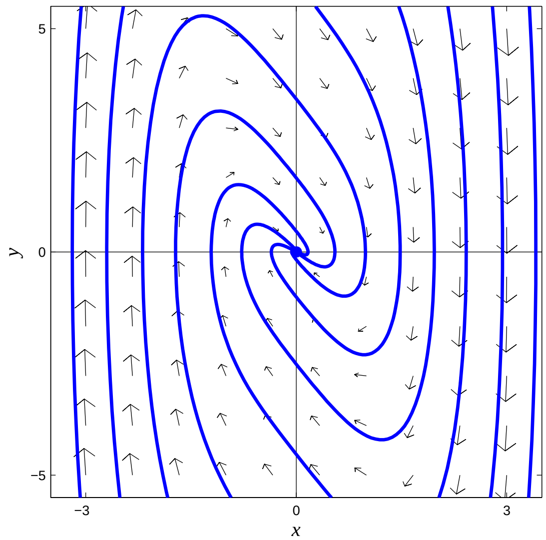
8. The second $-y - x^2 = 0$ equation gives $y = -x^2$ at a critical point. Substitution of this in the first equation $x - y - x^2 + xy = 0$ then gives $x - x^3 = 0$, so $x = -1$, $x = 0$, or $x = 1$. The only figure among Figs. 9.1.11 through 9.1.18 showing three critical points at
$(-1, -1)$, $(0, 0)$, and $(1, -1)$ is Fig. 9.1.16. Thus the critical points of the given autonomous system are the spiral point (-1, -1), the saddle point (0, 0), and the node (1, -1) shown in Figure 9.1.16 in the text.

In each of Problems 9-12 we need only set $x' = x'' = 0$ and solve the resulting equation for x.

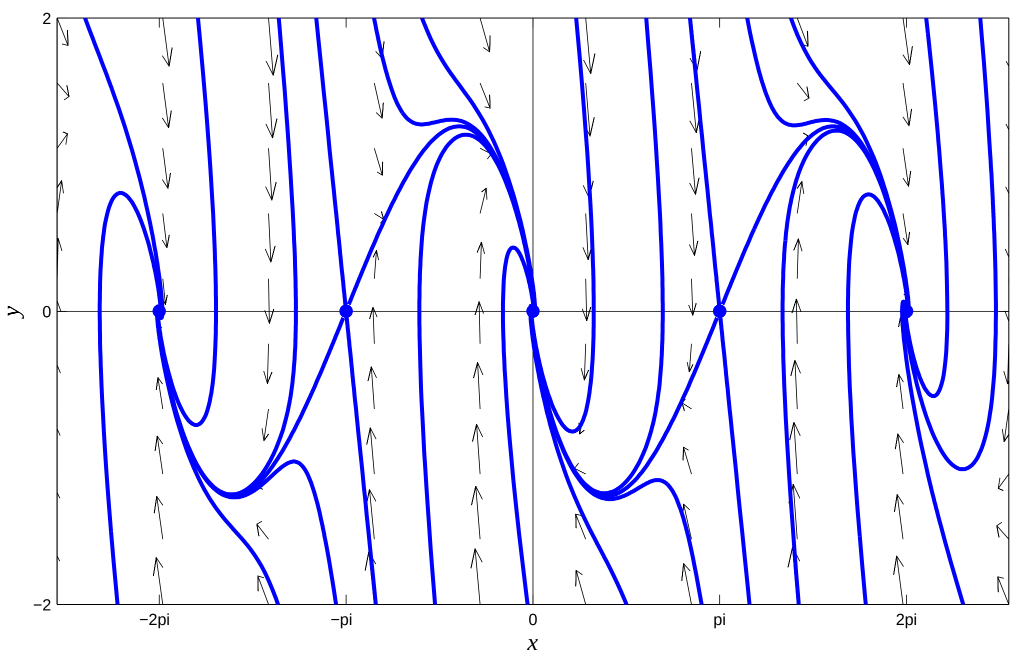
9. The equation $4x - x^3 = x(4 - x^2) = 0$ has the three solutions $x = 0, \pm 2$. This gives the three equilibrium solutions $x(t) \equiv 0$, $x(t) \equiv 2$, $x(t) \equiv -2$ of the given 2nd-order differential equation. A phase plane portrait for the equivalent 1st-order system $x' = y$, $y' = -4x + x^3$ is shown in the figure below. We observe that the critical point $(0, 0)$ in the phase plane appears to be a center, whereas the points $(\pm 2, 0)$ appear to be saddle points.



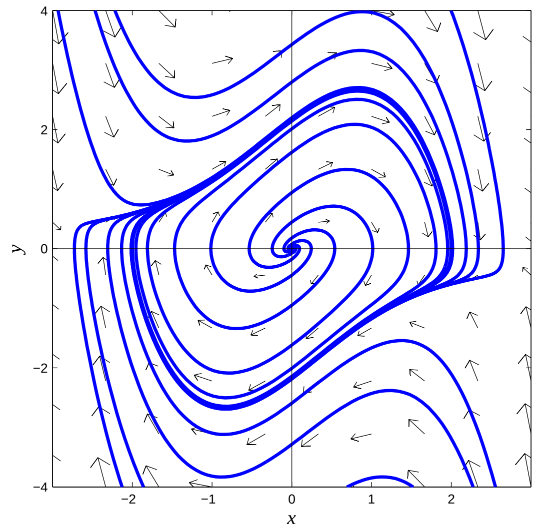
10. The equation $x + 4x^3 = x(1 + 4x^2) = 0$ has the single real solution $x = 0$. This gives the single equilibrium solution $x(t) \equiv 0$ of the given 2nd-order differential equation. A phase plane portrait for the equivalent 1st-order system $x' = y$, $y' = -2y - x - 4x^3$ is shown in the figure below. We observe that the critical point $(0,0)$ in the phase plane appears to be a spiral sink.



11. The equation $4\sin x = 0$ is satisfied by $x = n\pi$ for any integer n. Thus the given 2nd-order equation has infinitely many equilibrium solutions: $x(t) \equiv n\pi$ for any integer n. A phase portrait for the equivalent 1st-order system $x' = y$, $y' = -3y - 4\sin x$ is shown below. We observe that the critical point $(n\pi, 0)$ in the phase plane looks like a spiral sink if n is even, but a saddle point if n is odd.



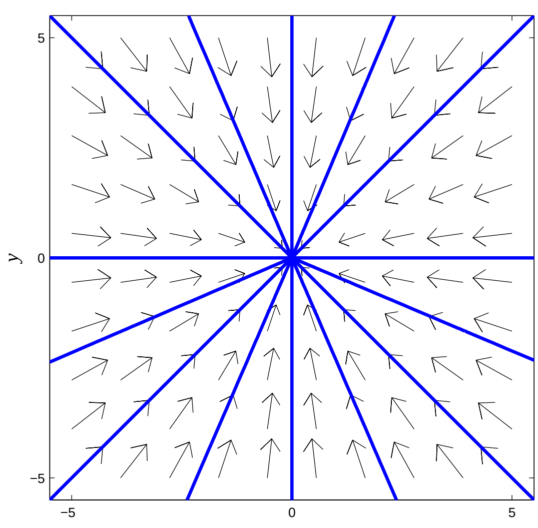
12. We immediately get the single solution $x = 0$ and thus the single equilibrium solution $x(t) \equiv 0$. A phase plane portrait for the equivalent 1st-order system $x' = y,\ y' = -(x^2 - 1)y - x$ is shown below. We observe that the critical point $(0,0)$ in the phase plane looks like a spiral source, with the solution curves emanating from this source spiraling outward toward a closed curve trajectory.



In Problems 13–16, the given *x*- and *y*-equations are independent exponential differential equations that we can solve immediately by inspection.

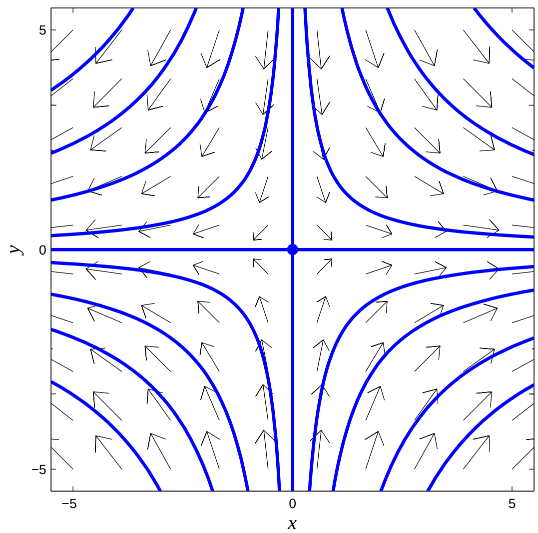
13. Solution: $x(t) = x_0 e^{-2t},\ y(t) = y_0 e^{-2t}$

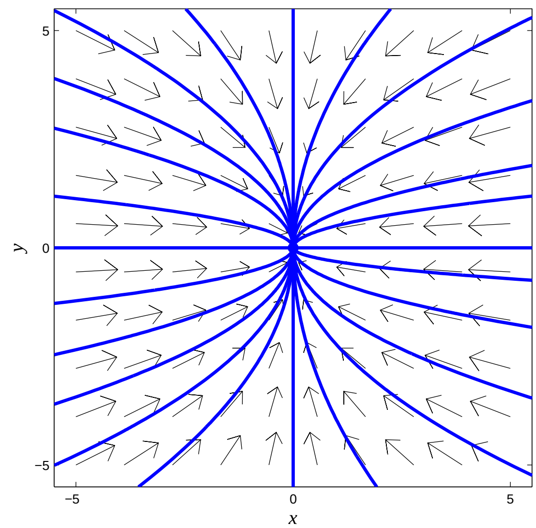
Then $y = (y_0 / x_0)x = kx,$ so the trajectories are straight lines through the origin. Clearly $x(t),\ y(t) \to 0$ as $t \to +\infty,$ so the origin is a stable proper node like the one shown below.



14. Solution: $x(t) = x_0e^{2t}, \quad y(t) = y_0e^{-2t}$

Then $xy = x_0y_0 = k$, so the trajectories are rectangular hyperbolas. Thus the origin is an unstable saddle point like the one in the left-hand figure below.



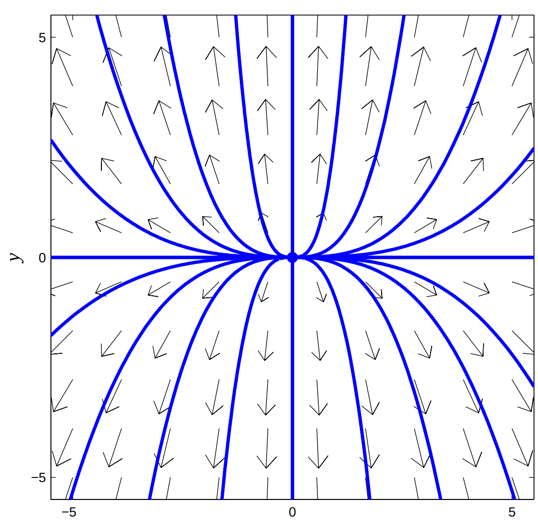


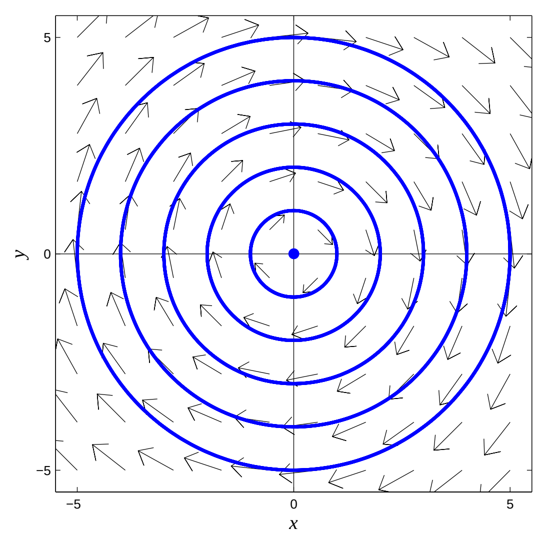
15. Solution: $x(t) = x_0e^{-2t}, \; y(t) = y_0e^{-t}$

Then $x = (x_0 / y_0^2)(y_0e^{-t})^2 = ky^2$, so the trajectories are parabolas of the form $x = ky^2$, and clearly $x(t), y(t) \to 0$ as $t \to +\infty$. Thus the origin is a stable improper node like the one shown in the right-hand figure above.

16. Solution: $x(t) = x_0e^{t}, \quad y(t) = y_0e^{3t}$

The origin is an unstable improper node. The trajectories consist of the y-axis and curves of the form $y = kx^3$, departing from the origin as in the left-hand figure below.





17. Differentiation of the first equation and substitution using the second one gives

$$x'' = y' = x, \quad \text{so} \quad x'' + x = 0.$$

We therefore get the general solution

$$\begin{aligned} x(t) &= A\cos t + B\sin t \\ y(t) &= B\cos t - A\sin t \qquad (y = x'). \end{aligned}$$

Then

$$\begin{aligned} x^2 + y^2 &= (A\cos t + B\sin t)^2 + (B\cos t - A\sin t)^2 \\ &= (A^2 + B^2)\cos^2 t + (A^2 + B^2)\sin^2 t = A^2 + B^2. \end{aligned}$$

Therefore the trajectories are clockwise-oriented circles centered at the origin, and the origin is a stable center as in the right-hand figure at the bottom of the preceding page.

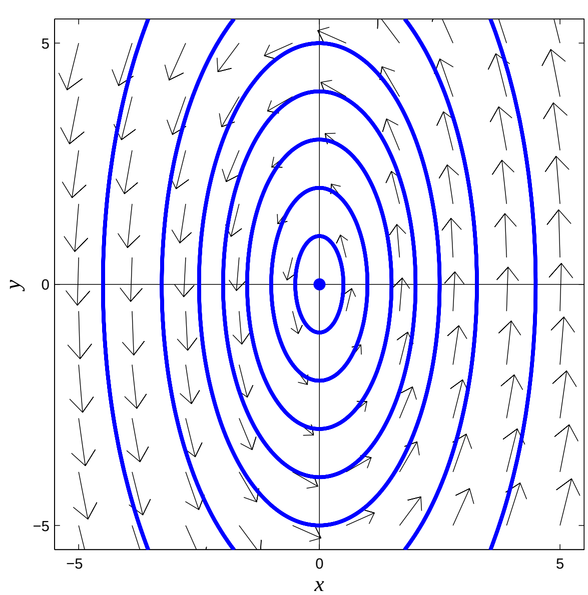
18. Elimination of y as in Problem 17 gives $x'' + 4x = 0$, so we get the general solution

$$\begin{aligned} x(t) &= A\cos 2t + B\sin 2t, \\ y(t) &= -2B\cos 2t + 2A\sin 2t \qquad (y = -x'). \end{aligned}$$

It follows readily that

$$4x^2 + y^2 = 4A^2 + 4B^2, \quad \text{so} \quad \frac{x^2}{(b/2)^2} + \frac{y^2}{b^2} = 1$$

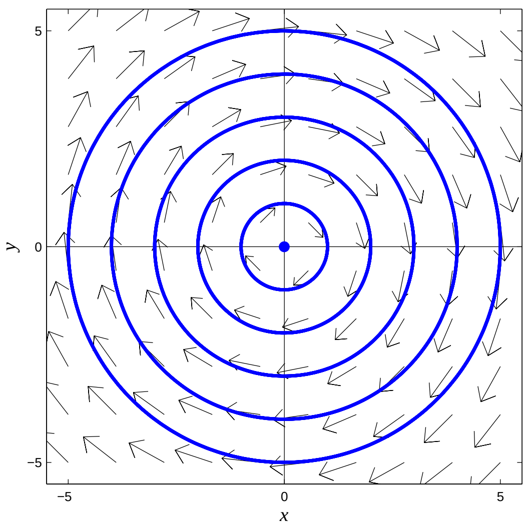
where $b^2 = 4A^2 + 4B^2$. Hence the origin is a stable center like the one illustrated in the figure below, and the vertical semiaxis of each ellipse is twice its horizontal semiaxis.

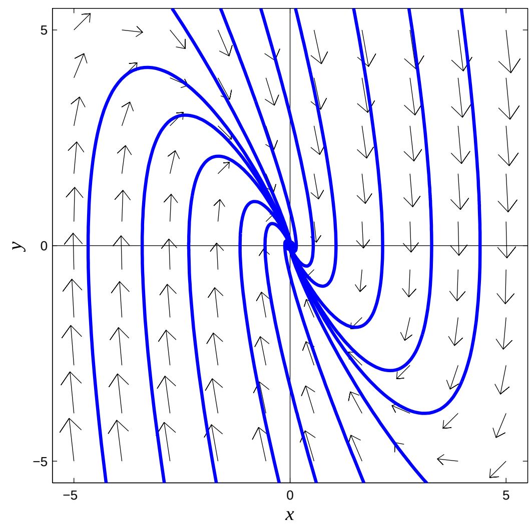


19. Elimination of y as in Problem 17 gives $x'' + 4x = 0$, so we get the general solution

$$\begin{aligned} x(t) &= A\cos 2t + B\sin 2t, \\ y(t) &= B\cos 2t - A\sin 2t \qquad (y = \tfrac{1}{2}x'). \end{aligned}$$

Then $x^2 + y^2 = A^2 + B^2$, so the origin is a stable center, and the trajectories are clockwise-oriented circles centered at (0, 0), as in the left-hand figure below.





20. Substitution of $y' = x''$ from the first equation into the second one gives $x'' = -5x - y = -5x - 4x'$, so $x'' + 4x' + 5x = 0$. The characteristic roots of this equation are $r = -2 \pm i$, so we get the general solution

$$\begin{aligned} x(t) &= e^{-2t}(A\cos t + B\sin \mathrm{t}), \\ y(t) &= e^{-2t}[(-2A + B)\cos t - (A + 2B)\sin t] \end{aligned}$$

(the latter because $y = x'$). Clearly $x(t), y(t) \to 0$ as $t \to +\infty$, so the origin is an asymptotically stable spiral point with trajectories approaching (0,0), as in the right-hand figure above.

21. We want to solve the system

$$\begin{aligned} -ky + x(1 - x^2 - y^2) &= 0 \\ kx + y(1 - x^2 - y^2) &= 0. \end{aligned}$$

If we multiply the first equation by $-y$ and the second one by x, then add the two results, we get $k(x^2 + y^2) = 0$. It therefore follows that $x = y = 0$.

22. After separation of variables, a partial-fractions decomposition gives

$$t = \int \frac{dr}{r(1-r^2)} = \int\left(\frac{1}{r} - \frac{1}{2(r+1)} - \frac{1}{2(r-1)}\right)$$
$$= \ln r - \frac{1}{2}\ln(r+1) - \frac{1}{2}\ln(r-1) + \frac{1}{2},$$

so

$$2t = \ln\frac{Cr^2}{r^2-1}$$

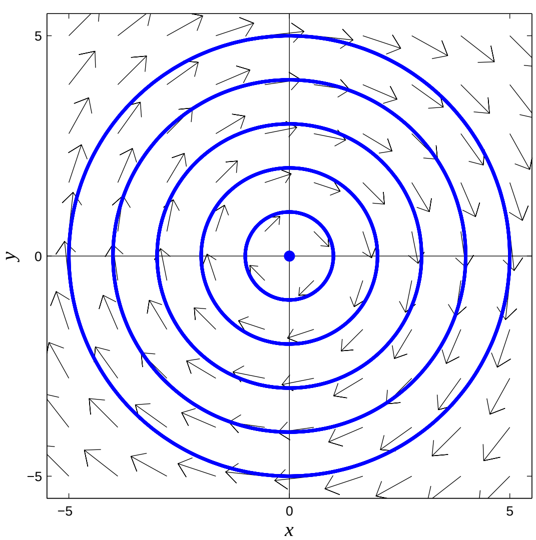
(assuming that $r > 1$, for instance). The initial condition $r(0) = r_0$ then gives

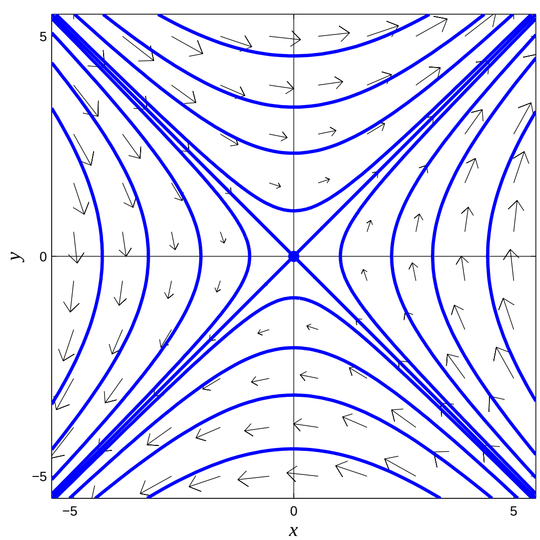
$$2t = \ln\frac{r^2(r_0^2-1)}{r_0^2(r^2-1)}, \quad \text{so} \quad e^{2t} = \frac{r^2(r_0^2-1)}{r_0^2(r^2-1)}.$$

We now solve readily for

$$r^2 = \frac{r_0^2 e^{2t}}{r_0^2 e^{2t} + (1-r_0^2)} = \frac{r_0^2}{r_0^2 + (1-r_0^2)e^{-2t}}.$$

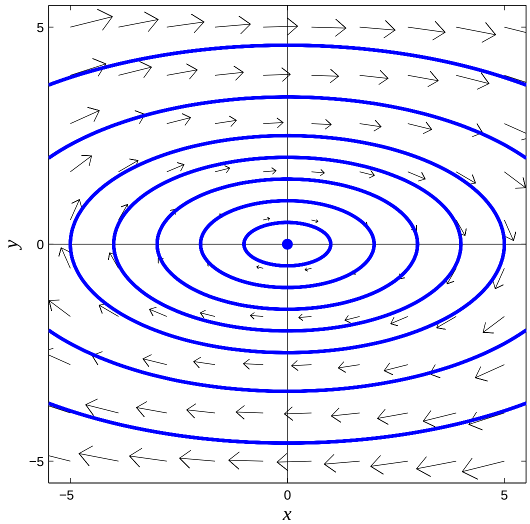
23. The equation $dy/dx = -x/y$ separates to $x\,dx + y\,dy = 0$, so $x^2 + y^2 = C$. Thus the trajectories consist of the origin $(0, 0)$ and the circles $x^2 + y^2 = C > 0$, as shown in the left-hand figure below.

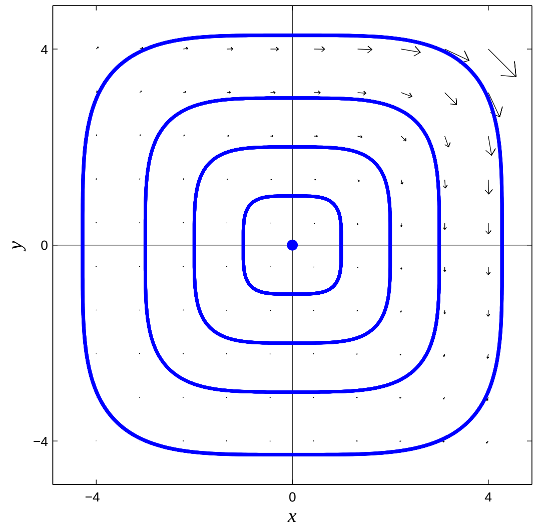




24. The equation $dy/dx = x/y$ separates to $y\,dy - x\,dx = 0$, so $y^2 - x^2 = C$. Thus the trajectories consist of the origin $(0, 0)$ and the hyperbolas $y^2 - x^2 = C$, as shown in the right-hand figure above.

25. The equation $dy/dx = -x/4y$ separates to $x\,dx + 4y\,dy = 0$, so $x^2 + 4y^2 = C$. Thus the trajectories consist of the origin $(0, 0)$ and the ellipses $x^2 + 4y^2 = C > 0$, as shown in the left-hand figure below.





26. The equation $dy/dx = -x^3/y^3$ separates to $x^3\,dx + y^3\,dy = 0$, so $x^4 + y^4 = C$. Thus the trajectories consist of the origin $(0, 0)$ and the ovals of the form $x^4 + y^4 = C$, as illustrated in the right-hand figure above.

27. If $\phi(t) = x(t+\gamma)$ and $\psi(t) = y(t+\gamma)$ then

$$\phi'(t) = x'(t+\gamma) = y(t+\gamma) = \psi(t),$$

but

$$\psi(\mathrm{t}) = y'(t+\gamma) = x(t+\gamma)\cdot(t+\gamma) = t\,\phi(t) + \gamma\,\phi(t) \neq t\,\phi(t).$$

28. If $\phi(t) = x(t+\gamma)$ and $\psi(t) = y(t+\gamma)$ then

$$\phi'(\mathrm{t}) = x'(t+\gamma) = F(x(t+\gamma), \quad y(t+\gamma)) = F(\phi(t), \psi(t)),$$

and

$$\psi'(t) = G(\phi(t), \psi(t))$$

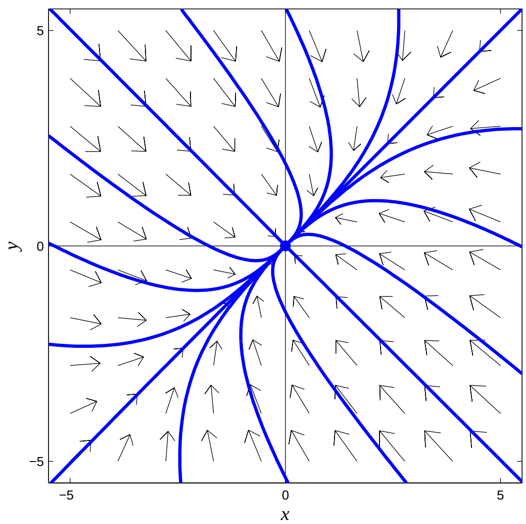
similarly. Therefore $\phi(t)$ and $\psi(t)$ satisfy the given differential equations.

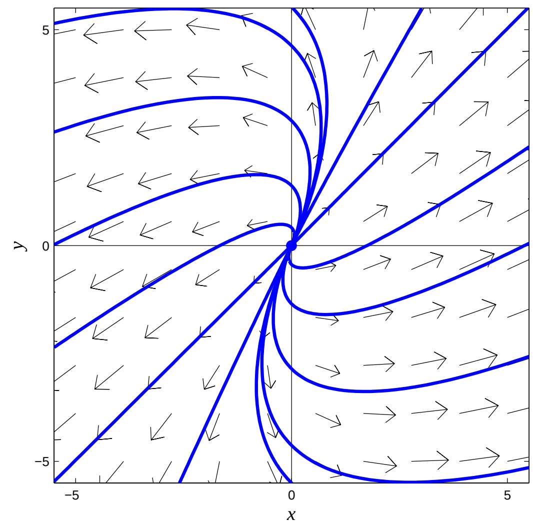
SECTION 9.2

LINEAR AND ALMOST LINEAR SYSTEMS

In Problems 1–10 we first find the roots λ_1 and λ_2 of the characteristic equation of the coefficient matrix of the given linear system. We can then read the type and stability of the critical point (0,0) from Theorem 1 and the table of Figure 9.2.9 in the text.

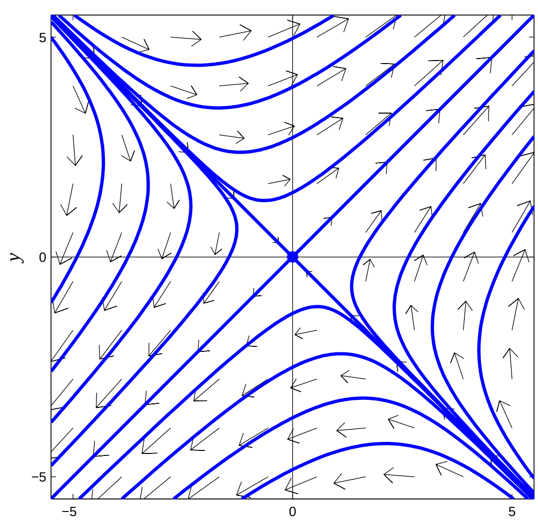
1. The roots $\lambda_1 = -1$ and $\lambda_2 = -3$ of the characteristic equation $\lambda^2 + 4\lambda + 3 = 0$ are both negative, so $(0,0)$ is an asymptotically stable node as shown on the left below.

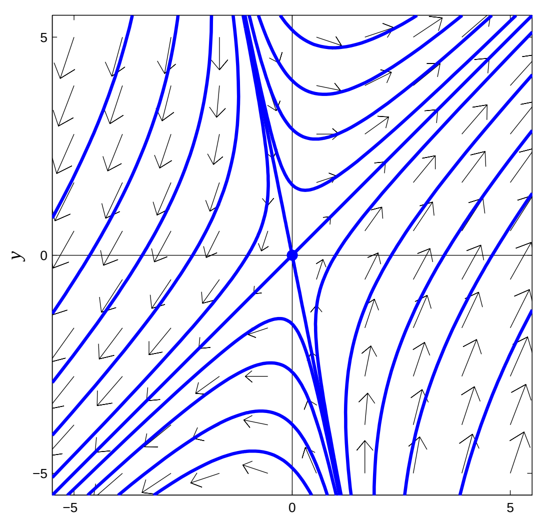




2. The roots $\lambda_1 = 2$ and $\lambda_2 = 3$ of the characteristic equation $\lambda^2 - 5\lambda + 6 = 0$ are both positive, so $(0,0)$ is an unstable improper node as shown on the right above.

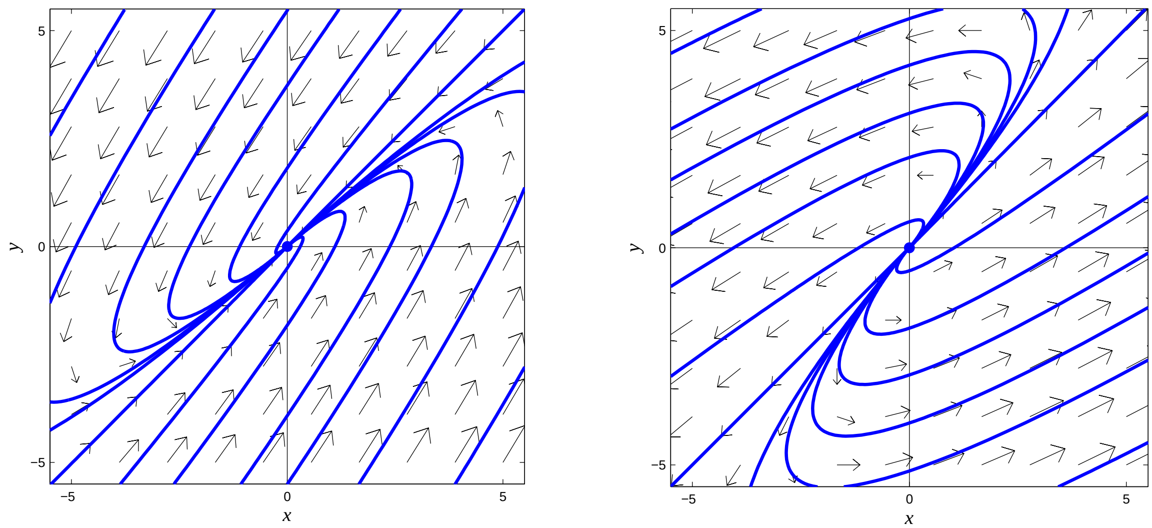
3. The roots $\lambda_1 = -1$ and $\lambda_2 = 3$ of the characteristic equation $\lambda^2 - 2\lambda - 3 = 0$ have different signs, so $(0,0)$ is an unstable saddle point as shown on the left below.





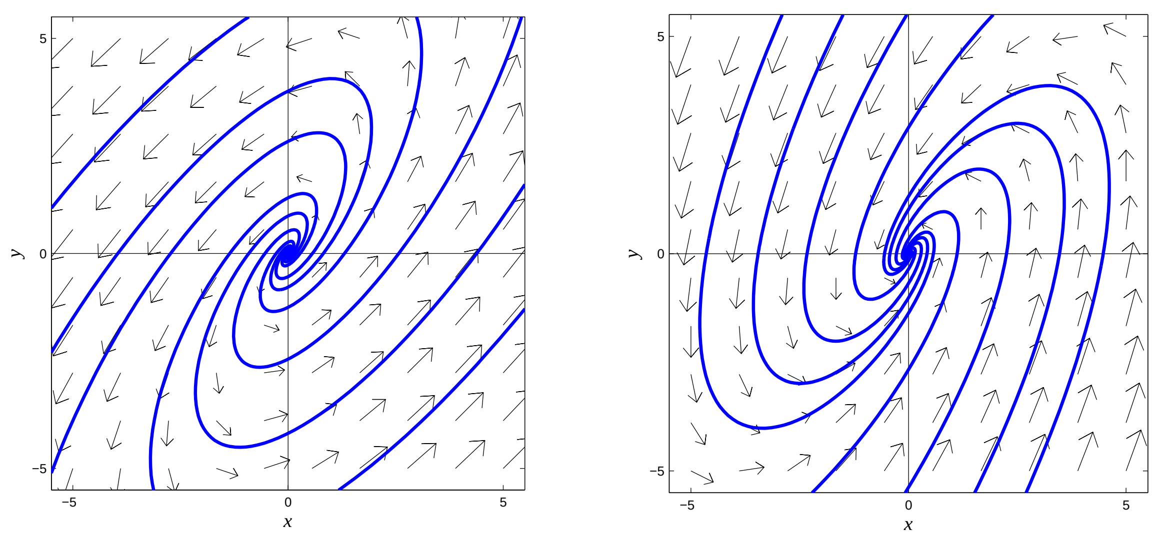
4. The roots $\lambda_1 = -2$ and $\lambda_2 = 4$ of the characteristic equation $\lambda^2 - 2\lambda - 3 = 0$ have different signs, so $(0,0)$ is an unstable saddle point as shown on the right above.

5. The roots $\lambda_1 = \lambda_2 = -1$ of the characteristic equation $\lambda^2 + 2\lambda + 1 = 0$ are negative and equal, so $(0,0)$ is an asymptotically stable node as in the left-hand figure below.



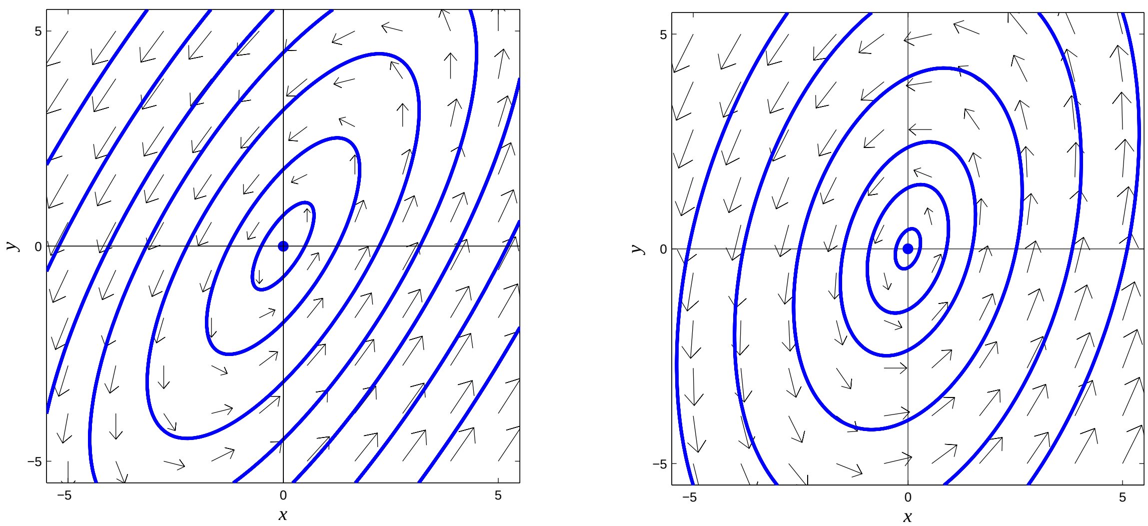
6. The roots $\lambda_1 = \lambda_2 = 2$ of the characteristic equation $\lambda^2 - 4\lambda + 4 = 0$ are positive and equal, so $(0,0)$ is an unstable node as in the right-hand figure above.

7. The roots $\lambda_1, \lambda_2 = 1 \pm 2\,i$ of the characteristic equation $\lambda^2 - 2\lambda + 5 = 0$ are complex conjugates with positive real part, so $(0,0)$ is an unstable spiral point as shown in the left-hand figure below.



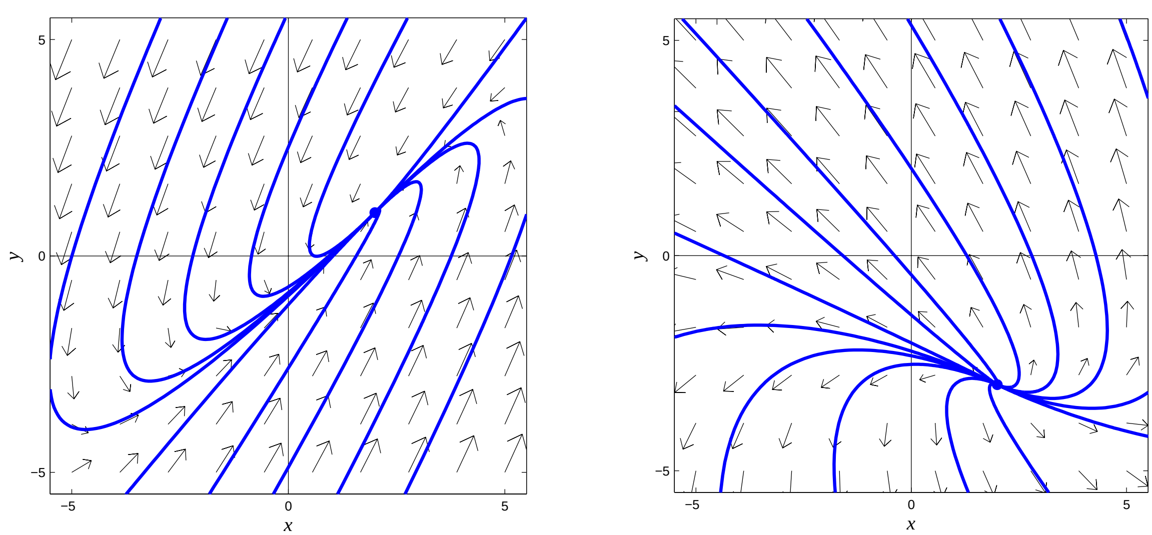
8. The roots $\lambda_1, \lambda_2 = -2 \pm 3\,i$ of the characteristic equation $\lambda^2 + 4\lambda + 13 = 0$ are complex conjugates with negative real part, so $(0,0)$ is an asymptotically stable spiral point as shown in the right-hand figure above.

9. The roots $\lambda_1, \lambda_2 = \pm 2\,i$ of the characteristic equation $\lambda^2 + 4 = 0$ are pure imaginary, so $(0,0)$ is a stable (but not asymptotically stable) center as in the left-hand figure below.



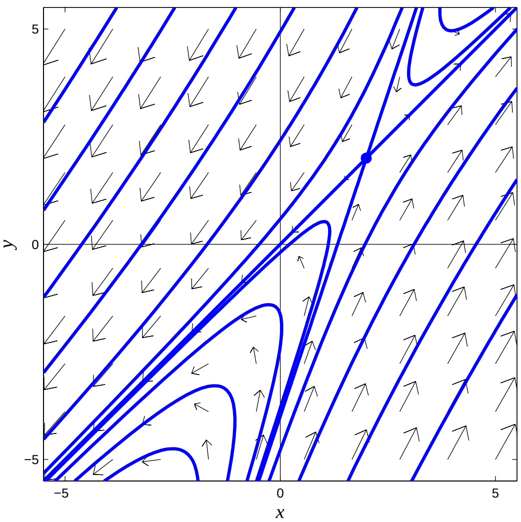
10. The roots $\lambda_1, \lambda_2 = \pm 3\,i$ of the characteristic equation $\lambda^2 + 9 = 0$ are pure imaginary, so $(0,0)$ is a stable (but not asymptotically stable) center as in the right-hand figure above.

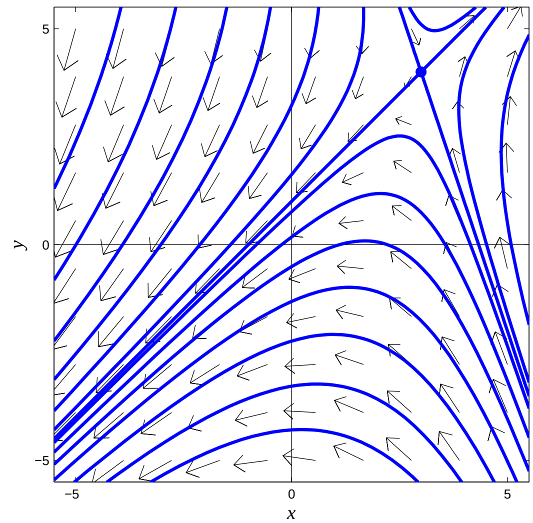
11. The Jacobian matrix $\mathbf{J} = \begin{bmatrix} 1 & -2 \\ 3 & -4 \end{bmatrix}$ has characteristic equation $\lambda^2 + 3\lambda + 2 = 0$ and eigenvalues $\lambda_1 = -1,\ \lambda_2 = -2$ that are both negative. Hence the critical point $(2, 1)$ is an asymptotically stable node as in the left-hand figure below.



12. The Jacobian matrix $\mathbf{J} = \begin{bmatrix} 1 & -2 \\ 1 & 4 \end{bmatrix}$ has characteristic equation $\lambda^2 - 5\lambda + 6 = 0$ and eigenvalues $\lambda_1 = 2,\ \lambda_2 = 3$ that are both positive. Hence the critical point $(2,-3)$ is an unstable node as in the right-hand figure above.

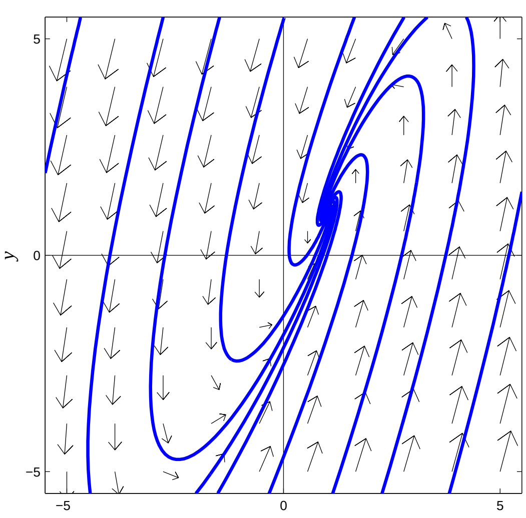
13. The Jacobian matrix $\mathbf{J} = \begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix}$ has characteristic equation $\lambda^2 - 1 = 0$ and eigenvalues $\lambda_1 = -1,\ \lambda_2 = +1$ having different signs. Hence the critical point $(2, 2)$ is an unstable saddle point as in the left-hand figure below.

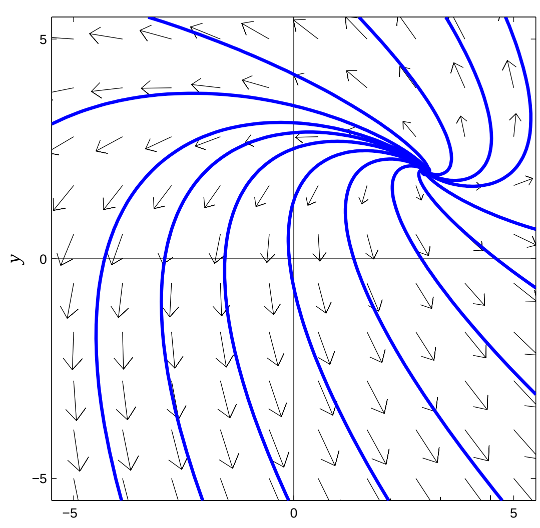




14. The Jacobian matrix $\mathbf{J} = \begin{bmatrix} 1 & 1 \\ 3 & -1 \end{bmatrix}$ has characteristic equation $\lambda^2 - 4 = 0$ and eigenvalues $\lambda_1 = -2,\ \lambda_2 = 2$ that are real with different signs. Hence the critical point $(3, 4)$ is an unstable saddle point as in the right-hand figure above.

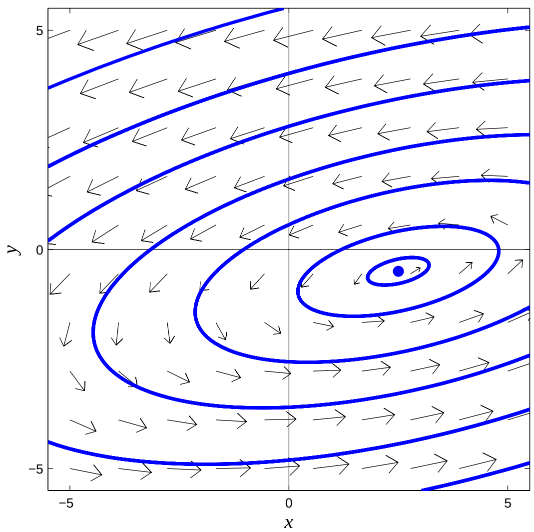
15. The Jacobian matrix $\mathbf{J} = \begin{bmatrix} 1 & -1 \\ 5 & -3 \end{bmatrix}$ has characteristic equation $\lambda^2 + 2\lambda + 2 = 0$ and eigenvalues $\lambda_1, \lambda_2 = -1 \pm i$ that are complex conjugates with negative real part. Hence the critical point $(1, 1)$ is an asymptotically stable spiral point as shown in the figure on the left below.

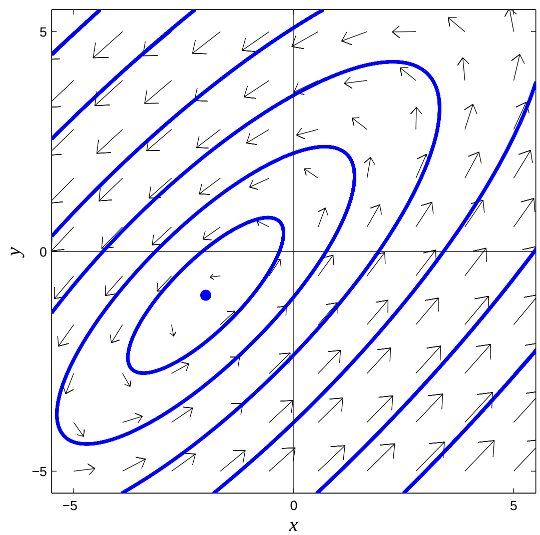




16. The Jacobian matrix $\mathbf{J} = \begin{bmatrix} 1 & -2 \\ 1 & 3 \end{bmatrix}$ has characteristic equation $\lambda^2 - 4\lambda + 5 = 0$ and eigenvalues $\lambda_1, \lambda_2 = 2 \pm i$ that are complex conjugates with positive real part. Hence the critical point $(3, 2)$ is an unstable spiral point as shown in the right-hand figure at the bottom of the preceding page.

17. The Jacobian matrix $\mathbf{J} = \begin{bmatrix} 1 & -5 \\ 1 & -1 \end{bmatrix}$ has characteristic equation $\lambda^2 + 4 = 0$ and pure imaginary eigenvalues $\lambda_1, \lambda_2 = \pm 2i$. Hence $(5/2, -1/2)$ is a stable (but not asymptotically stable) center as shown on the left below.





18. The Jacobian matrix $\mathbf{J} = \begin{bmatrix} 4 & -5 \\ 5 & -4 \end{bmatrix}$ has characteristic equation $\lambda^2 + 9 = 0$ and pure imaginary eigenvalues $\lambda_1, \lambda_2 = \pm 3i$. Hence $(-2, -1)$ is a stable (but not asymptotically stable) center as shown on the right above.

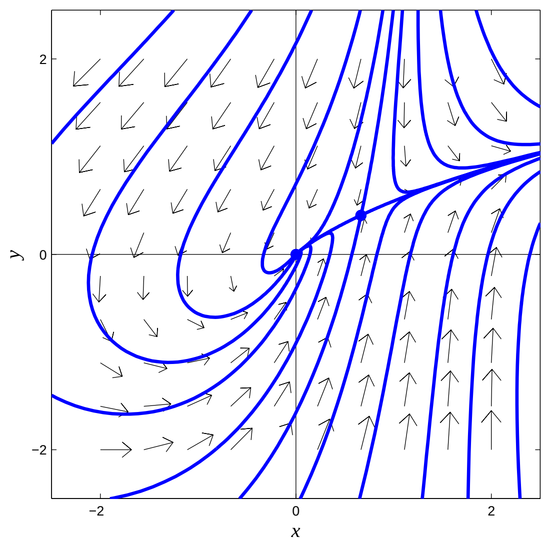
In each of Problems 19–28 we first calculate the Jacobian matrix $\mathbf{J}$ and its eigenvalues at (0,0) and at each of the other critical points we observe in our phase portrait for the given system. Then we apply Theorem 2 to determine as much as we can about the type and stability of each of these critical points of the given almost linear system. Finally we a phase portrait that

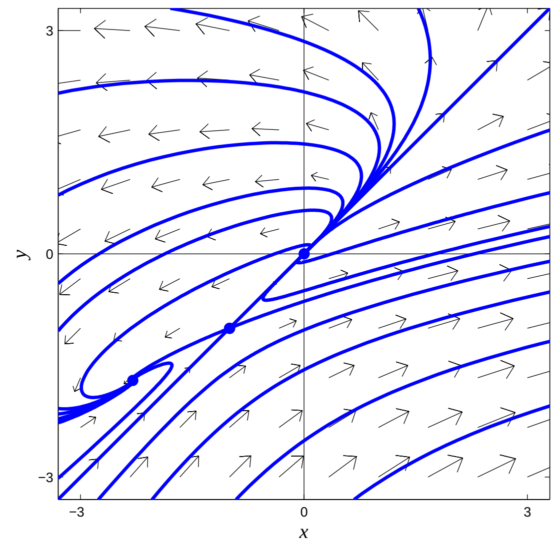
19. $\mathbf{J} = \begin{bmatrix} 1 + 2y & -3 + 2x \\ 4 - y & -6 - x \end{bmatrix}$

At (0,0): The Jacobian matrix $\mathbf{J} = \begin{bmatrix} 1 & -3 \\ 4 & -6 \end{bmatrix}$ has characteristic equation $\lambda^2 + 5\lambda + 6 = 0$ and eigenvalues $\lambda_1 = -3$, $\lambda_2 = -2$ that are both negative. Hence $(0,0)$ is an asymptotically stable node of the given almost linear system.

At (2/3, 2/5): The Jacobian matrix $\mathbf{J} = \begin{bmatrix} 9/5 & -5/3 \\ 18/5 & -20/3 \end{bmatrix}$ has characteristic equation $\lambda^2 + \frac{73}{15}\lambda - 6 = 0$ and approximate eigenvalues $\lambda_1 \approx -5.89$, $\lambda_2 \approx 1.02$ with different signs. Hence (2/3, 2/5) is a saddle point.

The left-hand figure below shows both these critical points.





20. $\mathbf{J} = \begin{bmatrix} 6+2x & -5 \\ 2 & -1+2y \end{bmatrix}$

At (0,0): The Jacobian matrix $\mathbf{J} = \begin{bmatrix} 6 & -5 \\ 2 & -1 \end{bmatrix}$ has characteristic equation $\lambda^2 - 5\lambda + 4 = 0$ and eigenvalues $\lambda_1 = 1$, $\lambda_2 = 4$ that are both positive. Hence (0,0) is an unstable node of the given almost linear system.

At (–1,–1): The Jacobian matrix $\mathbf{J} = \begin{bmatrix} 4 & -5 \\ 2 & -3 \end{bmatrix}$ has characteristic equation $\lambda^2 - \lambda - 2 = 0$ and eigenvalues $\lambda_1 = -1$, $\lambda_2 = 2$ with different signs. Hence (–1,–1) is a saddle point.

At (–2.30,–1.70): The Jacobian matrix $\mathbf{J} \approx \begin{bmatrix} 1.40 & -5 \\ 2 & -4.40 \end{bmatrix}$ has complex conjugate eigenvalues $\lambda_1 \approx -1.5 + 1.25i$, $\lambda_2 \approx -1.5 - 1.25i$ with negative real parts. Hence (–2.30,–1.70) is a spiral sink.

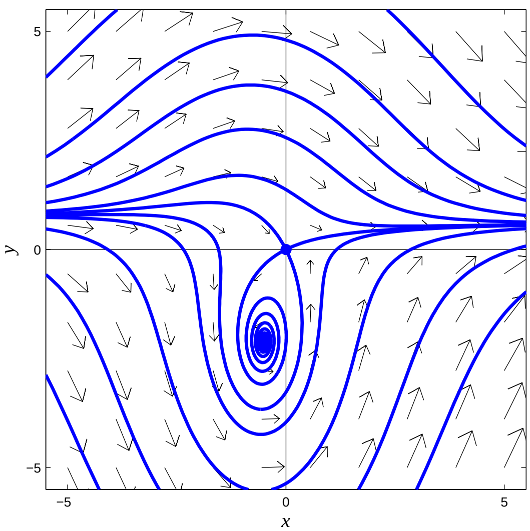
The figure on the right above shows these three critical points.

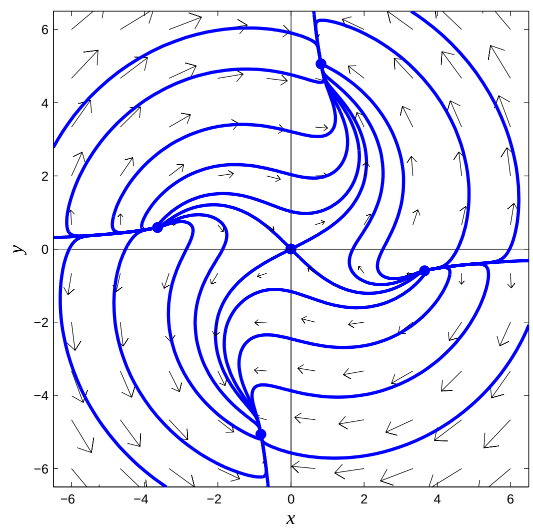
21. $\mathbf{J} = \begin{bmatrix} 1+2x & 2+2y \\ 2-3y & -2-3x \end{bmatrix}$

At (0,0): The Jacobian matrix $\mathbf{J} = \begin{bmatrix} 1 & 2 \\ 2 & -2 \end{bmatrix}$ has characteristic equation $\lambda^2 + \lambda - 6 = 0$ and eigenvalues $\lambda_1 = -3,\ \lambda_2 = 2$ with different signs. Hence (0,0) is a saddle point of the given almost linear system.

At (–0.51,–2.12): The Jacobian matrix $\mathbf{J} \approx \begin{bmatrix} -0.014 & -2.236 \\ 8.354 & -0.479 \end{bmatrix}$ has complex conjugate eigenvalues $\lambda_1 \approx -0.25 + 4.32i,\ \lambda_2 \approx -0.25 + 4.32i$ with negative real parts. Hence (–0.51,–2.12) is a spiral sink.

The figure on the left below shows these two critical points.





22. $\mathbf{J} = \begin{bmatrix} 1-y^2 & 4-2xy \\ 2+2xy & -1+x^2 \end{bmatrix}$

At (0,0): The Jacobian matrix $\mathbf{J} = \begin{bmatrix} 1 & 4 \\ 2 & -1 \end{bmatrix}$ has characteristic equation $\lambda^2 - 9 = 0$ and eigenvalues $\lambda_1 = -3,\ \lambda_2 = 3$ that have different signs. Hence (0,0) is a saddle point of the given almost linear system.

At $(\pm 3.65, \mp 0.59)$: The Jacobian matrix $\mathbf{J} \approx \begin{bmatrix} 0.649 & 8.325 \\ -2.325 & 12.325 \end{bmatrix}$ has positive real eigenvalues $\lambda_1 \approx 2.649,\ \lambda_2 \approx 10.325$. Hence these critical points are both nodal sources.

At $(\pm 0.82, \pm 5.06)$: The Jacobian matrix $\mathbf{J} \approx \begin{bmatrix} -24.649 & -4.325 \\ 10.325 & -0.325 \end{bmatrix}$ has negative real eigenvalues $\lambda_1 \approx -22.649,\ \lambda_2 \approx -2.325$. Hence these critical points are both nodal sinks.

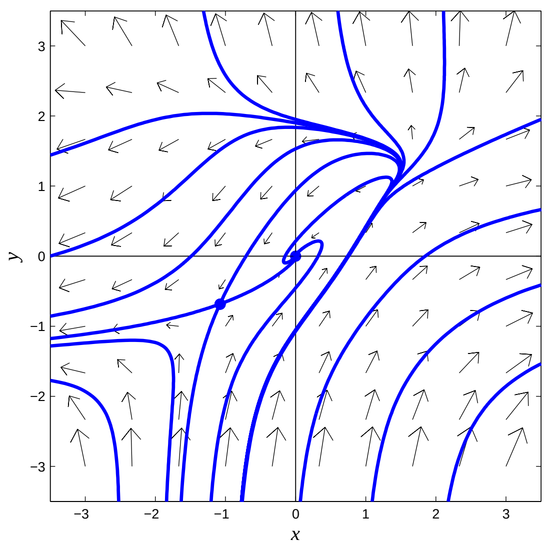
The figure on the right above shows these five critical points.

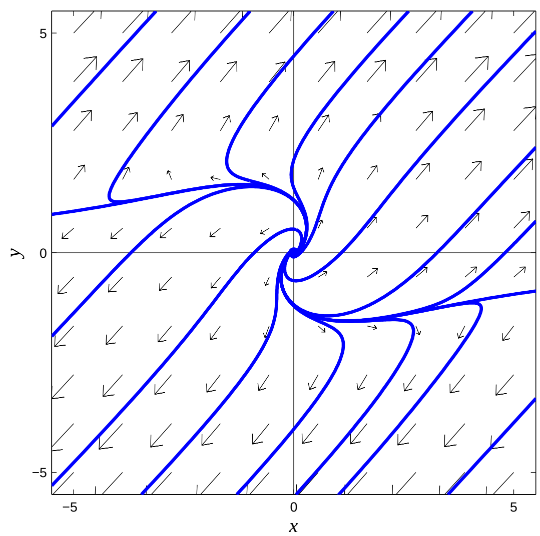
23. $\mathbf{J} = \begin{bmatrix} 2+3x^2 & -5 \\ 4 & -6+4y^3 \end{bmatrix}$

At (0,0): The Jacobian matrix $\mathbf{J} = \begin{bmatrix} 2 & -5 \\ 4 & -6 \end{bmatrix}$ has characteristic equation $\lambda^2 + 4\lambda + 8 = 0$ and complex conjugate eigenvalues $\lambda_1 = -2+2i,\ \lambda_2 = -2-2i$ with negative real part. Hence $(0,0)$ is a spiral sink of the given almost linear system.

At $(-1.08, -0.68)$: The Jacobian matrix $\mathbf{J} \approx \begin{bmatrix} 5.495 & -5 \\ 4 & -7.276 \end{bmatrix}$ has eigenvalues $\lambda_1 \approx -5.45,\ \lambda_2 \approx 3.67$ with different signs. Hence $(-1.08, -0.68)$ is a saddle point.

The figure on the left below shows these two critical points.





24. $\mathbf{J} = \begin{bmatrix} 5+2xy & -3+x^2+3y^2 \\ 5+2xy & x^2+3y^2 \end{bmatrix}$

At (0,0): The Jacobian matrix $\mathbf{J} = \begin{bmatrix} 5 & -3 \\ 5 & 0 \end{bmatrix}$ has characteristic equation $\lambda^2 - 5\lambda + 15 = 0$ and complex conjugate eigenvalues $\lambda_1 \approx 2.5+2.96i,\ \lambda_2 \approx 2.5-2.96i$ with positive real part. Hence $(0,0)$ is a spiral source of the given almost linear system. The figure on the right above shows this critical point.

25. $\mathbf{J} = \begin{bmatrix} 1+3y & -2+3x \\ 2-2x & -3-2y \end{bmatrix}$

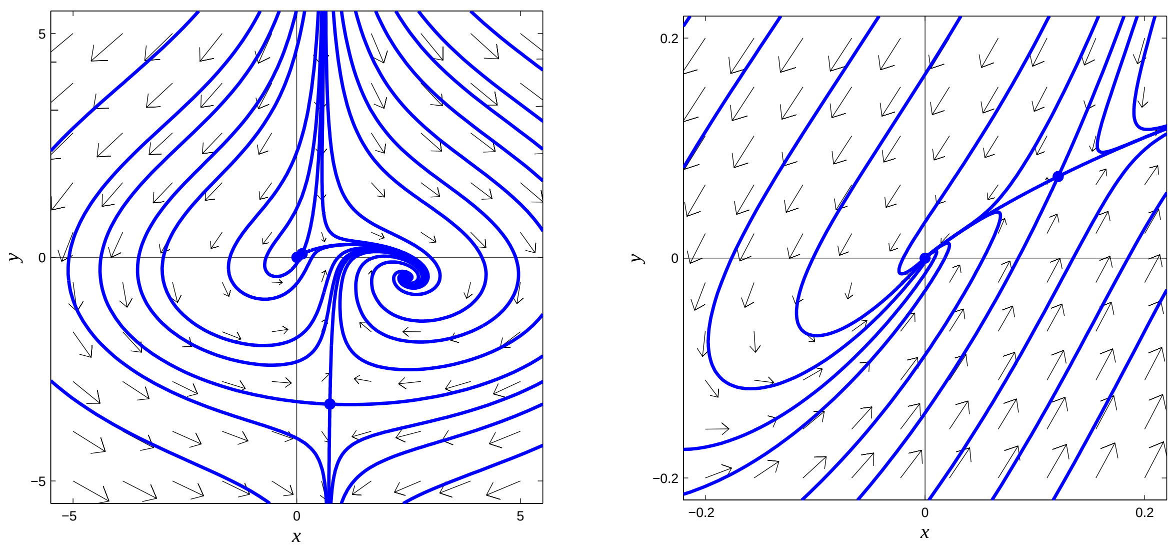
At (0,0): The Jacobian matrix $\mathbf{J} = \begin{bmatrix} 1 & -2 \\ 2 & -3 \end{bmatrix}$ has characteristic equation $\lambda^2 + 2\lambda + 1 = 0$ and equal negative eigenvalues $\lambda_1 = -1,\ \lambda_2 = -1$. Hence $(0,0)$ is either a nodal sink or a spiral sink of the given almost linear system.

At $(0.74,-3.28)$: The Jacobian matrix $\mathbf{J} \approx \begin{bmatrix} -8.853 & 0.226 \\ 0.516 & 3.568 \end{bmatrix}$ has real eigenvalues $\lambda_1 \approx -8.86,\ \lambda_2 \approx 3.58$ with different signs. Hence $(0.74,-3.28)$ is a saddle point.

At $(2.47,-0.46)$: The Jacobian matrix $\mathbf{J} \approx \begin{bmatrix} -0.370 & 5.410 \\ -2.940 & -2.087 \end{bmatrix}$ has complex conjugate eigenvalues $\lambda_1 \approx -1.23+3.89i,\ \lambda_2 \approx -1.23+3.89i$ with negative real part. Hence $(2.47,-0.46)$ is a spiral sink.

At $(0.121,0.074)$: The Jacobian matrix $\mathbf{J} \approx \begin{bmatrix} 1.222 & -1.636 \\ 1.758 & -3.148 \end{bmatrix}$ has real eigenvalues $\lambda_1 \approx -2.34,\ \lambda_2 \approx 0.42$ with different signs. Hence $(0.121,0.074)$ is a saddle point.

The left-hand figure below shows clearly the first three of these critical points. The right-hand figure is a close-up near the origin with the final critical point now visible.



26. $\mathbf{J} = \begin{bmatrix} 3-2x & -2-2y \\ 2-3y & -1-3x \end{bmatrix}$

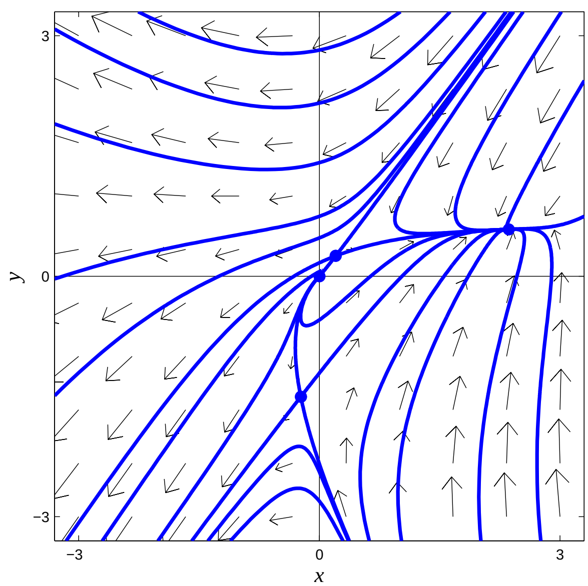
At (0,0): The Jacobian matrix $\mathbf{J} = \begin{bmatrix} 3 & -2 \\ 2 & -1 \end{bmatrix}$ has characteristic equation $\lambda^2 - 2\lambda + 1 = 0$ and equal positive eigenvalues $\lambda_1 = 1,\ \lambda_2 = 1$. Hence (0,0) is either a nodal source or a spiral source of the given almost linear system.

At $(0.203,0.253)$: The Jacobian matrix $\mathbf{J} \approx \begin{bmatrix} 2.592 & -2.506 \\ 1.241 & -1.611 \end{bmatrix}$ has real eigenvalues $\lambda_1 \approx -0.65,\ \lambda_2 \approx 1.63$ with different signs. Hence $(0.203,0.253)$ is a saddle point.

At $(-0.231,-1.504)$: The Jacobian matrix $\mathbf{J} \approx \begin{bmatrix} 3.462 & 1.008 \\ 6.511 & -0.307 \end{bmatrix}$ has real eigenvalues $\lambda_1 \approx -1.60,\ \lambda_2 \approx 4.76$ with different signs. Hence $(-0.231,-1.504)$ is a saddle point.

At $(2.360, 0.584)$: The Jacobian matrix $\mathbf{J} \approx \begin{bmatrix} -1.721 & -3.168 \\ -0.247 & -8.081 \end{bmatrix}$ has unequal negative eigenvalues $\lambda_1 \approx -7.96,\ \lambda_2 \approx -1.85$. Hence $(2.47,-0.46)$ is a nodal sink.

The figure below shows these four critical points.



27. $\mathbf{J} = \begin{bmatrix} 1+4x^3 & -1-2y \\ 2-2x & -1+4y^3 \end{bmatrix}$

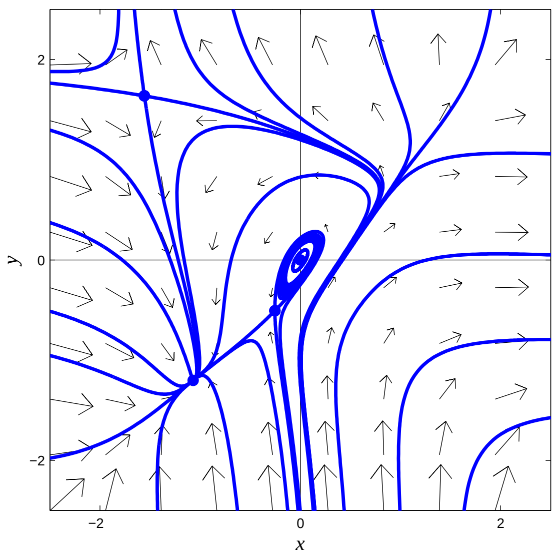
At (0,0): The Jacobian matrix $\mathbf{J} = \begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix}$ has characteristic equation $\lambda^2 + 1 = 0$ and equal positive eigenvalues $\lambda_1 = -i,\ \lambda_2 = +i$. Hence $(0,0)$ is either a center or a spiral point, but its stability is not determined by Theorem 2.

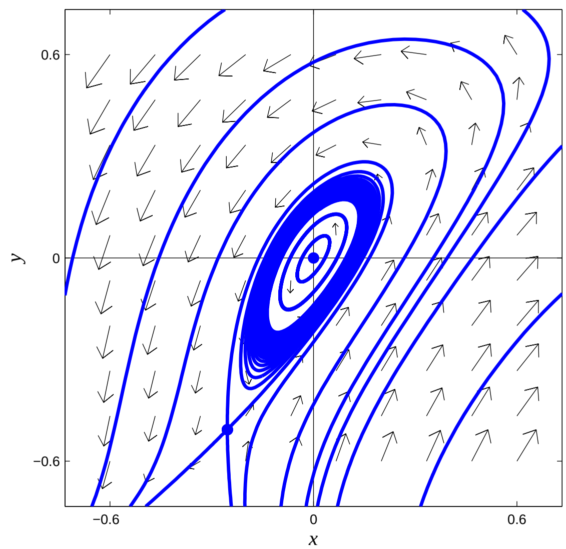
At $(-0.254,-0.507)$: The Jacobian matrix $\mathbf{J} \approx \begin{bmatrix} 0.934 & 0.014 \\ 2.508 & -1.521 \end{bmatrix}$ has real eigenvalues $\lambda_1 \approx -1.53,\ \lambda_2 \approx 0.95$ with different signs. Hence $(-0.254,-0.507)$ is a saddle point.

At $(-1.557,-1.637)$: The Jacobian matrix $\mathbf{J} \approx \begin{bmatrix} -14.087 & -4.273 \\ 5.113 & 16.532 \end{bmatrix}$ has real eigenvalues $\lambda_1 \approx -13.36,\ \lambda_2 \approx 15.80$ with different signs. Hence $(-1.557,-1.637)$ is a saddle point.

At $(-1.070, -1.202)$: The Jacobian matrix $\mathbf{J} \approx \begin{bmatrix} -3.905 & 1.403 \\ 4.141 & -7.940 \end{bmatrix}$ has unequal negative eigenvalues $\lambda_1 \approx -9.07$, $\lambda_2 \approx -2.78$. Hence $(-1.070, -1.202)$ is a nodal sink.

The left-hand figure below shows these four critical points. The close-up on the right suggests that the origin may (but may not) be a stable center.





28. $\mathbf{J} = \begin{bmatrix} 3+3x^2 & -1+3y^2 \\ 13+3y & -3+3x \end{bmatrix}$

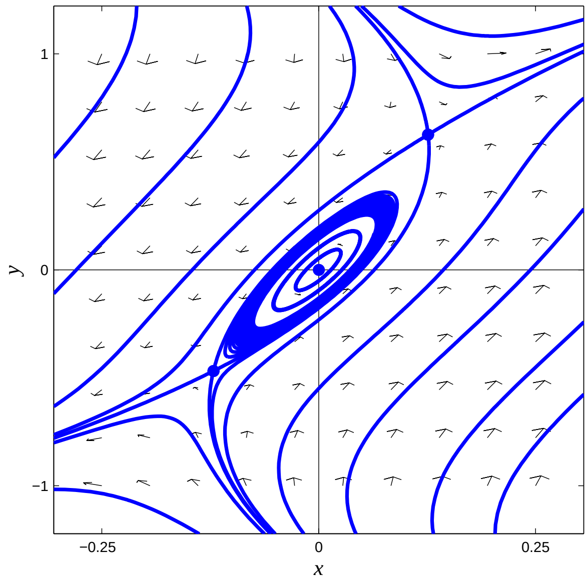
At (0,0): The Jacobian matrix $\mathbf{J} = \begin{bmatrix} 3 & -1 \\ 13 & -3 \end{bmatrix}$ has characteristic equation $\lambda^2 + 4 = 0$ and equal positive eigenvalues $\lambda_1 = -2i$, $\lambda_2 = +2i$. Hence (0,0) is either a center or a spiral point, but its stability is not determined by Theorem 2.

At $(-0.121, -0.469)$: The Jacobian matrix $\mathbf{J} \approx \begin{bmatrix} 3.044 & -0.340 \\ 11.593 & -3.364 \end{bmatrix}$ has real eigenvalues $\lambda_1 \approx -2.67$, $\lambda_2 \approx 2.35$ with different signs. Hence $(-0.121, -0.469)$ is a saddle point.

At $(0.126, 0.626)$: The Jacobian matrix $\mathbf{J} \approx \begin{bmatrix} 3.048 & 0.176 \\ 14.878 & -2.621 \end{bmatrix}$ has real eigenvalues $\lambda_1 \approx -3.05$, $\lambda_2 \approx 3.48$ with different signs. Hence $(0.126, 0.626)$ is a saddle point.

At $(5.132, -5.382)$: The Jacobian matrix $\mathbf{J} \approx \begin{bmatrix} 82.000 & 85.903 \\ -3.146 & 12.395 \end{bmatrix}$ has unequal positive eigenvalues $\lambda_1 \approx 16.52$, $\lambda_2 \approx 77.87$. Hence $(5.132, -5.382)$ is a nodal source.

The first three of these critical points are shown in the figure at the top of the next page.

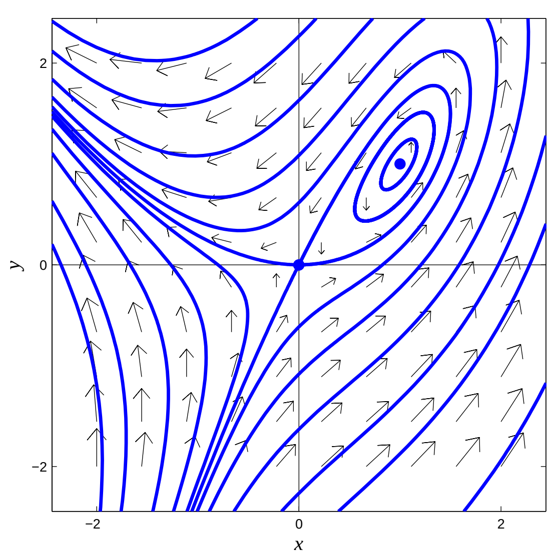


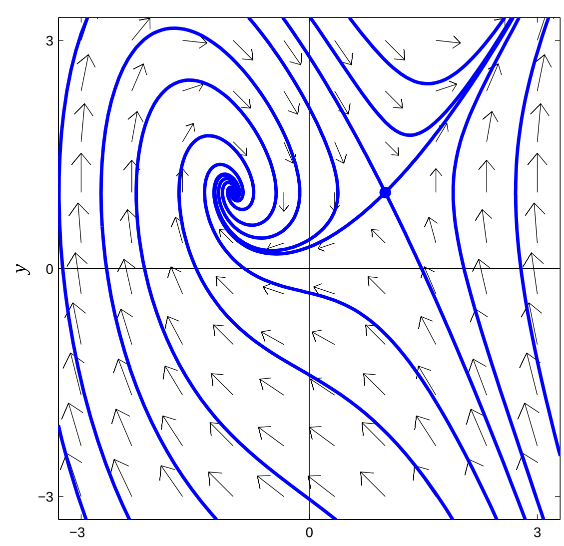
29. $\mathbf{J} = \begin{bmatrix} 1 & -1 \\ 2x & -1 \end{bmatrix}$

At (0,0): The Jacobian matrix $\mathbf{J} = \begin{bmatrix} 1 & -1 \\ 0 & -1 \end{bmatrix}$ has characteristic equation $\lambda^2 - 1 = 0$ and real eigenvalues $\lambda_1 \approx -1,\ \lambda_2 \approx +1$ with different signs. Hence (0,0) is a saddle point.

At $(1,1)$: The Jacobian matrix $\mathbf{J} = \begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix}$ has characteristic equation $\lambda^2 + 1 = 0$ and pure imaginary eigenvalues $\lambda_1 \approx +i,\ \lambda_2 \approx -i$. Hence $(1,1)$ is either a center or a spiral point, but its stability is not determined by Theorem 2.

The left-hand figure below suggests that $(1,1)$ is a stable center.





30. $\mathbf{J} = \begin{bmatrix} 0 & 1 \\ 2x & -1 \end{bmatrix}$

At $(1,1)$: The Jacobian matrix $\mathbf{J} = \begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix}$ has characteristic equation $\lambda^2 + \lambda - 2 = 0$ and real eigenvalues $\lambda_1 \approx -2$, $\lambda_2 \approx +1$ with different signs. Hence $(1,1)$ is a saddle point.

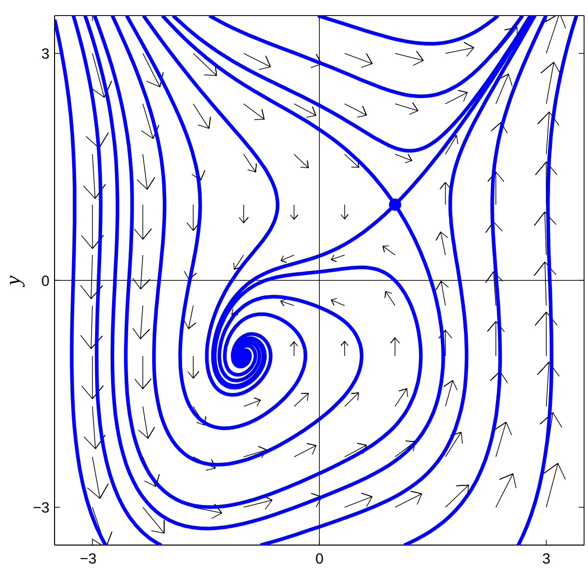
At $(1,-1)$: The Jacobian matrix $\mathbf{J} = \begin{bmatrix} 0 & 1 \\ -2 & -1 \end{bmatrix}$ has characteristic equation $\lambda^2 + \lambda + 2 = 0$ and complex conjugate eigenvalues $\lambda_1 \approx -0.5 + 1.323i$, $\lambda_2 \approx -0.5 - 1.323i$ with negative real part. Hence $(1,-1)$ is a spiral sink as in the right-hand figure on the preceding page.

31. $\mathbf{J} = \begin{bmatrix} 0 & 2y \\ 3x^2 & -1 \end{bmatrix}$

At $(1,1)$: The Jacobian matrix $\mathbf{J} = \begin{bmatrix} 0 & 2 \\ 3 & -1 \end{bmatrix}$ has characteristic equation $\lambda^2 + \lambda - 6 = 0$ and real eigenvalues $\lambda_1 = -3$, $\lambda_2 = +2$ with different signs. Hence $(1,1)$ is a saddle point.

At $(-1,-1)$: The Jacobian matrix $\mathbf{J} = \begin{bmatrix} 0 & -2 \\ 3 & -1 \end{bmatrix}$ has characteristic equation $\lambda^2 + \lambda + 6 = 0$ and complex conjugate eigenvalues $\lambda_1 \approx -0.5 + 2.398i$, $\lambda_2 \approx -0.5 - 2.398i$ with negative real part. Hence $(-1,-1)$ is a spiral sink.

These two critical points are shown in the figure below.

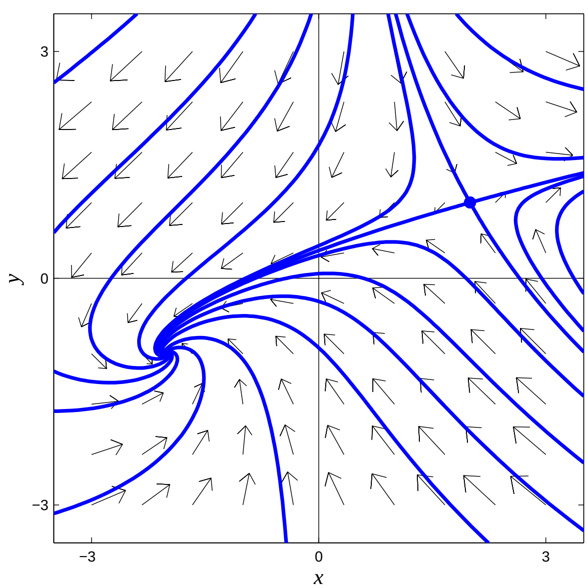


32. $\mathbf{J} = \begin{bmatrix} y & x \\ 1 & -2 \end{bmatrix}$

At $(2,1)$: The Jacobian matrix $\mathbf{J} = \begin{bmatrix} 1 & 2 \\ 1 & -2 \end{bmatrix}$ has characteristic equation $\lambda^2 + \lambda - 4 = 0$ and real eigenvalues $\lambda_1 \approx -2.56$, $\lambda_2 \approx +1.56$ with different signs. Hence $(1,1)$ is a saddle point.

At $(-2,-1)$: The Jacobian matrix $\mathbf{J} = \begin{bmatrix} -1 & -2 \\ 1 & -2 \end{bmatrix}$ has characteristic equation $\lambda^2 + 3\lambda + 4 = 0$ and complex conjugate eigenvalues $\lambda_1 \approx -1.5 + 1.323i$, $\lambda_2 \approx -1.5 - 1.323i$ with negative real part. Hence (–2,–1) is a spiral sink.

These two critical points are shown in the figure below.



33. The characteristic equation of the given linear system is

$$(\lambda - \varepsilon)^2 + 1 = 0$$

with characteristic roots $\lambda_1, \lambda_2 = \varepsilon \pm i$.

(a) So if $\varepsilon < 0$ then λ_1, λ_2 are complex conjugates with negative real part, and hence $(0, 0)$ is an asymptotically stable spiral point.

(b) If $\varepsilon = 0$ then $\lambda_1, \lambda_2 = \pm i$ (pure imaginary), so $(0,0)$ is a stable center.

(c) If $\varepsilon > 0$, the situation is the same as in (a) except that the real part is positive, so $(0, 0)$ is an unstable spiral point.

34. The characteristic equation of the given linear system is

$$(\lambda + 1)^2 - \varepsilon = 0.$$

(a) If $\varepsilon < 0$ then $\lambda_1, \lambda_2 = -1 \pm i\sqrt{-\varepsilon}$. Thus the characteristic roots are complex conjugates with negative real part, so it follows that $(0,0)$ is an asymptotically stable spiral point.

(b) If $\varepsilon = 0$ then the characteristic roots $\lambda_1 = \lambda_2 = -1$ are equal and negative, so $(0,0)$ is an asymptotically stable node. If $0 < \varepsilon < 1$ then $\lambda_1, \lambda_2 = -1 \pm \sqrt{\varepsilon}$ are both negative, so $(0,0)$ is an asymptotically stable improper node.

35. **(a)** If $h = 0$ we have the familiar system $x' = y$, $y' = -x$ with circular trajectories about the origin, which is therefore a center.

(b) The change to polar coordinates as in Example 6 of Section 9.1 is routine, yielding $r' = hr^3$ and $\theta' = -1$.

(c) If $h = -1$, then $r' = -r^3$ integrates to give $2r^2 = 1/(t + C)$ where C is a positive constant, so clearly $r \to 0$ as $t \to +\infty$, and thus the origin is a stable spiral point.

(d) If $h = +1$, then $r' = r^3$ integrates to give $2r^2 = -1/(t + C)$ where $C = -B$ is a positive constant. It follows that $2r^2 = 1/(B - t)$, so now r increases as t starts at 0 and increases.

36. **(a)** Again, the change of variables is essentially the same as in Example 6 of Section 9.1.

(b) If $\varepsilon = -a^2$ then the equation $r' = -r(a^2 + r^2)$ integrates to give the equation

$$t + C = -\frac{\ln r}{a^2} + \frac{\ln(a^2 + r^2)}{2a^2}$$

that (after exponentiating) we readily solve for

$$r^2 = \frac{a^2 \exp(-2ta^2 - 2Ca^2)}{1 - \exp(-2ta^2 - 2Ca^2)}.$$

This makes it clear that $r \to 0$ as $t \to +\infty$, so the origin is an asymptotically stable spiral point in this case.

(c) If $\varepsilon = a^2$ then the equation $r' = r(a^2 - r^2)$ integrates to give the equation

$$t+C = \frac{2\ln r - \ln(a-r) - \ln(a+r)}{2a^2}$$

that (after exponentiating) we solve for

$$r^2 = \frac{a^2}{1+\exp(-2ta^2 - 2Ca^2)}.$$

It therefore follows that $r \to a$ as $t \to +\infty$.

37. The substitution $y = vx$ in the homogeneous first-order equation

$$\frac{dy}{dx} = \frac{y(2x^3 - y^3)}{x(x^3 - 2y^3)}$$

yields

$$x\frac{dv}{dx} = -\frac{v^4 + v}{2v^3 - 1}.$$

Separating the variables and integrating by partial fractions, we get

$$\int\left(-\frac{1}{v} + \frac{1}{v+1} + \frac{2v-1}{v^2 - v + 1}\right)dv = -\int\frac{dx}{x}$$

$$\ln((v+1)(v^2 - v + 1)) = \ln v - \ln x + \ln C$$

$$(v+1)(v^2 - v + 1) = \frac{Cv}{x}$$

$$v^3 + 1 = \frac{Cv}{x}.$$

Finally, the replacement $v = y/x$ yields $x^3 + y^3 = Cxy$.

38. The roots of the characteristic equation $\lambda^2 - T\lambda + D = 0$ are given by

$$\lambda_1, \lambda_2 = \frac{T \pm \sqrt{T^2 - 4D}}{2}.$$

We examine the various possibilities individually.

- If the point (T, D) lies above the parabola $T^2 = 4D$ in the trace-determinant plane but off the D-axis, so the radicand $T^2 - 4D$ is negative, then λ_1 and λ_2 have nonzero imaginary part and nonzero real part $T/2$. Hence we have a spiral source if $T > 0$, a spiral sink if $T < 0$.

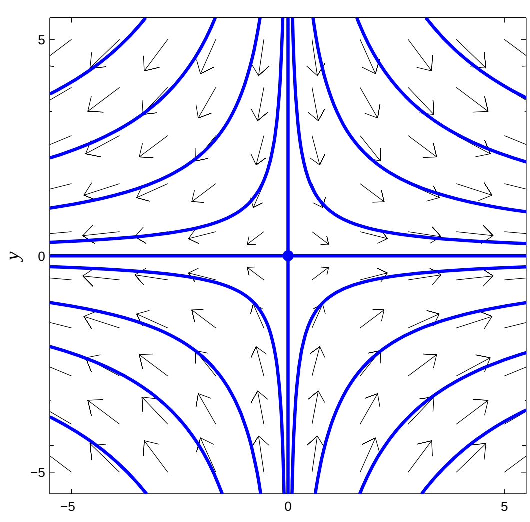
- If the point (T, D) lies on the positive D-axis, so $T = 0$ but $D > 0$, then $\lambda_1 = \lambda_2 = i\sqrt{D}$, pure imaginary, so we have a stable center.
- If the point (T, D) lies beneath the T-axis, then $\lambda_1, \lambda_2 = \frac{1}{2}\left(T \pm \sqrt{T^2 + 4|D|}\right)$ because $D < 0$. It follows that λ_1 and λ_2 are real with different signs, so we have a saddle point.
- If the point (T, D) lies between the T-axis and the parabola $T^2 = 4D$, then the radicand $T^2 - 4D$ is positive but less than T^2. It follows that λ_1 and λ_2 are real and both have the same sign as T, so we have a nodal source if $T > 0$, a nodal sink if $T < 0$.

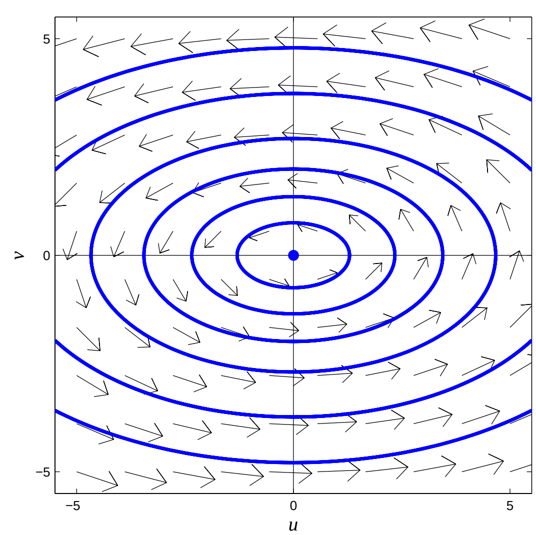
SECTION 9.3

ECOLOGICAL APPLICATIONS: PREDATORS AND COMPETITORS

1. $\mathbf{J} = \begin{bmatrix} 200 - 4y & -4x \\ 2y & -150 + 2x \end{bmatrix}$

At $(0,0)$: The Jacobian matrix $\mathbf{J} = \begin{bmatrix} 200 & 0 \\ 0 & -150 \end{bmatrix}$ has characteristic equation $(200 - \lambda)(-150 - \lambda) = 0$ and real eigenvalues $\lambda_1 = -150$, $\lambda_2 = 200$ with different signs. Hence $(0,0)$ is a saddle point of the linearized system $x' = 200x$, $y' = -150y$. See the left-hand figure below.





At $(75,50)$: The Jacobian matrix $\mathbf{J} = \begin{bmatrix} 0 & -300 \\ 100 & 0 \end{bmatrix}$ has characteristic equation $\lambda^2 + 30000 = 0$ and pure imaginary eigenvalues $\lambda_1, \lambda_2 = \pm 100i\sqrt{3}$. Hence $(75,50)$ is a stable center of the linearization $u' = -300v,\ v' = 100u$. See the right-hand figure at the bottom of the preceding page.

2. Upon separation of variables, the equation

$$\frac{dy}{dx} = \frac{-150y + 2xy}{200x - 4xy} = \frac{y(-150 + 2x)}{x(200 - 4y)}$$

yields

$$\int \left(\frac{200}{y} - 4 \right) dy = \int \left(2 - \frac{150}{x} \right) dx,$$

$$200 \ln y - 4y = 2x - 150 \ln x + C$$

assuming that $x, y > 0$.

3. The effect of using the insecticide is to replace b by $b + f$ and a by $a - f$ in the predator-prey equations, while leaving p and q unchanged. Hence the new harmful population is $(b + f)/q > b/q = x_E$, and the new benign population is $(a - f)/p < a/p = y_E$.

Problems 4–7 deal with the competition system

$$x' = 60x - 4x^2 - 3xy, \qquad y' = 42y - 2y^2 - 3xy \tag{2}$$

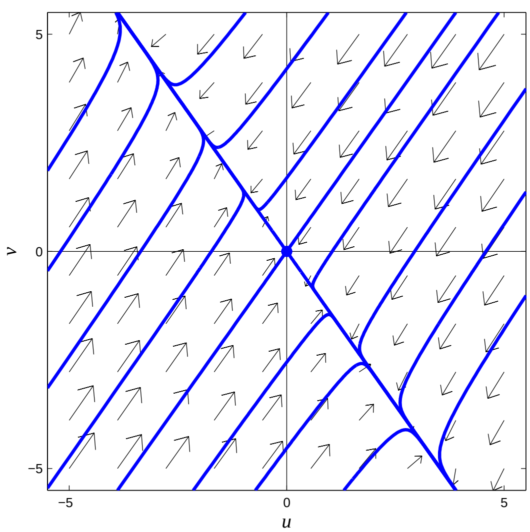
that has Jacobian matrix $\mathbf{J} = \begin{bmatrix} 60 - 8x - 3y & -3x \\ -3y & 42 - 4y - 3x \end{bmatrix}$.

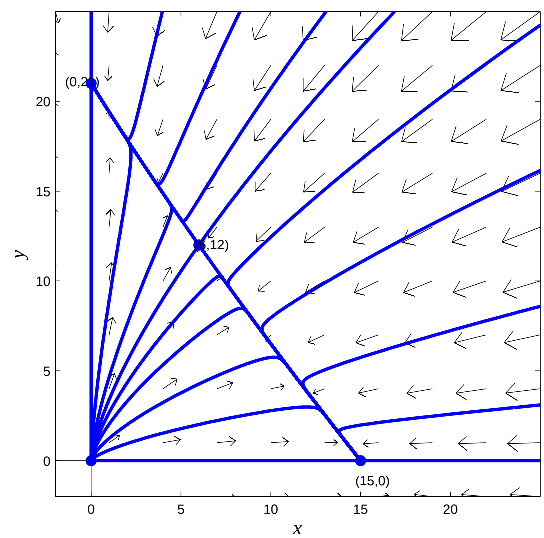
4. At $(0,0)$ the Jacobian matrix $\mathbf{J} = \begin{bmatrix} 60 & 0 \\ 0 & 42 \end{bmatrix}$ has characteristic equation $(60 - \lambda)(42 - \lambda) = 0$ and positive real eigenvalues $\lambda_1 = 42,\ \lambda_2 = 60$. Hence $(0,0)$ is a nodal source of the linearized system $x' = 60x,\ y' = 42y$.

5. At $(0,21)$ the Jacobian matrix $\mathbf{J} = \begin{bmatrix} -3 & 0 \\ -63 & -42 \end{bmatrix}$ has characteristic equation $(-3 - \lambda)(-42 - \lambda) = 0$ and negative real eigenvalues $\lambda_1 = -42,\ \lambda_2 = -3$. Hence $(0,21)$ is a nodal sink of the linearized system $u' = -3u,\ v' = -63u - 42v$.

6. At $(15,0)$ the Jacobian matrix $\mathbf{J} = \begin{bmatrix} -60 & -45 \\ 0 & -3 \end{bmatrix}$ has characteristic equation $(-60-\lambda)(-3-\lambda) = 0$ and negative real eigenvalues $\lambda_1 = -60,\ \lambda_2 = -3$. Hence $(15,0)$ is a nodal sink of the linearized system $u' = -60u - 45v,\ v' = -3v$.

7. At $(6,12)$ the Jacobian matrix $\mathbf{J} = \begin{bmatrix} -24 & -18 \\ -36 & -24 \end{bmatrix}$ has characteristic equation $(-24-\lambda)^2 - (-36)(-18) = 0$ and real eigenvalues $\lambda_1 = -24 + 18\sqrt{2} > 0$, $\lambda_2 = -24 - 18\sqrt{2} < 0$ with different signs. Hence $(6,12)$ is a saddle point of the linearized system $u' = -24u - 18v,\quad v' = -36u - 24v$. The figure on the left below illustrates this saddle point. The figure on the right shows all four critical points of the system.





Problems 8–10 deal with the competition system

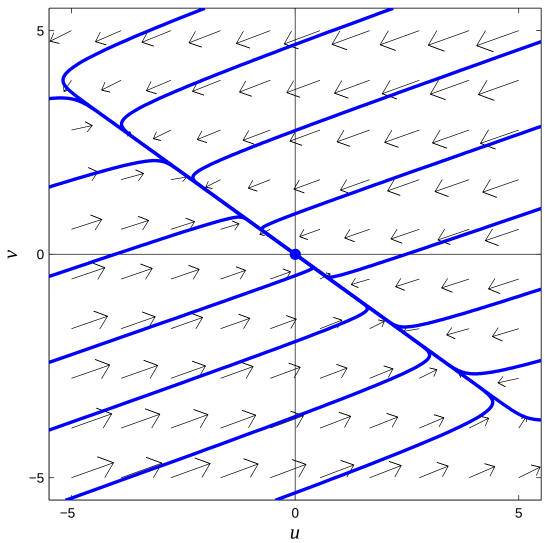
$$x' = 60x - 3x^2 - 4xy, \quad y' = 42y - 3y^2 - 2xy \tag{3}$$

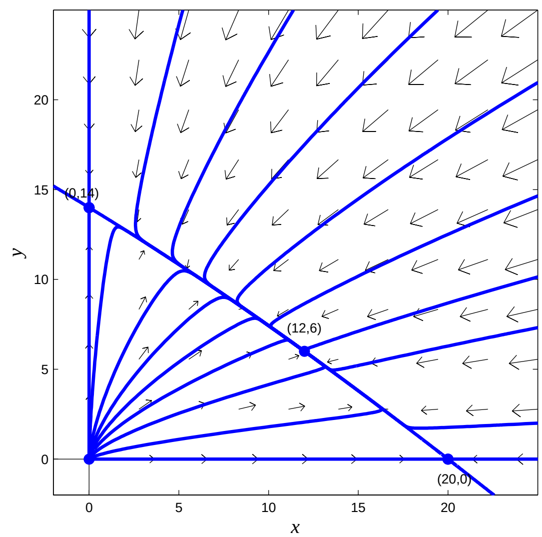
that has Jacobian matrix $\mathbf{J} = \begin{bmatrix} 60 - 6x - 4y & -4x \\ -2y & 42 - 6y - 2x \end{bmatrix}$.

8. At $(0,14)$ the Jacobian matrix $\mathbf{J} = \begin{bmatrix} 4 & 0 \\ -28 & -42 \end{bmatrix}$ has characteristic equation $(4-\lambda)(-42-\lambda) = 0$ and real eigenvalues $\lambda_1 = -42,\ \lambda_2 = 4$ with different signs. Hence $(0,14)$ is a saddle point of the linearized system $u' = 4u,\ v' = -28u - 42v$.

9. At $(20,0)$ the Jacobian matrix $\mathbf{J} = \begin{bmatrix} -60 & -80 \\ 0 & 2 \end{bmatrix}$ has characteristic equation $(-60-\lambda)(2-\lambda)=0$ and real eigenvalues $\lambda_1 = -60,\ \lambda_2 = 2$ with different signs. Hence $(20,0)$ is a saddle point of the linearized system $u' = -60u - 80v,\ v' = 2v$.

10. At $(12,6)$ the Jacobian matrix $\mathbf{J} = \begin{bmatrix} -36 & -48 \\ -12 & -18 \end{bmatrix}$ has characteristic equation $(-36-\lambda)(-18-\lambda)-(-12)(-48)=0$ and negative real eigenvalues $\lambda_1, \lambda_2 = -27 \pm 3\sqrt{73}$. Hence $(12,6)$ is a nodal sink of the linearized system $u' = -36u - 48v,\ v' = -12u - 18v$. The figure on the left below illustrates this sink. The figure on the right shows all four critical points of the system.





Problems 11–13 deal with the predator-prey system

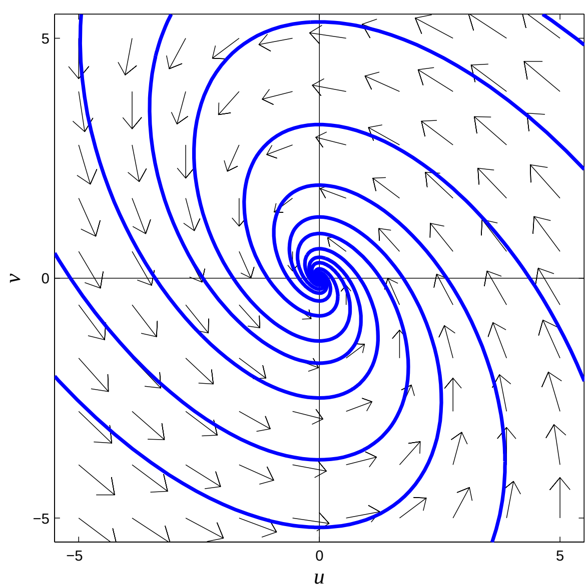
$$x' = 5x - x^2 - xy, \qquad y' = -2y + xy \tag{4}$$

that has Jacobian matrix $\mathbf{J} = \begin{bmatrix} 5-2x-y & -x \\ y & -2+x \end{bmatrix}$.

11. At $(0,0)$ the Jacobian matrix $\mathbf{J} = \begin{bmatrix} 5 & 0 \\ 0 & -2 \end{bmatrix}$ has characteristic equation $(5-\lambda)(-2-\lambda)=0$ and real eigenvalues $\lambda_1 = -2,\ \lambda_2 = 5$ with different signs. Hence $(0,0)$ is a saddle point of the linearized system $x' = 5x,\ y' = -2y$.

12. At $(5,0)$ the Jacobian matrix $\mathbf{J} = \begin{bmatrix} -5 & -5 \\ 0 & 3 \end{bmatrix}$ has characteristic equation $(-5-\lambda)(3-\lambda) = 0$ and real eigenvalues $\lambda_1 = -5,\ \lambda_2 = 3$ with different signs. Hence $(5,0)$ is a saddle point of the linearized system $u' = -5u - 5v,\ v' = 3v$.

13. At $(2,3)$ the Jacobian matrix $\mathbf{J} = \begin{bmatrix} -2 & -2 \\ 3 & 0 \end{bmatrix}$ has characteristic equation $(-2-\lambda)(-\lambda) - (3)(-2) = \lambda^2 + 2\lambda + 6 = 0$ and complex conjugate eigenvalues $\lambda_1, \lambda_2 = -1 \pm i\sqrt{5}$ with negative real part. Hence $(2,3)$ is a spiral sink of the linearized system $u' = -2u - 2v,\ v' = 3u$ (illustrated below).



Problems 14–17 deal with the predator-prey system

$$x' = x^2 - 2x - xy, \quad y' = y^2 - 4y + xy \tag{5}$$

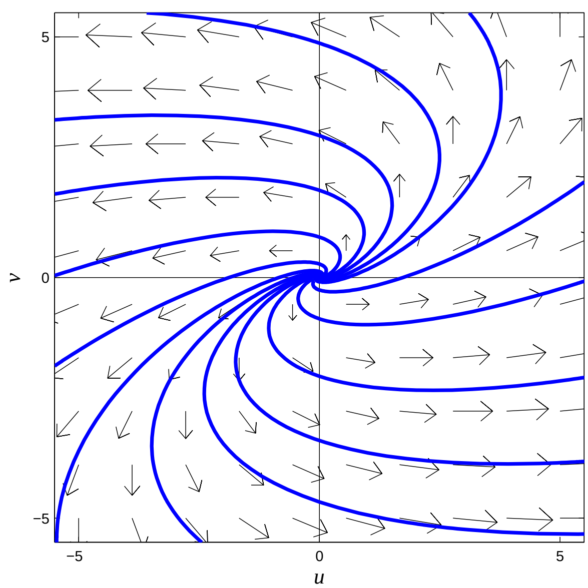
that has Jacobian matrix $\mathbf{J} = \begin{bmatrix} 2x-2-y & -x \\ y & 2y-4+x \end{bmatrix}$.

14. At $(0,0)$ the Jacobian matrix $\mathbf{J} = \begin{bmatrix} -2 & 0 \\ 0 & -4 \end{bmatrix}$ has characteristic equation $(-2-\lambda)(-4-\lambda) = 0$ and negative real eigenvalues $\lambda_1 = -4,\ \lambda_2 = -2$. Hence $(0,0)$ is a nodal sink of the linearized system $x' = -2x,\ y' = -4y$.

15. At $(0,4)$ the Jacobian matrix $\mathbf{J} = \begin{bmatrix} -6 & 0 \\ 4 & 4 \end{bmatrix}$ has characteristic equation $(-6-\lambda)(4-\lambda) = 0$ and real eigenvalues $\lambda_1 = -6,\ \lambda_2 = 4$ with different signs. Hence $(0,4)$ is a saddle point of the linearized system $u' = -6u,\ v' = 4u + 4v$.

16. At $(2,0)$ the Jacobian matrix $\mathbf{J} = \begin{bmatrix} 2 & -2 \\ 0 & -2 \end{bmatrix}$ has characteristic equation $(2-\lambda)(-2-\lambda) = 0$ and real eigenvalues $\lambda_1 = -2,\ \lambda_2 = 2$ with different signs. Hence $(2,0)$ is a saddle point of the linearized system $u' = 2u - 2v,\ v' = -2v$.

17. At $(3,1)$ the Jacobian matrix $\mathbf{J} = \begin{bmatrix} 3 & -3 \\ 1 & 1 \end{bmatrix}$ has characteristic equation $(3-\lambda)(1-\lambda) - (1)(-3) = \lambda^2 - 4\lambda + 6 = 0$ and complex conjugate eigenvalues $\lambda_1, \lambda_2 = 2 \pm i\sqrt{2}$ with positive real part. Hence $(3,1)$ is a spiral source of the linearized system $u' = 3u - 3v,\ \ v' = u + v$ (illustrated below).



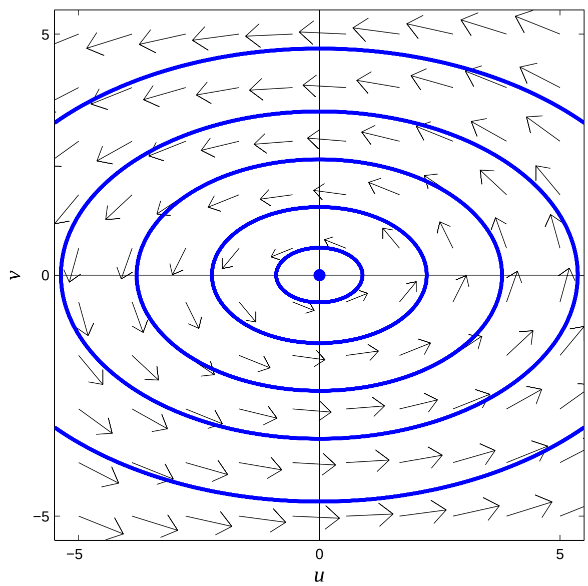
Problems 18 and 19 deal with the predator-prey system

$$x' = 2x - xy, \qquad y' = -5y + xy \tag{7}$$

that has Jacobian matrix $\mathbf{J} = \begin{bmatrix} 2-y & -x \\ y & -5+x \end{bmatrix}$.

18. At $(0,0)$ the Jacobian matrix $\mathbf{J} = \begin{bmatrix} 2 & 0 \\ 0 & -5 \end{bmatrix}$ has characteristic equation $(2-\lambda)(-5-\lambda) = 0$ and real eigenvalues $\lambda_1 = -5,\ \lambda_2 = 2$ with different signs. Hence $(0,0)$ is a saddle point of the linearized system $x' = 2x,\ y' = -5y$.

19. At $(5,2)$ the Jacobian matrix $\mathbf{J} = \begin{bmatrix} 0 & -5 \\ 2 & 0 \end{bmatrix}$ has characteristic equation $(-\lambda)(-\lambda) - (2)(-5) = \lambda^2 + 10 = 0$ and pure imaginary roots $\lambda = \pm i\sqrt{10}$, so the origin is a stable center for the linearized system $u' = -5v,\ v' = 2u$. This is the indeterminate case, but the figure below suggests that $(5,2)$ is also a stable center for the original system in (7).



Problems 20–22 deal with the predator-prey system

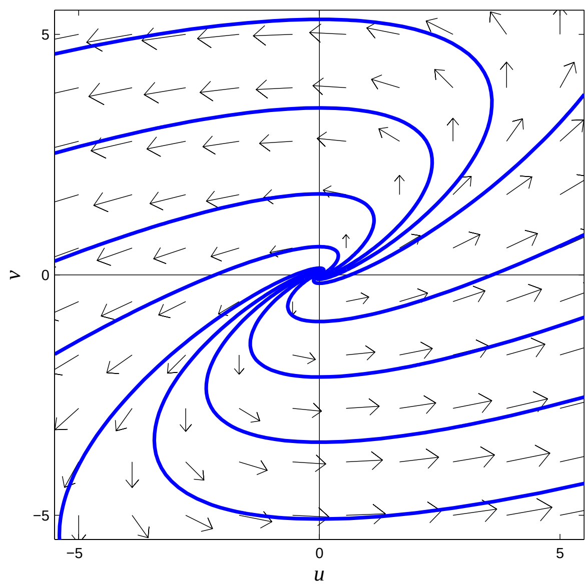
$$x' = -3x + x^2 - xy, \qquad y' = -5y + xy \tag{8}$$

that has Jacobian matrix $\mathbf{J} = \begin{bmatrix} -3+2x-y & -x \\ y & -5+x \end{bmatrix}$.

20. At $(0,0)$ the Jacobian matrix $\mathbf{J} = \begin{bmatrix} -3 & 0 \\ 0 & -5 \end{bmatrix}$ has characteristic equation $(-3-\lambda)(-5-\lambda) = 0$ and negative real eigenvalues $\lambda_1 = -5,\ \lambda_2 = -3$. Hence $(0,0)$ is a nodal sink of the linearized system $x' = -3x,\ y' = -5y$.

21. At $(3,0)$ the Jacobian matrix $\mathbf{J} = \begin{bmatrix} 3 & -3 \\ 0 & -2 \end{bmatrix}$ has characteristic equation $(3-\lambda)(-2-\lambda) = 0$ and real eigenvalues $\lambda_1 = -2,\ \lambda_2 = 3$ with different signs. Hence $(3,0)$ is a saddle point of the linearized system $u' = 3u - 3v,\ v' = -2v$.

22. At $(5,2)$ the Jacobian matrix $\mathbf{J} = \begin{bmatrix} 5 & -5 \\ 2 & 0 \end{bmatrix}$ has characteristic equation $(5-\lambda)(-\lambda) - (2)(-5) = \lambda^2 - 5\lambda + 10 = 0$ and complex conjugate eigenvalues $\lambda_1, \lambda_2 = \frac{1}{2}\left(5 \pm i\sqrt{15}\right)$ with positive real part. Hence $(5,2)$ is a spiral source of the linearized system $u' = 5u - 5v,\ v' = 2u$ (illustrated below).



Problems 23–25 deal with the predator-prey system

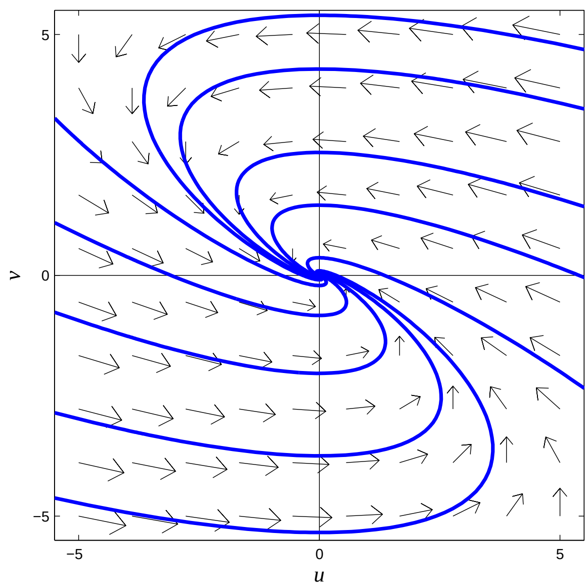
$$x' = 7x - x^2 - xy, \qquad y' = -5y + xy \tag{9}$$

that has Jacobian matrix $\mathbf{J} = \begin{bmatrix} 7-2x-y & -x \\ y & -5+x \end{bmatrix}$.

23. At $(0,0)$ the Jacobian matrix $\mathbf{J} = \begin{bmatrix} 7 & 0 \\ 0 & -5 \end{bmatrix}$ has characteristic equation $(7-\lambda)(-5-\lambda) = 0$ and real eigenvalues $\lambda_1 = -5,\ \lambda_2 = 7$ with different signs. Hence $(0,0)$ is a saddle point of the linearized system $x' = 7x,\ y' = -5y$.

24. At $(7,0)$ the Jacobian matrix $\mathbf{J} = \begin{bmatrix} -7 & -7 \\ 0 & 2 \end{bmatrix}$ has characteristic equation $(-7-\lambda)(2-\lambda) = 0$ and real eigenvalues $\lambda_1 = -7$, $\lambda_2 = 2$ with different signs. Hence $(7,0)$ is a saddle point of the linearized system $u' = -7u - 7v$, $v' = 2v$.

25. At $(5,2)$ the Jacobian matrix $\mathbf{J} = \begin{bmatrix} -5 & -5 \\ 2 & 0 \end{bmatrix}$ has characteristic equation $(-5-\lambda)(-\lambda) - (2)(-5) = \lambda^2 + 5\lambda + 10 = 0$ and complex conjugate eigenvalues $\lambda_1, \lambda_2 = \frac{1}{2}\left(-5 \pm i\sqrt{15}\right)$ with negative real part. Hence $(5,2)$ is a spiral sink of the linearized system $u' = -5u - 5v$, $v' = 2u$ (illustrated below).

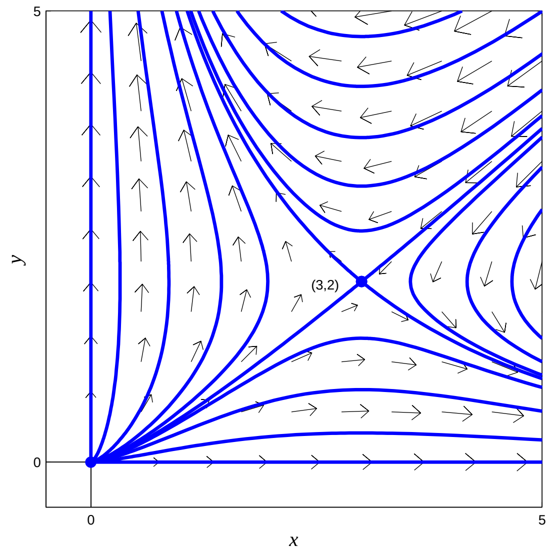


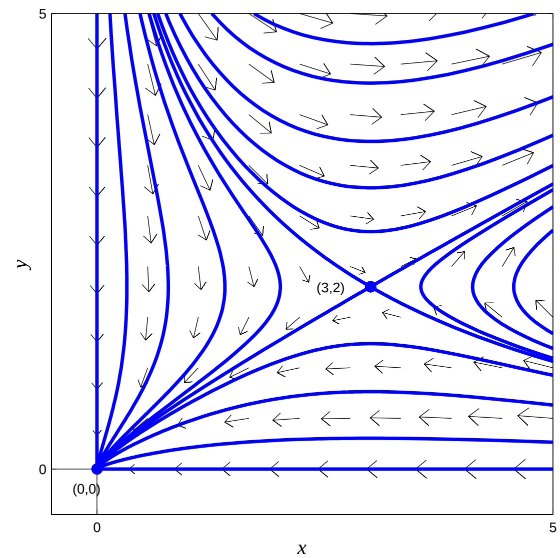
26. $\mathbf{J} = \begin{bmatrix} 2-y & -x \\ -y & 3-x \end{bmatrix}$

At $(0,0)$: The Jacobian matrix $\mathbf{J} = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$ has characteristic equation $\lambda^2 - 5\lambda + 6 = 0$ and positive real eigenvalues $\lambda_1 = 2$, $\lambda_2 = 3$. Hence $(0,0)$ is a nodal source.

At $(3,2)$: The Jacobian matrix $\mathbf{J} = \begin{bmatrix} 0 & -3 \\ -2 & 0 \end{bmatrix}$ has characteristic equation $\lambda^2 - 6 = 0$ and real eigenvalues $\lambda_1, \lambda_2 = \pm\sqrt{6}$ with different signs. Hence $(3,2)$ is a saddle point.

If the initial point (x_0, y_0) lies above the southwest-northeast separatrix through $(3,2)$, then $\big(x(t), y(t)\big) \to (0,\infty)$ as $t \to \infty$. But if (x_0, y_0) lies below this separatrix, then $\big(x(t), y(t)\big) \to (\infty, 0)$ as $t \to \infty$. See the left-hand figure below.





27. $\mathbf{J} = \begin{bmatrix} 2y-4 & 2x \\ y & x-3 \end{bmatrix}$

At $(0,0)$: The Jacobian matrix $\mathbf{J} = \begin{bmatrix} -4 & 0 \\ 0 & -3 \end{bmatrix}$ has characteristic equation $\lambda^2 + 7\lambda + 12 = 0$ and negative real eigenvalues $\lambda_1 = -4,\ \lambda_2 = -3$. Hence $(0,0)$ is a nodal sink.

At $(3,2)$: The Jacobian matrix $\mathbf{J} = \begin{bmatrix} 0 & 6 \\ 2 & 0 \end{bmatrix}$ has characteristic equation $\lambda^2 - 12 = 0$ and real eigenvalues $\lambda_1, \lambda_2 = \pm 2\sqrt{3}$ with different signs. Hence $(3,2)$ is a saddle point.

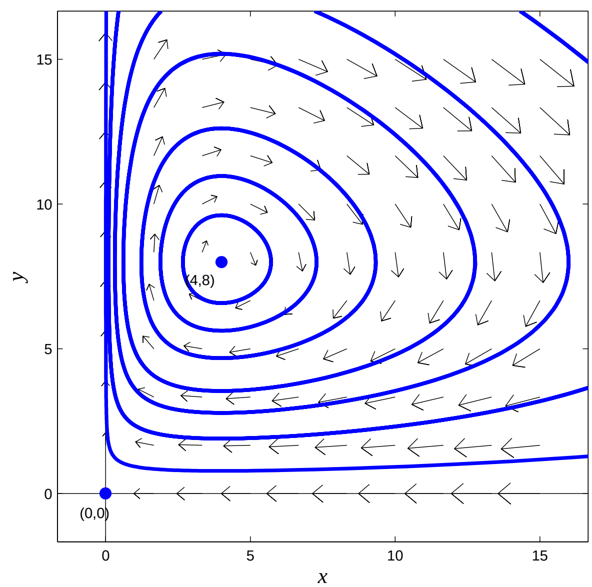
If the initial point (x_0, y_0) lies below the northwest-southeast separatrix through $(3,2)$, then $\big(x(t), y(t)\big) \to (0,0)$ as $t \to \infty$. But if (x_0, y_0) lies above this separatrix, then $\big(x(t), y(t)\big) \to (\infty,\infty)$ as $t \to \infty$. See the right-hand figure above.

28. $\mathbf{J} = \begin{bmatrix} 2y-16 & 2x \\ -y & 4-x \end{bmatrix}$

At $(0,0)$: The Jacobian matrix $\mathbf{J} = \begin{bmatrix} -16 & 0 \\ 0 & 4 \end{bmatrix}$ has characteristic equation $\lambda^2 + 12\lambda - 64 = 0$ and real eigenvalues $\lambda_1 = -16,\ \lambda_2 = 4$ with opposite signs. Hence $(0,0)$ is a saddle point.

At $(4,8)$: The Jacobian matrix $\mathbf{J} = \begin{bmatrix} 0 & 8 \\ -8 & 0 \end{bmatrix}$ has characteristic equation $\lambda^2 + 64 = 0$ and conjugate imaginary eigenvalues $\lambda_1, \lambda_2 = \pm 8i$. This is the indeterminate case, but the figure in the answers section of the textbook indicates that $(4,8)$ is a stable center for the original nonlinear system.

As $t \to \infty$, each solution point $(x(t), y(t))$ with nonzero initial conditions encircles the stable center $(4,8)$ periodically in a clockwise direction. See the figure below.



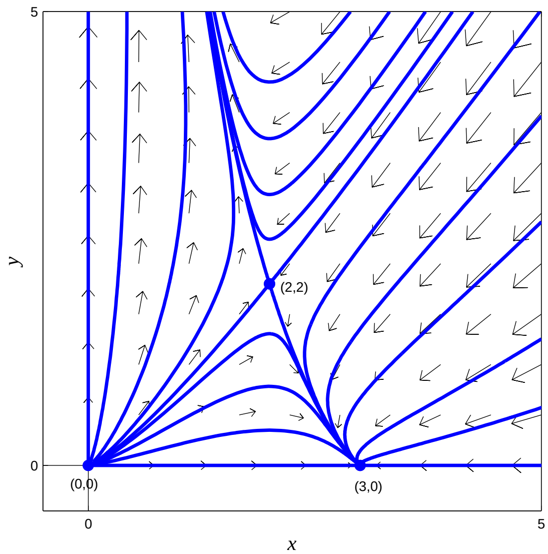
29. $\mathbf{J} = \begin{bmatrix} -2x - \frac{1}{2}y + 3 & -\frac{1}{2}x \\ -2y & 4 - 2x \end{bmatrix}$

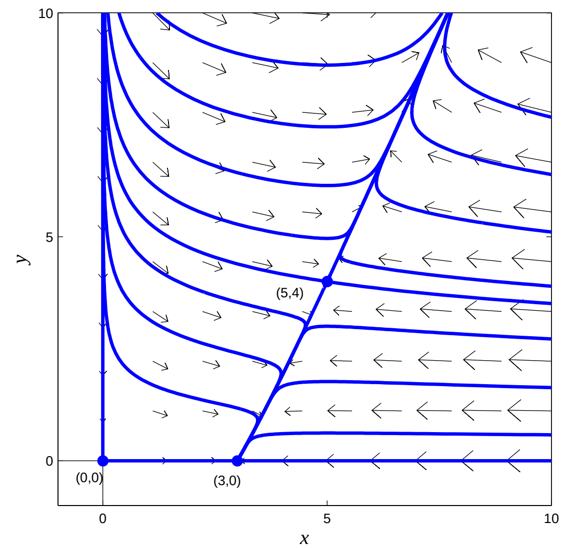
At $(0,0)$: The Jacobian matrix $\mathbf{J} = \begin{bmatrix} 3 & 0 \\ 0 & 4 \end{bmatrix}$ has characteristic equation $\lambda^2 - 7\lambda + 12 = 0$ and positive real eigenvalues $\lambda_1 = 3,\ \lambda_2 = 4$. Hence $(0,0)$ is a nodal source.

At $(3,0)$: The Jacobian matrix $\mathbf{J} = \begin{bmatrix} -3 & -\frac{3}{2} \\ 0 & -2 \end{bmatrix}$ has characteristic equation $\lambda^2 + 5\lambda + 6 = 0$ and negative real eigenvalues $\lambda_1 = -3,\ \lambda_2 = -2$. Hence $(3,0)$ is a nodal sink.

At $(2,2)$: The Jacobian matrix $\mathbf{J} = \begin{bmatrix} -2 & -1 \\ -4 & 0 \end{bmatrix}$ has characteristic equation $\lambda^2 + 2\lambda - 4 = 0$ and real eigenvalues $\lambda_1 \approx -3.2361, \lambda_2 = 1.2361$ with different signs. Hence $(2,2)$ is a saddle point.

If the initial point (x_0, y_0) lies above the southwest-northeast separatrix through $(2,2)$, then $(x(t), y(t)) \to (0,\infty)$ as $t \to \infty$. But if (x_0, y_0) lies below this separatrix, then $(x(t), y(t)) \to (3,0)$ as $t \to \infty$. See the left-hand figure below.





30. $\mathbf{J} = \begin{bmatrix} -2x + \frac{1}{2}y + 3 & \frac{1}{2}x \\ \frac{1}{5}y & \frac{1}{5}x - 1 \end{bmatrix}$

At $(0,0)$: The Jacobian matrix $\mathbf{J} = \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix}$ has characteristic equation $\lambda^2 - 2\lambda - 3 = 0$ and real eigenvalues $\lambda_1 = -1,\ \lambda_2 = 3$ of opposite sign. Hence $(0,0)$ is a saddle point.

At $(3,0)$: The Jacobian matrix $\mathbf{J} = \begin{bmatrix} -3 & -\frac{3}{2} \\ 0 & -2 \end{bmatrix}$ has characteristic equation $\lambda^2 + \frac{17}{5}\lambda + \frac{17}{5} = 0$ and negative real eigenvalues $\lambda_1 = -3,\ \lambda_2 = -\frac{17}{5}$. Hence $(3,0)$ is a nodal sink.

At $(5,4)$: The Jacobian matrix $\mathbf{J} = \begin{bmatrix} -5 & \frac{5}{2} \\ \frac{4}{5} & 0 \end{bmatrix}$ has characteristic equation $\lambda^2 + 5\lambda - 2 = 0$ and real eigenvalues $\lambda_1 \approx -5.3723, \lambda_2 = 0.3723$ with different signs. Hence $(5,4)$ is a saddle point.

If the initial point (x_0, y_0) lies above the northwest-southeast separatrix through $(5,4)$, then $(x(t), y(t)) \to (\infty,\infty)$ as $t \to \infty$. But if (x_0, y_0) lies below this separatrix, then $(x(t), y(t)) \to (3,0)$ as $t \to \infty$. See the right-hand figure above.

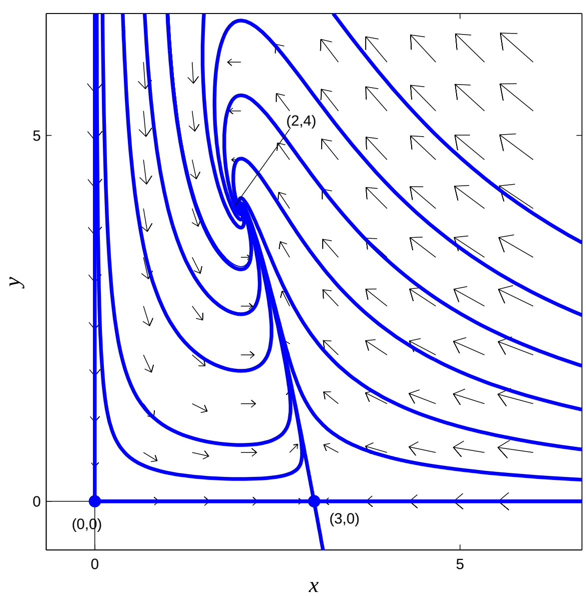
31. $\mathbf{J} = \begin{bmatrix} -2x - \frac{1}{4}y + 3 & -\frac{1}{4}x \\ y & x - 2 \end{bmatrix}$

At $(0,0)$: The Jacobian matrix $\mathbf{J} = \begin{bmatrix} 3 & 0 \\ 0 & -2 \end{bmatrix}$ has characteristic equation $\lambda^2 - \lambda - 6 = 0$ and real eigenvalues $\lambda_1 = -2,\ \lambda_2 = 3$ of opposite sign. Hence $(0,0)$ is a saddle point.

At $(3,0)$: The Jacobian matrix $\mathbf{J} = \begin{bmatrix} -3 & -\frac{3}{4} \\ 0 & 1 \end{bmatrix}$ has characteristic equation $\lambda^2 + 2\lambda - 3 = 0$ and real eigenvalues $\lambda_1 = -3,\ \lambda_2 = 1$ of opposite sign. Hence $(3,0)$ is a saddle point.

At $(2,4)$: The Jacobian matrix $\mathbf{J} = \begin{bmatrix} -2 & -\frac{1}{2} \\ 4 & 0 \end{bmatrix}$ has characteristic equation $\lambda^2 + 2\lambda + 2 = 0$ and complex conjugate eigenvalues $\lambda_1, \lambda_2 = -1 \pm i$ with negative real part. Hence $(2,4)$ is a spiral sink.

As $t \to \infty$, each solution point $\big(x(t), y(t)\big)$ with nonzero initial conditions approaches the spiral sink $(2,4)$, as indicated by the direction arrows in the figure below.



32. $\mathbf{J} = \begin{bmatrix} -6x + y + 30 & x \\ 4y & 4x - 6y + 60 \end{bmatrix}$

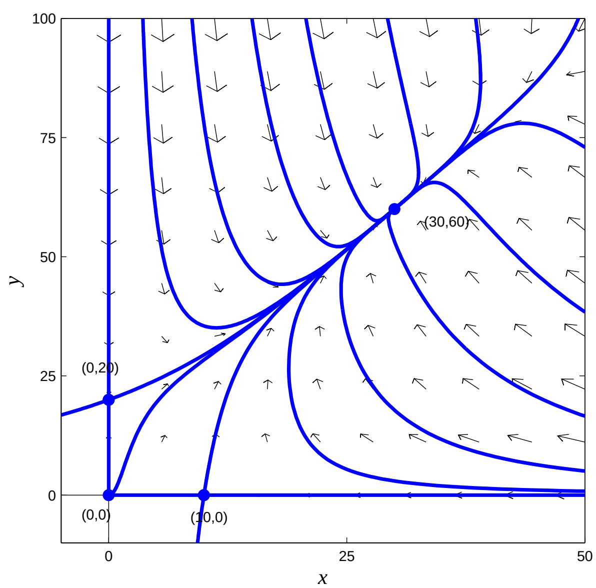
At $(0,0)$: The Jacobian matrix $\mathbf{J} = \begin{bmatrix} 30 & 0 \\ 0 & 60 \end{bmatrix}$ has characteristic equation $\lambda^2 - 90\lambda + 1800 = 0$ and positive real eigenvalues $\lambda_1 = 30,\ \lambda_2 = 60$. Hence $(0,0)$ is a nodal source.

At $(0,20)$: The Jacobian matrix $\mathbf{J}=\begin{bmatrix}50 & 0\\ 80 & -60\end{bmatrix}$ has characteristic equation $\lambda^2+10\lambda-3000=0$ and real eigenvalues $\lambda_1=-60,\ \lambda_2=50$ of opposite sign. Hence $(0,20)$ is a saddle point.

At $(10,0)$: The Jacobian matrix $\mathbf{J}=\begin{bmatrix}-30 & 10\\ 0 & 100\end{bmatrix}$ has characteristic equation $\lambda^2-70\lambda-3000=0$ and real eigenvalues $\lambda_1=-30,\ \lambda_2=100$ of opposite sign. Hence $(10,0)$ is a saddle point.

At $(30,60)$: The Jacobian matrix $\mathbf{J}=\begin{bmatrix}-90 & 30\\ 240 & -180\end{bmatrix}$ has characteristic equation $\lambda^2+240\lambda+9000=0$ and negative real eigenvalues $\lambda_1\approx-231.05, \lambda_2=-38.95$. Hence $(30,60)$ is a nodal sink.

As $t\to\infty$, each solution point $\left(x(t),y\left(t\right)\right)$ with nonzero initial conditions approaches the nodal sink $(30,60)$. See the figure below.



33. $\mathbf{J}=\begin{bmatrix}-6x+y+30 & x\\ 4y & 4x-6y+60\end{bmatrix}$

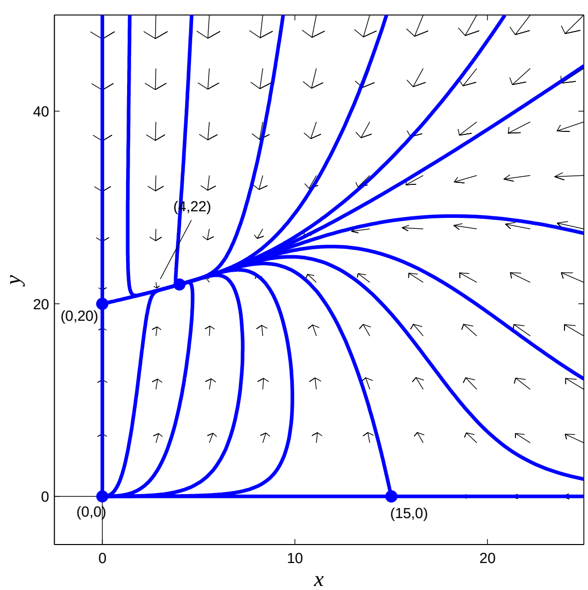
At $(0,0)$: The Jacobian matrix $\mathbf{J}=\begin{bmatrix}30 & 0\\ 0 & 80\end{bmatrix}$ has characteristic equation $\lambda^2-110\lambda+2400=0$ and positive real eigenvalues $\lambda_1=30,\ \lambda_2=80$. Hence $(0,0)$ is a nodal source.

At $(0,20)$: The Jacobian matrix $\mathbf{J} = \begin{bmatrix} 10 & 0 \\ 40 & -80 \end{bmatrix}$ has characteristic equation $\lambda^2 + 70\lambda - 800 = 0$ and real eigenvalues $\lambda_1 = -80,\ \lambda_2 = 10$ of opposite sign. Hence $(0,20)$ is a saddle point.

At $(15,0)$: The Jacobian matrix $\mathbf{J} = \begin{bmatrix} -30 & 15 \\ 0 & 110 \end{bmatrix}$ has characteristic equation $\lambda^2 - 80\lambda - 3300 = 0$ and real eigenvalues $\lambda_1 = -30,\ \lambda_2 = 110$ of opposite sign. Hence $(15,0)$ is a saddle point.

At $(4,22)$: The Jacobian matrix $\mathbf{J} = \begin{bmatrix} -8 & -4 \\ 44 & -88 \end{bmatrix}$ has characteristic equation $\lambda^2 + 96\lambda + 880 = 0$ and negative real eigenvalues $\lambda_1 \approx -85.736, \lambda_2 = -10.264$. Hence $(4,22)$ is a nodal sink.

As $t \to \infty$, each solution point $\left(x(t), y(t)\right)$ with nonzero initial conditions approaches the nodal sink $(4,22)$. See the figure below.



34. $\mathbf{J} = \begin{bmatrix} -4x - y + 30 & -x \\ 2y & 2x - 8y + 20 \end{bmatrix}$

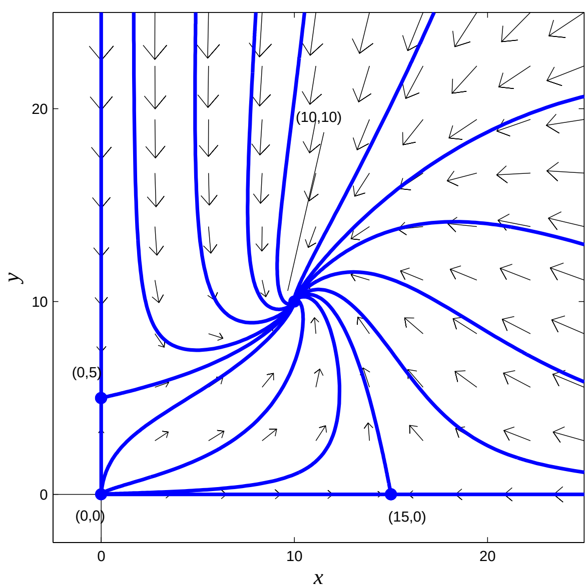
At $(0,0)$: The Jacobian matrix $\mathbf{J} = \begin{bmatrix} 30 & 0 \\ 0 & 20 \end{bmatrix}$ has characteristic equation $\lambda^2 - 50\lambda + 600 = 0$ and positive real eigenvalues $\lambda_1 = 20,\ \lambda_2 = 30$. Hence $(0,0)$ is a nodal source.

At $(0,5)$: The Jacobian matrix $\mathbf{J} = \begin{bmatrix} 25 & 0 \\ 10 & -20 \end{bmatrix}$ has characteristic equation $\lambda^2 - 5\lambda - 500 = 0$ and real eigenvalues $\lambda_1 = -20,\ \lambda_2 = 25$ of opposite sign. Hence $(0,5)$ is a saddle point.

At $(15,0)$: The Jacobian matrix $\mathbf{J} = \begin{bmatrix} -30 & -15 \\ 0 & 50 \end{bmatrix}$ has characteristic equation $\lambda^2 - 20\lambda - 1500 = 0$ and real eigenvalues $\lambda_1 = -30,\ \lambda_2 = 50$ of opposite sign. Hence $(15,0)$ is a saddle point.

At $(10,10)$: The Jacobian matrix $\mathbf{J} = \begin{bmatrix} -20 & -10 \\ 20 & -40 \end{bmatrix}$ has characteristic equation $\lambda^2 + 60\lambda + 1000 = 0$ and complex conjugate eigenvalues $\lambda_1, \lambda_2 \approx -30 \pm 10i$ with negative real part. Hence $(10,10)$ is a spiral sink.

As $t \to \infty$, each solution point $\left(x(t), y(t)\right)$ with nonzero initial conditions approaches the nodal sink $(10,10)$. See the figure below.



SECTION 9.4

NONLINEAR MECHANICAL SYSTEMS

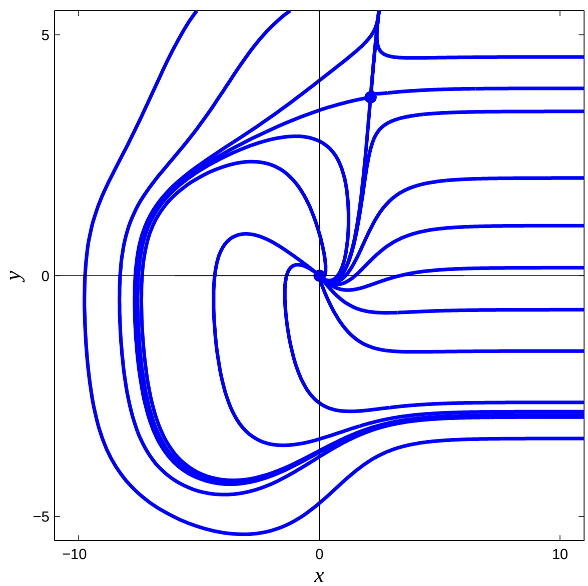
In each of Problems 1–4 we need only substitute the familiar power series for the exponential, sine, and cosine functions, and then discard all higher-order terms. For each problem we give the corresponding linear system, the eigenvalues λ_1 and λ_2, and the type of this critical point.

1. $x' = 1-\left(1+x+\frac{1}{2}x^2+\cdots\right)+2y \approx -x+2y$

$y' = -x-4\left(y-\frac{1}{6}y^3+\cdots\right) \approx -x-4y$

The coefficient matrix $\mathbf{A} = \begin{bmatrix} -1 & 2 \\ -1 & -4 \end{bmatrix}$ has negative eigenvalues $\lambda_1 = -2$ and $\lambda_2 = -3$ indicating a stable nodal sink as illustrated in the figure below. Alternatively, we can calculate the Jacobian matrix

$$\mathbf{J}(x,y) = \begin{bmatrix} -e^x & 2 \\ -1 & -4\cos x \end{bmatrix}, \quad \text{so} \quad \mathbf{J}(0,0) = \begin{bmatrix} -1 & 2 \\ -1 & -4 \end{bmatrix}.$$



2. $x' = 2\left(x-\frac{1}{6}x^3+\cdots\right)+\left(y-\frac{1}{6}y^3+\cdots\right) \approx 2x+y$

$y' = \left(x-\frac{1}{6}x^3+\cdots\right)+2\left(y-\frac{1}{6}y^3+\cdots\right) \approx x+2y$

The coefficient matrix $\mathbf{A} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ has positive eigenvalues $\lambda_1 = 1$ and $\lambda_2 = 3$ indicating an unstable nodal source. Alternatively, we can calculate the Jacobian matrix

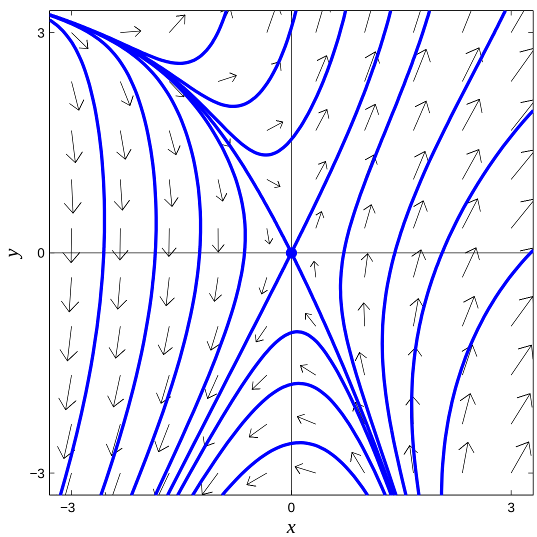
$$\mathbf{J}(x,y) = \begin{bmatrix} 2\cos x & \cos y \\ \cos x & 2\cos y \end{bmatrix}, \quad \text{so} \quad \mathbf{J}(0,0) = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}.$$

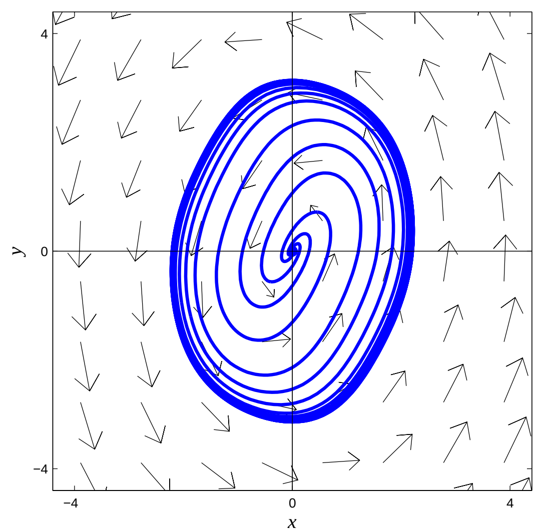
3. $x' = \left(1+x+\frac{1}{2}x^2+\cdots\right)+2y-1 \approx x+2y$

$y' = 8x+\left(1+y+\frac{1}{2}y^2+\cdots\right)-1 \approx 8x+y$

The coefficient matrix $\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 8 & 1 \end{bmatrix}$ has real eigenvalues $\lambda_1 = -3$ and $\lambda_2 = 5$ of opposite sign, indicating an unstable saddle point as illustrated in the left-hand figure below. Alternatively, we can calculate the Jacobian matrix

$$\mathbf{J}(x,y) = \begin{bmatrix} e^x & 2 \\ 8 & e^y \end{bmatrix}, \quad \text{so} \quad \mathbf{J}(0,0) = \begin{bmatrix} 1 & 2 \\ 8 & 1 \end{bmatrix}.$$





4. The linear system is $x' = x - 2y,\ y' = 4x - 3y$ because

$$\sin x \cos y = (x - x^3/3! + \cdots)(1 - y^2/2! + \cdots) = x + \cdots,$$

and $\cos x \sin y \approx y$ similarly. The coefficient matrix $\mathbf{A} = \begin{bmatrix} 1 & -2 \\ 4 & -3 \end{bmatrix}$ has complex conjugate eigenvalues $\lambda_1, \lambda_2 = -1 \pm 2\,i$ with negative real part, indicating a stable spiral point as illustrated in the right-hand figure above. Alternatively, we can calculate the Jacobian matrix

$$\mathbf{J}(x,y) = \begin{bmatrix} \cos x \cos y & -\sin x \sin y - 2 \\ 3\sin x \sin y + 4 & -3\cos x \cos y \end{bmatrix}, \quad \text{so} \quad \mathbf{J}(0,0) = \begin{bmatrix} 1 & -2 \\ 4 & -3 \end{bmatrix}.$$

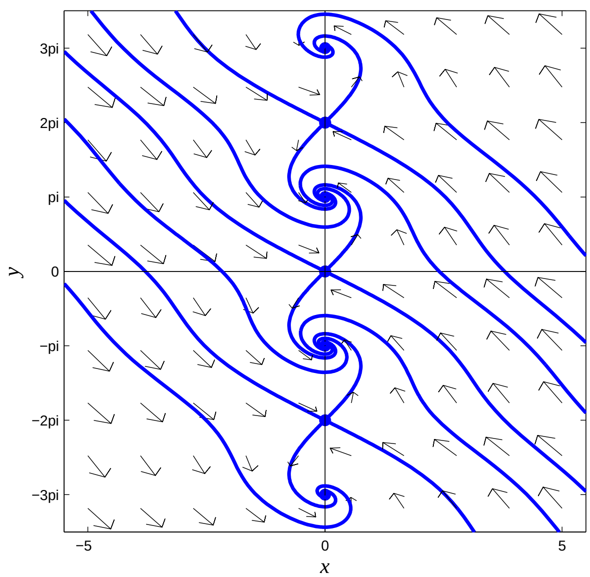
5. The critical points are of the form $(0, n\pi)$ where n is an integer, so we substitute $x = u,\quad y = v + n\pi$. Then

$$u' = x' = -u + \sin(v + n\pi) = -u + (\cos n\pi)v = -u + (-1)^n v.$$

Hence the linearized system at $(0, n\pi)$ is

$$u' = -u \pm v, \qquad v' = 2u$$

where we take the plus sign if n is even, the minus sign if n is odd. If n is even the eigenvalues are $\lambda_1 = 1$ and $\lambda_2 = -2$, so $(0, n\pi)$ is an unstable saddle point. If n is odd the eigenvalues are $\lambda_1, \lambda_2 = (-1 \pm i\sqrt{7})/2$, so $(0, n\pi)$ is a stable spiral point.



Alternatively, we can start by calculating the Jacobian matrix $\mathbf{J}(x,y) = \begin{bmatrix} -1 & \cos y \\ 2 & 0 \end{bmatrix}$.

At $(0, n\pi)$, n even: The Jacobian matrix $\mathbf{J} = \begin{bmatrix} -1 & 1 \\ 2 & 0 \end{bmatrix}$ has characteristic equation $\lambda^2 + \lambda - 2 = 0$ and real eigenvalues $\lambda_1 = -2$, $\lambda_2 = 1$ of opposite sign. Hence $(0, n\pi)$ is a saddle point if n is even, as we see in the figure above.

At $(0, n\pi)$, n odd: The Jacobian matrix $\mathbf{J} = \begin{bmatrix} -1 & -1 \\ 2 & 0 \end{bmatrix}$ has characteristic equation $\lambda^2 + \lambda + 2 = 0$ and complex conjugate eigenvalues $\lambda_1, \lambda_2 = \frac{1}{2}(-1 \pm i\sqrt{7})$ with negative real. Hence $(0, n\pi)$ is a spiral sink if n is odd, as indicated in the figure.

6. The critical points are of the form $(n, 0)$ where n is an integer, so we substitute $x = u + n$, $y = v$. Then

$$v' = y' = \sin \pi(u+n) - v = \cos n\pi \sin \pi u \approx (-1)^n \pi u - v,$$

Hence the linearized system at $(n, 0)$ is

$$u' = v, \quad v' = \pm\pi u - v$$

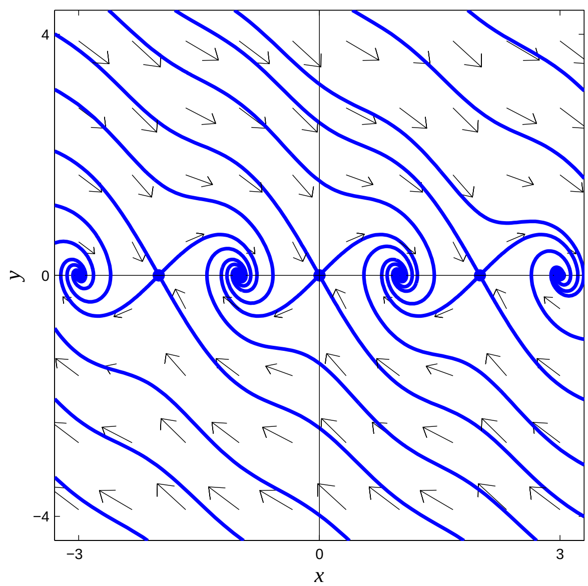
with coefficient matrix $\mathbf{A} = \begin{bmatrix} 0 & 1 \\ \pm\pi & -1 \end{bmatrix}$ where we take the plus sign if n is even, the minus sign if n is odd. The characteristic equation

$$\lambda^2 + \lambda - \pi = 0$$

has one positive and one negative root, so $(n, 0)$ is an unstable saddle point if n is even. The equation

$$\lambda^2 + \lambda + \pi = 0$$

has complex conjugate roots with negative real part, so $(n, 0)$ is a stable spiral point if n is odd.



Alternatively, we can start by calculating the Jacobian matrix $\mathbf{J}(x, y) = \begin{bmatrix} 0 & 1 \\ \pi\cos\pi x & -1 \end{bmatrix}$.

At $(n,0)$, n even : The Jacobian matrix $\mathbf{J} = \begin{bmatrix} 0 & 1 \\ \pi & -1 \end{bmatrix}$ has characteristic equation $\lambda^2 + \lambda - \pi = 0$ and real eigenvalues $\lambda_1 \approx -2.3416$, $\lambda_2 \approx 1.3416$ of opposite sign. Hence $(n,0)$ is a saddle point if n is even, as we see in the figure above.

At $(n,0)$, n odd : The Jacobian matrix $\mathbf{J} = \begin{bmatrix} 0 & 1 \\ -\pi & -1 \end{bmatrix}$ has characteristic equation $\lambda^2 + \lambda + \pi = 0$ and complex conjugate eigenvalues $\lambda_1, \lambda_2 \approx -0.5 \pm 1.7005i$ with negative real. Hence $(n,0)$ is a spiral sink if n is odd, as we see in the figure.

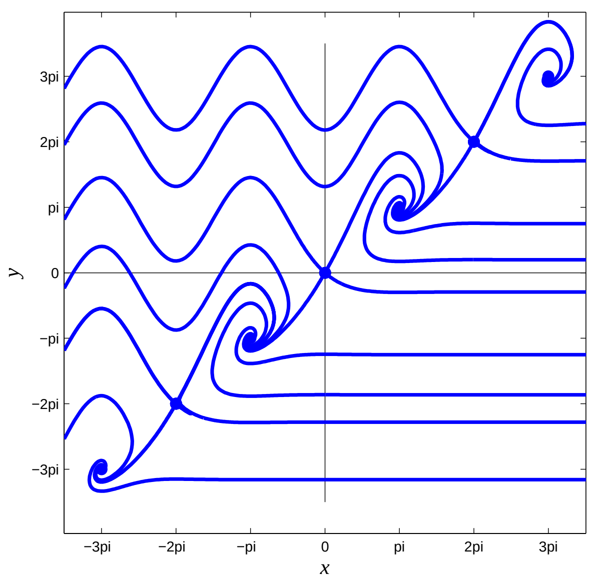
7. The critical points are of the form $(n\pi, n\pi)$ where n is an integer, so we substitute $x = u + n\pi$, $y = v + n\pi$. Then

$$u' = x' = 1 - e^{u-v} = 1 - \left(1 + (u-v) + \tfrac{1}{2}(u-v)^2 + \cdots\right) \approx -u + v,$$
$$v' = y' = 2\sin\left(u + n\pi\right) = 2\sin u \cos n\pi \approx 2(-1)^n u.$$

Hence the linearized system at $(n\pi, n\pi)$ is

$$u' = -u + v, \qquad v' = \pm 2u$$

and has coefficient matrix $\mathbf{A} = \begin{bmatrix} -1 & 1 \\ \pm 2 & 0 \end{bmatrix}$, where we take the plus sign if n is even, the minus sign if n is odd. With n even, The characteristic equation $\lambda^2 + \lambda - 2 = 0$ has real roots $\lambda_1 = 1$ and $\lambda_2 = -2$ of opposite sign, so $(n\pi, n\pi)$ is an unstable saddle point. With n odd, the characteristic equation $\lambda^2 + \lambda + 2 = 0$ has complex conjugate eigenvalues are $\lambda_1, \lambda_2 = (-1 \pm i\sqrt{7})/2$ with negative real part, so $(n\pi, n\pi)$ is a stable spiral point.



Alternatively, we can start by calculating the Jacobian matrix $\mathbf{J}(x,y) = \begin{bmatrix} -e^{x-y} & e^{x-y} \\ 2\cos x & 0 \end{bmatrix}$.

At $(n\pi, n\pi)$, n even : The Jacobian matrix $\mathbf{J} = \begin{bmatrix} -1 & 1 \\ 2 & 0 \end{bmatrix}$ has characteristic equation $\lambda^2 + \lambda - 2 = 0$ and real eigenvalues $\lambda_1 = -2$, $\lambda_2 = 1$ of opposite sign. Hence $(n\pi, n\pi)$ is a saddle point if n is even, as we see in the figure above.

At $(n\pi, n\pi)$, n odd: The Jacobian matrix $\mathbf{J} = \begin{bmatrix} -1 & 1 \\ -2 & 0 \end{bmatrix}$ has characteristic equation $\lambda^2 + \lambda + 2 = 0$ and complex conjugate eigenvalues $\lambda_1, \lambda_2 \approx -0.5 \pm 1.3229i$ with negative real. Hence $(n\pi, n\pi)$ is a spiral sink if n is odd, as we see in the figure.

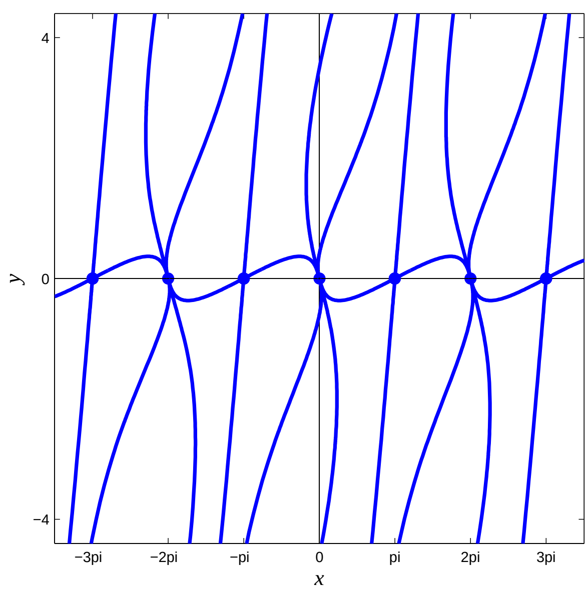
8. The critical points are of the form $(n\pi, 0)$ where n is an integer, so we substitute $x = u + n\pi$, $y = v$. Then

$$\begin{aligned} u' &= x' = 3\sin(u + n\pi) + v = 3\sin u \cos n\pi + v \approx 3(-1)^n u + v, \\ v' &= y' = \sin(u + n\pi) + 2v = \sin u \cos n\pi + 2v \approx (-1)^n u + 2v, \end{aligned}$$

Hence the linearized system at $(n\pi, 0)$ is

$$u' = \pm 3u + v, \qquad v' = \pm u + 2v$$

with coefficient matrix $\mathbf{A} = \begin{bmatrix} \pm 3 & 1 \\ \pm 1 & 2 \end{bmatrix}$, where we take the plus signs if n is even, the minus signs if n is odd. If n is even then the characteristic equation $\lambda^2 - 5\lambda + 5 = 0$ has roots $\lambda_1, \lambda_2 = (5 \pm \sqrt{5})/2$ that are both positive, so $(n\pi, 0)$ is an unstable nodal source. If n is odd then the characteristic equation $\lambda^2 - 5\lambda + 5 = 0$ has real roots $\lambda_1, \lambda_2 = (-1 \pm \sqrt{21})/2$ with opposite signs, so $(n\pi, 0)$ is an unstable saddle point.



Alternatively, we can start by calculating the Jacobian matrix $\mathbf{J}(x, y) = \begin{bmatrix} 3\cos x & 1 \\ \cos x & 2 \end{bmatrix}$.

At $(n\pi, 0)$, n even: The Jacobian matrix $\mathbf{J} = \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}$ has characteristic equation $\lambda^2 - 5\lambda + 5 = 0$ and positive real eigenvalues $\lambda_1 \approx 1.3812,\ \lambda_2 = 2.6180$. Hence $(n\pi, 0)$ is a nodal source if n is even, as we see in the figure on the preceding page.

At $(n\pi, 0)$, n odd: The Jacobian matrix $\mathbf{J} = \begin{bmatrix} -3 & 1 \\ -1 & 2 \end{bmatrix}$ has characteristic equation $\lambda^2 + \lambda - 5 = 0$ and real eigenvalues $\lambda_1 \approx -2.7913,\ \lambda_2 = 1.7913$ of opposite sign. Hence $(n\pi, 0)$ is a saddle point if n is odd, as we see in the figure.

As preparation for Problems 9–11, we first calculate the Jacobian matrix

$$\mathbf{J}(x, y) = \begin{bmatrix} 0 & 1 \\ -\omega^2 \cos x & -c \end{bmatrix}$$

of the damped pendulum system in (34) in the text. At the critical point $(n\pi, 0)$ we have

$$\mathbf{J}(n\pi, 0) = \begin{bmatrix} 0 & 1 \\ -\omega^2 \cos n\pi & -c \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \pm\omega^2 & -c \end{bmatrix},$$

where we take the plus sign if n is odd, the minus sign if n is even.

9. If n is odd then the characteristic equation $\lambda^2 + c\lambda - \omega^2 = 0$ has real roots

$$\lambda_1, \lambda_2 = \frac{-c \pm \sqrt{c^2 + 4\omega^2}}{2}$$

with opposite signs, so $(n\pi, 0)$ is an unstable saddle point.

10. If n is even then the characteristic equation $\lambda^2 + c\lambda + \omega^2 = 0$ has roots

$$\lambda_1, \lambda_2 = \frac{-c \pm \sqrt{c^2 - 4\omega^2}}{2}.$$

If $c^2 > 4\omega^2$ then λ_1 and λ_2 are both negative so $(n\pi, 0)$ is a stable nodal sink.

11. If n is even and $c^2 < 4\omega^2$ then the two eigenvalues

$$\lambda_1, \lambda_2 = \frac{-c \pm \sqrt{c^2 - 4\omega^2}}{2} = -\frac{c}{2} \pm \frac{i}{2}\sqrt{4\omega^2 - c^2}$$

are complex conjugates with negative real part, so $(n\pi, 0)$ is a stable spiral point.

Problems 12–16 call for us to find and classify the critical points of the first order-system $x' = y,\quad y' = -f(x, y)$ that corresponds to the given equation $x'' + f(x, x') = 0$. After finding the critical points $(x, 0)$ where $f(x, 0) = 0$, we first calculate the Jacobian matrix $\mathbf{J}(x, y)$.

12. $\mathbf{J}(x,y) = \begin{bmatrix} 0 & 1 \\ 15x^2 - 20 & 0 \end{bmatrix}.$

At $(0,0)$: The Jacobian matrix $\mathbf{J} = \begin{bmatrix} 0 & 1 \\ -20 & 0 \end{bmatrix}$ has characteristic equation $\lambda^2 + 20 = 0$ and pure imaginary eigenvalues $\lambda_1, \lambda_2 = \pm i\sqrt{20}$ consistent with the stable center we see at (0,0) in Fig. 9.4.4 in the textbook.

At $(\pm 2, 0)$: The Jacobian matrix $\mathbf{J} = \begin{bmatrix} 0 & 1 \\ 40 & 0 \end{bmatrix}$ has characteristic equation $\lambda^2 - 40 = 0$ and real eigenvalues $\lambda_1, \lambda_2 = \pm\sqrt{40}$ of opposite sign, consistent with the saddle points we see at $(\pm 2, 0)$ in Fig. 9.4.4.

13. $\mathbf{J}(x,y) = \begin{bmatrix} 0 & 1 \\ 15x^2 - 20 & -2 \end{bmatrix}.$

At $(0,0)$: The Jacobian matrix $\mathbf{J} = \begin{bmatrix} 0 & 1 \\ -20 & -2 \end{bmatrix}$ has characteristic equation $\lambda^2 + 2\lambda + 20 = 0$ and complex conjugate eigenvalues $\lambda_1, \lambda_2 = -1 \pm i\sqrt{19}$ consistent with the spiral node we see at (0,0) in Fig. 9.4.6 in the textbook.

At $(\pm 2, 0)$: The Jacobian matrix $\mathbf{J} = \begin{bmatrix} 0 & 1 \\ 40 & -2 \end{bmatrix}$ has characteristic equation $\lambda^2 + 2\lambda - 40 = 0$ and real eigenvalues $\lambda_1, \lambda_2 = -1 \pm \sqrt{41}$ of opposite sign, consistent with the saddle points we see at $(\pm 2, 0)$ in Fig. 9.4.6.

14. $\mathbf{J}(x,y) = \begin{bmatrix} 0 & 1 \\ 8 - 6x^2 & 0 \end{bmatrix}.$

At $(0,0)$: The Jacobian matrix $\mathbf{J} = \begin{bmatrix} 0 & 1 \\ 8 & 0 \end{bmatrix}$ has characteristic equation $\lambda^2 - 8 = 0$ and real eigenvalues $\lambda_1, \lambda_2 = \pm\sqrt{8}$ of opposite sign, consistent with the saddle point we see at (0,0) in Fig. 9.4.12 in the textbook.

At $(\pm 2, 0)$: The Jacobian matrix $\mathbf{J} = \begin{bmatrix} 0 & 1 \\ -16 & 0 \end{bmatrix}$ has characteristic equation $\lambda^2 + 16 = 0$ and pure imaginary eigenvalues $\lambda_1, \lambda_2 = \pm 4i$, consistent with the stable centers we see at $(\pm 2, 0)$ in Fig. 9.4.12.

15. $\mathbf{J}(x,y)=\begin{bmatrix} 0 & 1 \\ 2x-4 & 0 \end{bmatrix}.$

At $(0,0)$: The Jacobian matrix $\mathbf{J}=\begin{bmatrix} 0 & 1 \\ -4 & 0 \end{bmatrix}$ has characteristic equation $\lambda^2+4=0$ and pure imaginary eigenvalues $\lambda_1,\lambda_2=\pm 2i,$ consistent with the stable center we see at (0,0) in Fig. 9.4.13 in the textbook.

At $(4,0)$: The Jacobian matrix $\mathbf{J}=\begin{bmatrix} 0 & 1 \\ 4 & 0 \end{bmatrix}$ has characteristic equation $\lambda^2-4=0$ and real eigenvalues $\lambda_1,\lambda_2=\pm 2$ of opposite sign, consistent with the saddle point we see at $(4,0)$ in Fig. 9.4.13.

16. $\mathbf{J}(x,y)=\begin{bmatrix} 0 & 1 \\ -4+15x^2-5x^4 & 0 \end{bmatrix}.$

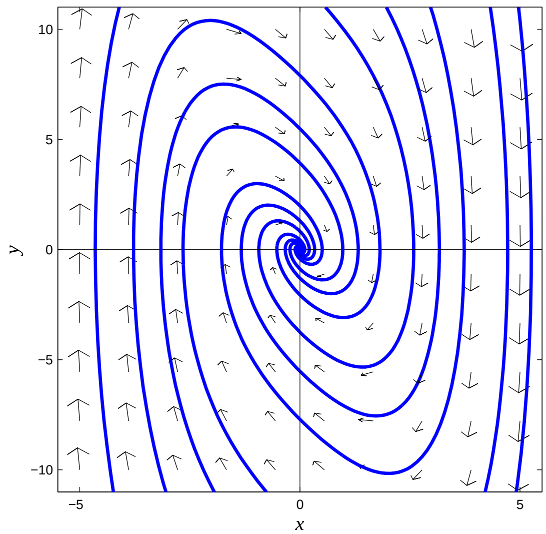
At $(0,0)$: The Jacobian matrix $\mathbf{J}=\begin{bmatrix} 0 & 1 \\ -4 & 0 \end{bmatrix}$ has characteristic equation $\lambda^2+4=0$ and pure imaginary eigenvalues $\lambda_1,\lambda_2=\pm 2i,$ consistent with the stable center we see at (0,0) in Fig. 9.4.14 in the textbook.

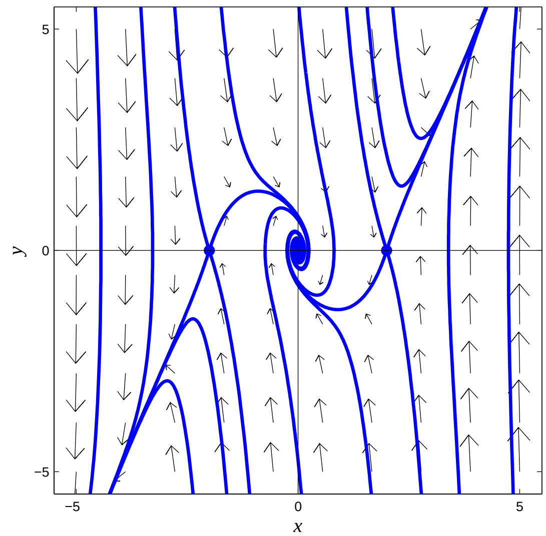
At $(\pm 1,0)$: The Jacobian matrix $\mathbf{J}=\begin{bmatrix} 0 & 1 \\ 6 & 0 \end{bmatrix}$ has characteristic equation $\lambda^2-6=0$ and real eigenvalues $\lambda_1,\lambda_2=\pm\sqrt{6}$ of opposite sign, consistent with the saddle points we see at $(\pm 1,0)$ in Fig. 9.4.14.

At $(\pm 2,0)$: The Jacobian matrix $\mathbf{J}=\begin{bmatrix} 0 & 1 \\ -24 & 0 \end{bmatrix}$ has characteristic equation $\lambda^2+24=0$ and pure imaginary eigenvalues $\lambda_1,\lambda_2=\pm i\sqrt{24},$ consistent with the stable centers we see at $(\pm 2,0)$ in Fig. 9.4.14.

17. $\mathbf{J}(x,y)=\begin{bmatrix} 0 & 1 \\ -5-\frac{15}{4}x^2 & -2 \end{bmatrix}.$

At $(0,0)$: The Jacobian matrix $\mathbf{J}=\begin{bmatrix} 0 & 1 \\ -5 & -2 \end{bmatrix}$ has characteristic equation $\lambda^2+2\lambda+5=0$ and complex conjugate eigenvalues $\lambda_1,\lambda_2=-1\pm 2i$ with negative real part, consistent with the spiral sink we see in the left-hand figure at the top of the next page.

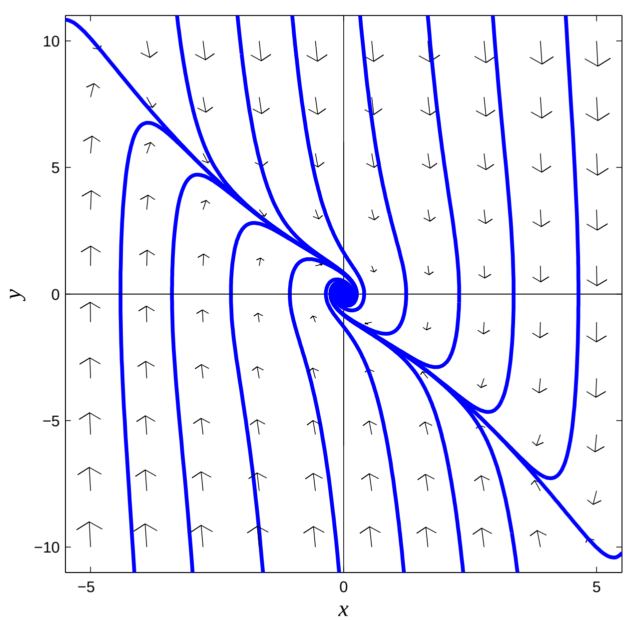




18. $\mathbf{J}(x,y) = \begin{bmatrix} 0 & 1 \\ -5+\frac{15}{4}x^2 & -4|y| \end{bmatrix}.$

At $(0,0)$: The Jacobian matrix $\mathbf{J} = \begin{bmatrix} 0 & 1 \\ -5 & 0 \end{bmatrix}$ has characteristic equation $\lambda^2 + 5 = 0$ and pure imaginary eigenvalues $\lambda_1, \lambda_2 = \pm i\sqrt{5}$. This corresponds to the indeterminate case of Theorem 2 in Section 9.3, but is not inconsistent with the spiral sink we see at the origin in the figure on the right above.

At $(\pm 2,0)$: The Jacobian matrix $\mathbf{J} = \begin{bmatrix} 0 & 1 \\ 10 & 0 \end{bmatrix}$ has characteristic equation $\lambda^2 - 10 = 0$ and real eigenvalues $\lambda_1, \lambda_2 = \pm\sqrt{10}$, consistent with the saddle points we see at $(\pm 2,0)$ in the right-hand figure above.



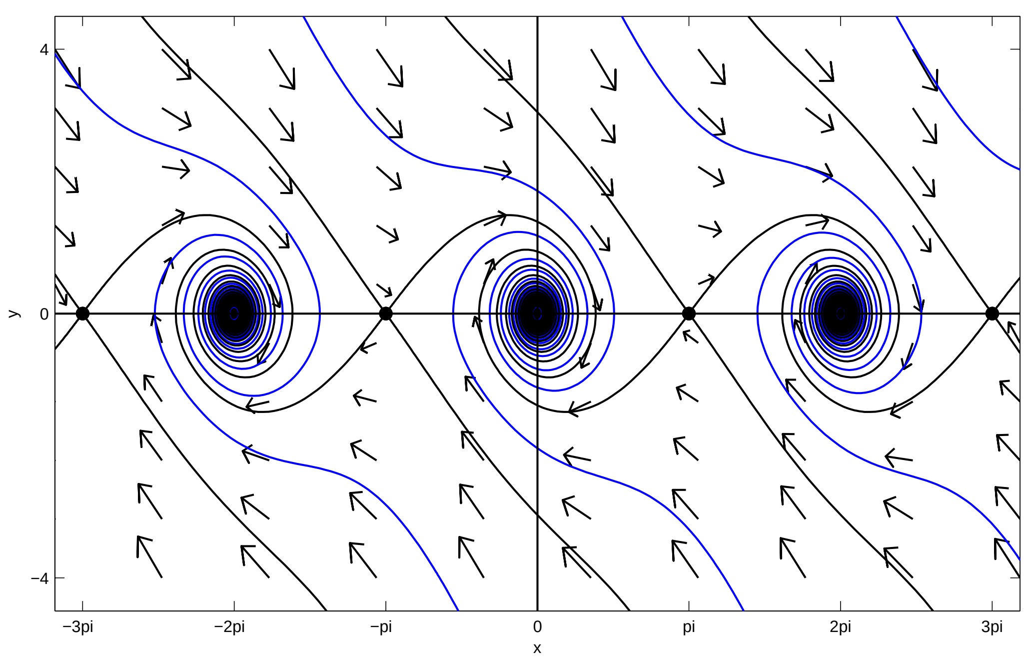
19. $\mathbf{J}(x,y) = \begin{bmatrix} 0 & 1 \\ -5-\frac{15}{4}x^2 & -4|y| \end{bmatrix}.$

At $(0,0)$: The Jacobian matrix $\mathbf{J} = \begin{bmatrix} 0 & 1 \\ -5 & 0 \end{bmatrix}$ has characteristic equation $\lambda^2 + 5 = 0$ and pure imaginary eigenvalues $\lambda_1, \lambda_2 = \pm i\sqrt{5}$. This corresponds to the indeterminate case of Theorem 2 in Section 9.3, but is not inconsistent with the spiral sink we see in the figure at the bottom of the preceding page.

20. $\mathbf{J}(x,y) = \begin{bmatrix} 0 & 1 \\ -\cos x & -\frac{1}{2}|y| \end{bmatrix}.$

At $(n\pi, 0)$, n even: The Jacobian matrix $\mathbf{J} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ has characteristic equation $\lambda^2 + 1 = 0$ and pure imaginary eigenvalues $\lambda_1, \lambda_2 = \pm i$. This corresponds to the indeterminate case of Theorem 2 in Section 9.3, but is not inconsistent with the spiral sinks we see in the figure below.

At $(n\pi, 0)$, n odd: The Jacobian matrix $\mathbf{J} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ has characteristic equation $\lambda^2 - 1 = 0$ and real eigenvalues $\lambda_1, \lambda_2 = \pm 1$ of opposite sign, consistent with the saddle points we see in the figure.



The statements of Problems 21–26 in the text include their answers and rather fully outline their solutions, which therefore are omitted here.

CHAPTER 10

LAPLACE TRANSFORM METHODS

SECTION 10.1

LAPLACE TRANSFORMS AND INVERSE TRANSFORMS

The objectives of this section are especially clearcut. They include familiarity with the definition of the Laplace transform $\mathcal{L}\{f(t)\} = F(s)$ that is given in Equation (1) in the textbook, the direct application of this definition to calculate Laplace transforms of simple functions (as in Examples 1–3), and the use of known transforms (those listed in Figure 10.1.2) to find Laplace transforms and inverse transforms (as in Examples 4-6). Perhaps students need to be told explicitly to memorize the transforms that are listed in the short table that appears in Figure 10.1.2.

1. $\mathcal{L}\{t\} = \int_0^\infty e^{-st}t\,dt \qquad (u = -st, \quad du = -s\,dt)$

$$= \int_0^{-\infty}\left[\frac{1}{s^2}\right]ue^u\,du = \frac{1}{s^2}\Big[(u-1)e^u\Big]_0^{-\infty} = \frac{1}{s^2}$$

2. We substitute $u = -st$ in the tabulated integral

$$\int u^2e^u\,du = e^u\left(u^2-2u+2\right)+C$$

(or, alternatively, integrate by parts) and get

$$\mathcal{L}\left\{t^2\right\} = \int_0^\infty e^{-st}t^2\,dt = \left[-e^{-st}\left(\frac{t^2}{s}+\frac{2t}{s^2}+\frac{2}{s^3}\right)\right]_{t=0}^\infty = \frac{2}{s^3}.$$

3. $\mathcal{L}\left\{e^{3t+1}\right\} = \int_0^\infty e^{-st}e^{3t+1}\,dt = e\int_0^\infty e^{-(s-3)t}\,dt = \dfrac{e}{s-3}$

4. With $a = -s$ and $b = 1$ the tabulated integral

$$\int e^{au}\cos bu\,du = e^{au}\left[\frac{a\cos bu + b\sin bu}{a^2+b^2}\right]+C$$

yields

$$\mathcal{L}\{\cos t\} = \int_0^\infty e^{-st}\cos t\,dt = \left[\frac{e^{-st}(-s\cos t+\sin t)}{s^2+1}\right]_{t=0}^\infty = \frac{s}{s^2+1}.$$

5. $\mathcal{L}\{\sinh t\} = \tfrac{1}{2}\mathcal{L}\{e^t - e^{-t}\} = \tfrac{1}{2}\int_0^\infty e^{-st}\left(e^t - e^{-t}\right)dt = \tfrac{1}{2}\int_0^\infty \left(e^{-(s-1)t} - e^{-(s+1)t}\right)dt$

$$= \frac{1}{2}\left[\frac{1}{s-1} - \frac{1}{s+1}\right] = \frac{1}{s^2-1}$$

6. $\mathcal{L}\{\sin^2 t\} = \int_0^\infty e^{-st}\sin^2 t\,dt = \tfrac{1}{2}\int_0^\infty e^{-st}\left(1-\cos 2t\right)dt$

$$= \frac{1}{2}\left[e^{-st}\left(-\frac{1}{s}\right) - e^{-st}\cdot\frac{-s\cos 2t + 2\sin 2t}{s^2+4}\right]_{t=0}^{\infty} = \frac{1}{2}\left[\frac{1}{s} - \frac{s}{s^2+4}\right]$$

7. $\mathcal{L}\{f(t)\} = \int_0^1 e^{-st}\,dt = \left[-\frac{1}{s}e^{-st}\right]_0^1 = \frac{1-e^{-s}}{s}$

8. $\mathcal{L}\{f(t)\} = \int_1^2 e^{-st}\,dt = \left[-\frac{e^{-st}}{s}\right]_1^2 = \frac{e^{-s} - e^{-2s}}{s}$

9. $\mathcal{L}\{f(t)\} = \int_0^1 e^{-st}t\,dt = \frac{1-e^{-s} - se^{-s}}{s^2}$

10. $\mathcal{L}\{f(t)\} = \int_0^1 (1-t)e^{-st}\,dt = \left[-e^{-st}\left(\frac{1}{s} - \frac{t}{s} - \frac{1}{s^2}\right)\right]_0^1 = \frac{1}{s} - \frac{1}{s^2} + \frac{e^{-s}}{s^2}$

11. $\mathcal{L}\{\sqrt{t} + 3t\} = \frac{\Gamma(3/2)}{s^{3/2}} + 3\cdot\frac{1}{s^2} = \frac{\sqrt{\pi}}{2s^{3/2}} + \frac{3}{s^2}$

12. $\mathcal{L}\{3t^{5/2} - 4t^3\} = 3\cdot\frac{\Gamma(7/2)}{s^{7/2}} - 4\cdot\frac{3!}{s^4} = \frac{45\sqrt{\pi}}{8s^{7/2}} - \frac{24}{s^2}$

13. $\mathcal{L}\{t - 2e^{3t}\} = \frac{1}{s^2} - \frac{2}{s-3}$

14. $\mathcal{L}\{t^{3/2} + e^{-10t}\} = \frac{\Gamma(5/2)}{s^{5/2}} + \frac{1}{s+10} = \frac{3\sqrt{\pi}}{4s^{5/2}} + \frac{1}{s+10}$

15. $\mathcal{L}\{1 + \cosh 5t\} = \frac{1}{s} + \frac{s}{s^2-25}$

16. $\mathcal{L}\{\sin 2t + \cos 2t\} = \dfrac{2}{s^2+4}+\dfrac{s}{s^2+4} = \dfrac{s+2}{s^2+4}$

17. $\mathcal{L}\{\cos^2 2t\} = \dfrac{1}{2}\mathcal{L}\{1+\cos 4t\} = \dfrac{1}{2}\left(\dfrac{1}{s}+\dfrac{s}{s^2+16}\right)$

18. $\mathcal{L}\{\sin 3t\cos 3t\} = \dfrac{1}{2}\mathcal{L}\{\sin 6t\} = \dfrac{1}{2}\cdot\dfrac{6}{s^2+36} = \dfrac{3}{s^2+36}$

19. $\mathcal{L}\left\{(1+t)^3\right\} = \mathcal{L}\left\{1+3t+3t^2+t^3\right\} = \dfrac{1}{s}+3\cdot\dfrac{1!}{s^2}+3\cdot\dfrac{2!}{s^3}+\dfrac{3!}{s^4} = \dfrac{1}{s}+\dfrac{3}{s^2}+\dfrac{6}{s^3}+\dfrac{6}{s^4}$

20. Integrating by parts with $u = t$, $dv = e^{-(s-1)t}dt$, we get

$$\begin{aligned}\mathcal{L}\{te^t\} &= \int_0^\infty e^{-st}\,te^t\,dt = \int_0^\infty te^{-(s-1)t}\,dt \\ &= \left[\frac{-te^{-(s-1)t}}{s-1}\right]_0^\infty + \frac{1}{s-1}\int_0^\infty e^{-st}e^t\,dt = \frac{1}{s-1}\mathcal{L}\{t\} = \frac{1}{(s-1)^2}.\end{aligned}$$

21. Integration by parts with $u = t$ and $dv = e^{-st}\cos 2t\,dt$ yields

$$\begin{aligned}\mathcal{L}\{t\cos 2t\} &= \int_0^\infty te^{-st}\cos 2t\,dt = -\frac{1}{s^2+4}\int_0^\infty e^{-st}\left(-s\cos 2t+2\sin 2t\right)dt \\ &= -\frac{1}{s^2+4}\left[-s\,\mathcal{L}\{\cos 2t\}+2\,\mathcal{L}\{\sin 2t\}\right] \\ &= -\frac{1}{s^2+4}\left[\frac{-s^2}{s^2+4}+\frac{4}{s^2+4}\right] = \frac{s^2-4}{\left(s^2+4\right)^2}.\end{aligned}$$

22. $\mathcal{L}\{\sinh^2 3t\} = \dfrac{1}{2}\mathcal{L}\{\cosh 6t-1\} = \dfrac{1}{2}\left(\dfrac{s}{s^2-36}-\dfrac{1}{s}\right)$

23. $\mathcal{L}^{-1}\left\{\dfrac{3}{s^4}\right\} = \mathcal{L}^{-1}\left\{\dfrac{1}{2}\cdot\dfrac{6}{s^4}\right\} = \dfrac{1}{2}t^3$

24. $\mathcal{L}^{-1}\left\{\dfrac{1}{s^{3/2}}\right\} = \mathcal{L}^{-1}\left\{\dfrac{2}{\sqrt{\pi}}\cdot\dfrac{\sqrt{\pi}}{2s^{3/2}}\right\} = \dfrac{2}{\sqrt{\pi}}\cdot t^{1/2} = 2\sqrt{\dfrac{t}{\pi}}$

25. $\mathcal{L}^{-1}\left\{\dfrac{1}{s}-\dfrac{2}{s^{5/2}}\right\} = \mathcal{L}^{-1}\left\{\dfrac{1}{s}-\dfrac{2}{\Gamma(5/2)}\cdot\dfrac{\Gamma(5/2)}{s^{5/2}}\right\} = 1-\dfrac{2}{\frac{3}{2}\cdot\frac{1}{2}\sqrt{\pi}}\cdot t^{3/2} = 1-\dfrac{8t^{3/2}}{3\sqrt{\pi}}$

26. $\mathcal{L}^{-1}\left\{\dfrac{1}{s+5}\right\} = = e^{-5t}$

27. $\mathcal{L}^{-1}\left\{\dfrac{3}{s-4}\right\} = 3\cdot\mathcal{L}^{-1}\left\{\dfrac{1}{s-4}\right\} = 3e^{4t}$

28. $\mathcal{L}^{-1}\left\{\dfrac{3s+1}{s^2+4}\right\} = 3\cdot\mathcal{L}^{-1}\left\{\dfrac{s}{s^2+4}\right\}+\dfrac{1}{2}\cdot\mathcal{L}^{-1}\left\{\dfrac{2}{s^2+4}\right\} = 3\cos 2t+\dfrac{1}{2}\sin 2t$

29. $\mathcal{L}^{-1}\left\{\dfrac{5-3s}{s^2+9}\right\} = \dfrac{5}{3}\cdot\mathcal{L}^{-1}\left\{\dfrac{3}{s^2+9}\right\}-3\cdot\mathcal{L}^{-1}\left\{\dfrac{s}{s^2+9}\right\} = \dfrac{5}{3}\sin 3t-3\cos 3t$

30. $\mathcal{L}^{-1}\left\{\dfrac{9+s}{4-s^2}\right\} = -\dfrac{9}{2}\cdot\mathcal{L}^{-1}\left\{\dfrac{2}{s^2-4}\right\}-\mathcal{L}^{-1}\left\{\dfrac{s}{s^2-4}\right\} = -\dfrac{9}{2}\sinh 2t-\cosh 2t$

31. $\mathcal{L}^{-1}\left\{\dfrac{10s-3}{25-s^2}\right\} = -10\cdot\mathcal{L}^{-1}\left\{\dfrac{s}{s^2-25}\right\}+\dfrac{3}{5}\cdot\mathcal{L}^{-1}\left\{\dfrac{5}{s^2-25}\right\} = -10\cosh 5t+\dfrac{3}{5}\sinh 5t$

32. $\mathcal{L}^{-1}\left\{2\cdot\dfrac{e^{-3s}}{s}\right\} = 2u(t-3) = 2u_3(t)$ [See Example 8 in the textbook.]

33. $\mathcal{L}\{\sin kt\} = \mathcal{L}\left\{\dfrac{e^{ikt}-e^{-ikt}}{2i}\right\} = \dfrac{1}{2i}\left(\dfrac{1}{s-ik}-\dfrac{1}{s+ik}\right)$

$= \dfrac{1}{2i}\cdot\dfrac{2ik}{(s-ik)(s-ik)} = \dfrac{k}{s^2+k^2}$ (because $i^2=-1$)

34. $\mathcal{L}\{\sinh kt\} = \mathcal{L}\left\{\dfrac{e^{kt}-e^{-kt}}{2}\right\} = \dfrac{1}{2}\left(\dfrac{1}{s-k}-\dfrac{1}{s+k}\right) = \dfrac{1}{2}\cdot\dfrac{2k}{s^2-k^2} = \dfrac{k}{s^2-k^2}$

35. Using the given tabulated integral with $a=-s$ and $b=k$, we find that

$$\mathcal{L}\{\cos kt\} = \int_0^\infty e^{-st}\cos kt\,dt = \left[\frac{e^{-st}}{s^2+k^2}(-s\cos kt+k\sin kt)\right]_{t=0}^{\infty}$$

$$= \lim_{t\to\infty}\left(\frac{e^{-st}}{s^2+k^2}(-s\cos kt+k\sin kt)\right)-\frac{e^0}{s^2+k^2}(-s\cdot 1+k\cdot 0) = \frac{s}{s^2+k^2}.$$

36. Evidently the function $f(t) = \sin(e^{t^2})$ is of exponential order because it is bounded; we can simply take $c = 0$ and $M = 1$ in Eq. (23) of this section in the text. However, its derivative $f'(t) = 2te^{t^2}\cos(e^{t^2})$ is *not* bounded by any exponential function e^{ct}, because $e^{t^2}/e^{ct} = e^{t^2-ct} \to \infty$ as $t \to \infty$.

37. $f(t) = 1 - u_a(t) = 1 - u(t - a)$ so

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{1\} - \mathcal{L}\{u_a(t)\} = \frac{1}{s} - \frac{e^{-as}}{s} = s^{-1}(1 - e^{-as}).$$

For the graph of f, note that $f(a) = 1 - u(a) = 1 - 1 = 0$.

38. $f(t) = u(t-a) - u(t-b)$, so

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{u_a(t)\} - \mathcal{L}\{u_b(t)\} = \frac{e^{-as}}{s} - \frac{e^{-bs}}{s} = s^{-1}\left(e^{-as} - e^{-bs}\right).$$

For the graph of f, note that $f(a) = u(0) - u(a-b) = 1 - 0 = 1$ because $a < b$, but $f(b) = u(b-a) - u(0) = 1 - 1 = 0$.

39. Use of the geometric series gives

$$\mathcal{L}\{f(t)\} = \sum_{n=0}^{\infty} \mathcal{L}\{u(t-n)\} = \sum_{n=0}^{\infty} \frac{e^{-ns}}{s} = \frac{1}{s}\left(1 + e^{-s} + e^{-2s} + e^{-3s} + \cdots\right)$$

$$= \frac{1}{s}\left(1 + (e^{-s}) + (e^{-s})^2 + (e^{-s})^3 + \cdots\right) = \frac{1}{s} \cdot \frac{1}{1 - e^{-s}} = \frac{1}{s\left(1 - e^{-s}\right)}.$$

40. Use of the geometric series gives

$$\mathcal{L}\{f(t)\} = \sum_{n=0}^{\infty} (-1)^n \mathcal{L}\{u(t-n)\} = \sum_{n=0}^{\infty} \frac{(-1)^n e^{-ns}}{s} = \frac{1}{s}\left(1 - e^{-s} + e^{-2s} - e^{-3s} + \cdots\right)$$

$$= \frac{1}{s}\left(1 + (-e^{-s}) + (-e^{-s})^2 + (-e^{-s})^3 + \cdots\right) = \frac{1}{s} \cdot \frac{1}{1 - (-e^{-s})} = \frac{1}{s\left(1 + e^{-s}\right)}.$$

41. By checking values at sample points, you can verify that $g(t) = 2f(t) - 1$ in terms of the square wave function $f(t)$ of Problem 40. Hence

$$\mathcal{L}\{g(t)\} = \mathcal{L}\{2f(t) - 1\} = \frac{2}{s\left(1 + e^{-s}\right)} - \frac{1}{s} = \frac{1}{s}\left(\frac{2}{1 + e^{-s}} - 1\right) = \frac{1}{s} \cdot \frac{1 - e^{-s}}{1 + e^{-s}}$$

$$= \frac{1}{s} \cdot \frac{1 - e^{-s}}{1 + e^{-s}} \cdot \frac{e^{s/2}}{e^{s/2}} = \frac{1}{s} \cdot \frac{e^{s/2} - e^{-s/2}}{e^{s/2} + e^{-s/2}} = \frac{1}{s} \cdot \frac{\frac{1}{2}\left(e^{s/2} - e^{-s/2}\right)}{\frac{1}{2}\left(e^{s/2} + e^{-s/2}\right)}$$

$$= \frac{1}{s}\cdot\frac{\sinh(s/2)}{\cosh(s/2)} = \frac{1}{s}\tanh\frac{s}{2}.$$

42. Let's refer to $(n-1,n]$ as an odd interval if the integer n is odd, and even interval if n is even. Then our function $h(t)$ has the value a on odd intervals, the value b on even intervals. Now the unit step function $f(t)$ of Problem 40 has the value 1 on odd intervals, the value 0 on even intervals. Hence the function $(a-b)f(t)$ has the value $(a-b)$ on odd intervals, the value 0 on even intervals. Finally, the function $(a-b)f(t)+b$ has the value $(a-b)+b=a$ on odd intervals, the value b on even intervals, and hence $(a-b)f(t)+b=h(t)$. Therefore

$$L\{h(t)\} = L\{(a-b)f(t)\}+L\{b\} = \frac{a-b}{s(1+e^{-s})}+\frac{b}{s} = \frac{a+be^{-s}}{s(1+e^{-s})}.$$

SECTION 10.2

TRANSFORMATION OF INITIAL VALUE PROBLEMS

The focus of this section is on the use of transforms of derivatives (Theorem 1) to solve initial value problems (as in Examples 1 and 2). Transforms of integrals (Theorem 2) appear less frequently in practice, and the extension of Theorem 1 at the end of Section 10.2 may be considered entirely optional (except perhaps for electrical engineering students).

In Problems 1–10 we give first the transformed differential equation, then the transform $X(s)$ of the solution, and finally the inverse transform $x(t)$ of $X(s)$.

1. $[s^2X(s) - 5s] + 4\{X(s)\} = 0$

$$X(s) = \frac{5s}{s^2+4} = 5\cdot\frac{s}{s^2+4}$$

$x(t) = \mathcal{L}^{-1}\{X(s)\} = 5\cos 2t$

2. $[s^2X(s) - 3s - 4] + 9[X(s)] = 0$

$$X(s) = \frac{3s+4}{s^2+9} = 3\cdot\frac{s}{s^2+9}+\frac{4}{3}\cdot\frac{3}{s^2+9}$$

$x(t) = \mathcal{L}^{-1}\{X(s)\} = 3\cos 3t + (4/3)\sin 3t$

3. $[s^2X(s) - 2] - [sX(s)] - 2[X(s)] = 0$

$$X(s) \;=\; \frac{2}{s^2 - s - 2} \;=\; \frac{2}{(s-2)(s+1)} \;= \frac{2}{3}\left(\frac{1}{s-2} - \frac{1}{s+1}\right)$$

$x(t) \;=\; (2/3)(e^{2t} - e^{-t})$

4. $[s^2X(s) - 2s + 3] + 8[s\,X(s) - 2] + 15[X(s)] \;=\; 0$

$$X(s) \;=\; \frac{2s+13}{s^2+8s+15} \;=\; \frac{7}{2}\cdot\frac{1}{s+3} \;-\frac{3}{2}\cdot\frac{1}{s+5}$$

$x(t) \;=\; \mathcal{L}^{-1}\{X(s)\} \;=\; (7/2)e^{-3t} - (3/2)e^{-5t}$

5. $[s^2X(s)] + [X(s)] \;=\; 2/(s^2 + 4)$

$$X(s) \;=\; \frac{2}{(s^2+1)(s^2+4)} \;=\; \frac{2}{3}\cdot\frac{1}{s^2+1} - \frac{1}{3}\cdot\frac{2}{s^2+4}$$

$x(t) \;=\; (2\sin t - \sin 2t)/3$

6. $[s^2X(s)] + 4[X(s)] \;=\; \mathcal{L}\{\cos t\} \;=\; s/(s^2 + 1)$

$$X(s) \;=\; \frac{2}{(s^2+1)(s^2+4)} \;=\; \frac{1}{3}\cdot\frac{s}{s^2+1} - \frac{1}{3}\cdot\frac{s}{s^2+4}$$

$x(t) \;=\; \mathcal{L}^{-1}\{X(s)\} \;=\; (\cos t - \cos 2t)/3$

7. $[s^2X(s) - s] + [X(s)] \;=\; s/s^2 + 9)$

$(s^2 + 1)X(s) \;=\; s + s/(s^2 + 9) \;=\; (s^3 + 10s)/(s^2 + 9)$

$$X(s) \;=\; \frac{s^2+10s}{(s^2+1)(s^2+9)} \;=\; \frac{9}{9}\cdot\frac{s}{s^2+1} - \frac{1}{8}\cdot\frac{s}{s^2+9}$$

$x(t) \;=\; (9\cos t - \cos 3t)/8$

8. $[s^2X(s)] + 9[X(s)] \;=\; \mathcal{L}\{1\} \;=\; 1/s$

$$X(s) \;=\; \frac{1}{s(s^2+9)} \;=\; \frac{1}{9}\cdot\frac{1}{s} - \frac{1}{9}\cdot\frac{s}{s^2+9}$$

$x(t) \;=\; \mathcal{L}^{-1}\{X(s)\} \;=\; (1 - \cos 3t)/9$

9. $s^2X(s) + 4sX(s) + 3X(s) \;=\; 1/s$

$$X(s) \;=\; \frac{1}{s(s^2+4s+3)} \;=\; \frac{1}{s(s+1)(s+3)} \;=\; \frac{1}{3}\cdot\frac{1}{s} - \frac{1}{2}\cdot\frac{1}{s+1} + \frac{1}{6}\cdot\frac{1}{s+3}$$

$x(t) \;=\; (2 - 3e^{-t} + e^{-3t})/6$

10. $[s^2X(s) - 2] + 3[sX(s)] + 2[X(s)] = \mathcal{L}\{t\} = 1/s^2$

$(s^2 + 3s + 2)X(s) = 2 + 1/s^2 = (2s^2 + 1)/s^2$

$$X(s) = \frac{2s^2+1}{s^2(s^2+3s+2)} = \frac{2s^2+1}{s^2(s+1)(s+2)} = -\frac{3}{4}\cdot\frac{1}{s}+\frac{1}{2}\cdot\frac{1}{s^2}+3\cdot\frac{1}{s+1}-\frac{9}{4}\cdot\frac{1}{s+2}$$

$x(t) = \mathcal{L}^{-1}\{X(s)\} = (-3 + 2t + 12e^{-t} - 9e^{-2t})/4$

11. The transformed equations are

$$sX(s) - 1 = 2X(s) + Y(s)$$
$$sY(s) + 2 = 6X(s) + 3Y(s).$$

We solve for the Laplace transforms

$$X(s) = \frac{s-5}{s(s-5)} = \frac{1}{s}$$
$$Y(s) = X(s) = \frac{-2s+10}{s(s-5)} = -\frac{2}{s}.$$

Hence the solution is given by

$$x(t) = 1, \qquad y(t) = -2.$$

12. The transformed equations are

$$s\,X(s) = X(s) + 2Y(s)$$
$$s\,Y(s) = X(s) + 1/(s+1),$$

which we solve for

$$X(s) = \frac{2}{(s-2)(s+1)^2} = \frac{2}{9}\left(\frac{1}{s-2}-\frac{1}{s+1}-3\cdot\frac{1}{(s+1)^2}\right)$$
$$Y(s) = \frac{s-1}{(s-2)(s+1)^2} = \frac{1}{9}\left(\frac{1}{s-2}-\frac{1}{s+1}-6\cdot\frac{1}{(s+1)^2}\right).$$

Hence the solution is

$$x(t) = (2/9)(e^{2t} - e^{-t} - 3t\,e^{-t})$$
$$y(t) = (1/9)(e^{2t} - e^{-t} + 6t\,e^{-t}).$$

13. The transformed equations are

$$\begin{aligned} sX(s) + 2[sY(s) - 1] + X(s) &= 0 \\ sX(s) - [sY(s) - 1] + Y(s) &= 0, \end{aligned}$$

which we solve for the transforms

$$X(s) = -\frac{2}{3s^2-1} = -\frac{2}{3}\cdot\frac{1}{s^2-1/3} = -\frac{2}{\sqrt{3}}\cdot\frac{1/\sqrt{3}}{s^2-\left(1/\sqrt{3}\right)^2}$$

$$X(s) = \frac{3s+1}{3s^2-1} = \frac{s+1/3}{s^2-1/3} = \frac{s}{s^2-\left(1/\sqrt{3}\right)^2}+\frac{1}{\sqrt{3}}\cdot\frac{1/\sqrt{3}}{s^2-\left(1/\sqrt{3}\right)^2}.$$

Hence the solution is

$$\begin{aligned} x(t) &= -\left(2/\sqrt{3}\right)\sinh\left(t/\sqrt{3}\right) \\ y(t) &= \cosh\left(t/\sqrt{3}\right)+\left(1/\sqrt{3}\right)\sinh\left(t/\sqrt{3}\right). \end{aligned}$$

14. The transformed equations are

$$\begin{aligned} s^2X(s) + 1 + 2X(s) + 4Y(s) &= 0 \\ s^2Y(s) + 1 + X(s) + 2Y(s) &= 0, \end{aligned}$$

which we solve for

$$X(s) = \frac{-s^2+2}{s^2(s^2+4)} = \frac{1}{4}\left(2\cdot\frac{1}{s^2}-3\cdot\frac{2}{s^2+4}\right)$$

$$Y(s) = \frac{-s^2-1}{s^2(s^2+4)} = -\frac{1}{8}\left(2\cdot\frac{1}{s^2}+3\cdot\frac{2}{s^2+4}\right).$$

Hence the solution is

$$\begin{aligned} x(t) &= (1/4)(2t - 3\sin 2t) \\ y(t) &= (-1/8)(2t + 3\sin 2t). \end{aligned}$$

15. The transformed equations are

$$\begin{aligned} [s^2X - s] + [sX - 1] + [sY - 1] + 2X - Y &= 0 \\ [s^2Y - s] + [sX - 1] + [sY - 1] + 4X - 2Y &= 0, \end{aligned}$$

which we solve for

$$X(s) = \frac{s^2+3s+2}{s^3+3s^2+3s} = \frac{1}{3}\left(\frac{2}{s}+\frac{s+3}{s^2+3s+3}\right) = \frac{1}{3}\left(\frac{2}{s}+\frac{s+3}{(s+3/2)^2+(3/4)}\right)$$

$$= \frac{1}{3}\left(\frac{2}{s}+\frac{s+3/2}{(s+3/2)^2+(\sqrt{3}/2)^2}+\sqrt{3}\cdot\frac{\sqrt{3}/2}{(s+3/2)^2+(\sqrt{3}/2)^2}\right)$$

$$Y(s) = \frac{-s^3-2s^2+2s+4}{s^3+3s^2+3s} = \frac{1}{21}\left(\frac{28}{s}-\frac{9}{s-1}+\frac{2s+15}{s^2+3s+3}\right)$$

$$= \frac{1}{21}\left(\frac{28}{s}-\frac{9}{s-1}+\frac{2s+15}{(s+3/2)^2+3/4}\right)$$

$$= \frac{1}{21}\left(\frac{28}{s}-\frac{9}{s-1}+2\cdot\frac{s+3/2}{(s+3/2)^2+(\sqrt{3}/2)^2}+8\sqrt{3}\cdot\frac{\sqrt{3}/2}{(s+3/2)^2+(\sqrt{3}/2)^2}\right).$$

Here we've used some fairly heavy-duty partial fractions (Section 7.3). The transforms

$$\mathcal{L}\left\{e^{at}\cos kt\right\} = \frac{s-a}{(s-a)^2+k^2}, \qquad \mathcal{L}\left\{e^{at}\sin kt\right\} = \frac{k}{(s-a)^2+k^2}$$

from the inside-front-cover table (with $a=-3/2,\ k=\sqrt{3}/2$) finally yield

$$x(t) = \frac{1}{3}\left\{2+e^{-3t/2}\left[\cos\left(\sqrt{3}t/2\right)+\sqrt{3}\sin\left(\sqrt{3}t/2\right)\right]\right\}$$

$$y(t) = \frac{1}{21}\left\{28-9e^t+e^{-3t/2}\left[2\cos\left(\sqrt{3}t/2\right)+8\sqrt{3}\sin\left(\sqrt{3}t/2\right)\right]\right\}.$$

16. The transformed equations are

$$\begin{aligned} s\,X(s) - 1 &= X(s) + Z(s) \\ s\,Y(s) &= X(s) + Y(s) \\ s\,Z(s) &= -2X(s) - Z(s), \end{aligned}$$

which we solve for

$$X(s) = \frac{s^2-1}{(s-1)(s^2+1)} = \frac{s+1}{s^2+1}$$

$$Y(s) = \frac{s+1}{(s-1)(s^2+1)} = \frac{1}{s-1}-\frac{s}{s^2+1}$$

$$Z(s) = \frac{-2s+2}{(s-1)(s^2+1)} = -\frac{2}{s^2+1}.$$

Hence the solution is

$$x(t) = \cos t + \sin t$$
$$y(t) = e^t - \cos t$$
$$z(t) = -2 \sin t.$$

17. $f(t) = \int_0^t e^{3\tau}\,d\tau = \left[\frac{1}{3}e^{3\tau}\right]_{\tau=0}^{t} = \frac{1}{3}\left(e^{3t}-1\right)$

18. $f(t) = \int_0^t 3e^{-5\tau}\,d\tau = \left[-\frac{3}{5}e^{-5\tau}\right]_{\tau=0}^{t} = \frac{3}{5}\left(1-e^{-5t}\right)$

19. $f(t) = \int_0^t \tfrac{1}{2}\sin 2\tau\,d\tau = \left[-\frac{1}{4}\cos 2\tau\right]_{\tau=0}^{t} = \frac{1}{4}\left(1-\cos 2t\right)$

20. $f(t) = \int_0^t \left(2\cos 3\tau + \tfrac{1}{3}\sin 3\tau\right)d\tau = \left[\frac{2}{3}\sin 3\tau - \frac{1}{9}\cos 3\tau\right]_{\tau=0}^{t} = \frac{1}{9}\left(6\sin 3t - \cos 3t + 1\right)$

21. $f(t) = \int_0^t \left[\int_0^\tau \sin t\,dt\right]d\tau = \int_0^t (1-\cos\tau)\,d\tau = \left[\tau - \sin\tau\right]_{\tau=0}^{t} = t - \sin t$

22. $f(t) = \int_0^t \tfrac{1}{3}\sinh 3\tau\,d\tau = \left[\frac{1}{9}\cosh 3\tau\right]_{\tau=0}^{t} = \frac{1}{9}\left(\cosh 3t - 1\right)$

23. $f(t) = \int_0^t \left[\int_0^\tau \sinh t\,dt\right]d\tau = \int_0^t (\cosh\tau - 1)\,d\tau = \left[\sinh\tau - \tau\right]_{\tau=0}^{t} = \sinh t - t$

24. $f(t) = \int_0^t \left(e^{-\tau} - e^{-2\tau}\right)d\tau = \left[-e^{-\tau} + \frac{1}{2}e^{-2\tau}\right]_{\tau=0}^{t} = \frac{1}{2}\left(e^{-2t} - 2e^{-t} + 1\right)$

25. With $f(t) = \cos kt$ and $F(s) = s/(s^2+k^2)$, Theorem 1 in this section yields

$$\mathcal{L}\{-k\sin kt\} = \mathcal{L}\{f'(t)\} = sF(s) - 1 = s\cdot\frac{s}{s^2+k^2} - 1 = -\frac{k^2}{s^2+k^2},$$

so division by $-k$ yields $\mathcal{L}\{\sin kt\} = k/(s^2+k^2)$.

26. With $f(t) = \sinh kt$ and $F(s) = k/(s^2 - k^2)$, Theorem 1 yields

$$\mathcal{L}\{f'(t)\} = \mathcal{L}\{k\cosh kt\} = ks/(s^2 - k^2) = sF(s),$$

so it follows upon division by k that $\mathcal{L}\{\cosh kt\} = s/(s^2 - k^2)$.

27. **(a)** With $f(t) = t^n e^{at}$ and $f'(t) = nt^{n-1}e^{at} + at^n e^{at}$, Theorem 1 yields

$$\mathcal{L}\{nt^{n-1}e^{at} + at^n e^{at}\} = s\,\mathcal{L}\{t^n e^{at}\}$$

so

$$n\,\mathcal{L}\{t^{n-1}e^{at}\} = (s - a)\mathcal{L}\{t^n e^{at}\}$$

and hence

$$\mathcal{L}\left\{t^n e^{at}\right\} = \frac{n}{s-a}\mathcal{L}\left\{t^{n-1}e^{at}\right\}.$$

(b) $n=1:\ \mathcal{L}\left\{t\,e^{at}\right\} = \dfrac{1}{s-a}\mathcal{L}\left\{e^{at}\right\} = \dfrac{1}{s-a}\cdot\dfrac{1}{s-a} = \dfrac{1}{(s-a)^2}$

$n=2:\ \mathcal{L}\left\{t^2\,e^{at}\right\} = \dfrac{2}{s-a}\mathcal{L}\left\{t\,e^{at}\right\} = \dfrac{2}{s-a}\cdot\dfrac{1}{(s-a)^2} = \dfrac{2!}{(s-a)^3}$

$n=3:\ \mathcal{L}\left\{t^3\,e^{at}\right\} = \dfrac{3}{s-a}\mathcal{L}\left\{t^2\,e^{at}\right\} = \dfrac{3}{s-a}\cdot\dfrac{2!}{(s-a)^3} = \dfrac{3!}{(s-a)^4}$

And so forth.

28. Problems 28 and 30 are the trigonometric and hyperbolic versions of essentially the same computation. For Problem 30 we let $f(t) = t\cosh kt$, so $f(0) = 0$. Then

$$f'(t) = \cosh kt + kt\sinh kt$$

$$f''(t) = 2k\sinh kt + k^2 t\cosh kt,$$

and thus $f'(0) = 1$, so Formula (5) in this section yields

$$\mathcal{L}\{2k\sinh kt + k^2 t\cosh kt\} = s^2\mathcal{L}\{t\cosh kt\} - 1,$$

$$2k\cdot\frac{k}{s^2-k^2} + k^2F(s) = s^2F(s) - 1.$$

We readily solve this last equation for

$$\mathcal{L}\{t\cosh kt\} = F(s) = \frac{s^2+k^2}{\left(s^2-k^2\right)^2}.$$

29. Let $f(t) = t\sinh kt$, so $f(0) = 0$. Then

$$f'(t) = \sinh kt + kt \cosh kt$$

$$f''(t) = 2k \cosh kt + k^2 t \sinh kt,$$

and thus $f'(0) = 0$, so Formula (5) in this section yields

$$\mathcal{L}\{2k \cosh kt + k^2 t \sinh kt\} = s^2 \mathcal{L}\{\sinh kt\},$$

$$2k \cdot \frac{s}{s^2 - k^2} + k^2 F(s) = s^2 F(s).$$

We readily solve this last equation for

$$\mathcal{L}\{t \cosh kt\} = F(s) = \frac{2ks}{\left(s^2 - k^2\right)^2}.$$

30. See Problem 28.

31. Using the known transform of $\sin kt$ and the Problem 28 transform of $t \cos kt$, we obtain

$$\mathcal{L}\left\{\frac{1}{2k^3}\left(\sin kt - kt \cos kt\right)\right\} = \frac{1}{2k^3} \cdot \frac{k}{s^2 + k^2} - \frac{k}{2k^3} \cdot \frac{s^2 - k^2}{\left(s^2 + k^2\right)^2}$$

$$= \frac{1}{2k^2}\left(\frac{1}{s^2 + k^2} - \frac{s^2 - k^2}{\left(s^2 + k^2\right)^2}\right) = \frac{1}{2k^2} \cdot \frac{2k^2}{\left(s^2 + k^2\right)^2} = \frac{1}{\left(s^2 + k^2\right)^2}$$

32. If $f(t) = u(t - a)$, then the only jump in $f(t)$ is $j_1 = 1$ at $t_1 = a$. Since $f(0) = 0$ and $f'(t) = 0$, Formula (21) in this section yields

$$0 = s\,F(\mathrm{s}) - 0 - e^{as}(1).$$

Hence $\mathcal{L}\{u(t - a)\} = F(s) = s^{-1} e^{-as}$.

33. $f(t) = u_a(t) - u_b(t) = u(t - a) - u(t - b)$, so the result of Problem 32 gives

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{u(t-a)\} - \mathcal{L}\{u(t-b)\} = \frac{e^{-as}}{s} - \frac{e^{-bs}}{s} = \frac{e^{-as} - e^{-bs}}{s}.$$

34. The square wave function of Figure 7.2.9 has a sequence $\{t_n\}$ of jumps with $t_n = n$ and $j_n = 2(-1)^n$ for $n = 1, 2, 3, \ldots$. Hence Formula (21) yields

$$0 = s\,F(s) - 1 - \sum_{n=1}^{\infty} e^{-ns} \cdot 2(-1)^n.$$

It follows that

$$
\begin{aligned}
s\,F(s) &= 1 + 2\sum_{n=1}^{\infty}(-1)^n e^{-ns} \\
&= -1 + 2(1 - e^{-s} + e^{-2s} - e^{-3s} + \cdots) \\
&= -1 + 2/(1 + e^{-s}) \\
&= (1 - e^{-s})/(1 + e^{-s}) \\
&= (e^{s/2} - e^{-s/2})/(e^{s/2} + e^{-s/2}) \\
s\,F(s) &= \tanh(s/2),
\end{aligned}
$$

because $2\cosh(s/2) = e^{s/2} + e^{-s/2}$ and $2\sinh(s/2) = e^{s/2} - e^{-s/2}$.

35. Let's write $g(t)$ for the on-off function of this problem to distinguish it from the square wave function of Problem 34. Then comparison of Figures 7.2.9 and 7.2.10 makes it clear that $g(t) = \tfrac{1}{2}\big(1 + f(t)\big)$, so (using the result of Problem 34) we obtain

$$
\begin{aligned}
G(s) &= \frac{1}{2s} + \frac{1}{2}F(s) = \frac{1}{2s} + \frac{1}{2s}\tanh\frac{s}{2} = \frac{1}{2s}\left(1 + \frac{e^{s/2} - e^{-s/2}}{e^{s/2} + e^{-s/2}} \cdot \frac{e^{-s/2}}{e^{-s/2}}\right) \\
&= \frac{1}{2s}\left(1 + \frac{1 - e^{-s}}{1 + e^{-s}}\right) = \frac{1}{2s} \cdot \frac{2}{1 + e^{-s}} = \frac{1}{s\left(1 + e^{-s}\right)}.
\end{aligned}
$$

36. If $g(t)$ is the triangular wave function of Figure 7.2.11 and $f(t)$ is the square wave function of Problem 34, then $g'(t) = f(t)$. Hence Theorem 1 and the result of Problem 34 yield

$$
\begin{aligned}
\mathcal{L}\{g'(t)\} &= s\,\mathcal{L}\{g(t)\} - g(0), \\
F(s) &= s\,G(s), \qquad \text{(because } g(0) = 0\text{)} \\
\mathcal{L}\{g(t)\} &= s^{-1}F(s) = s^{-2}\tanh(s/2).
\end{aligned}
$$

37. We observe that $f(0) = 0$ and that the sawtooth function has jump -1 at each of the points $t_n = n = 1, 2, 3, \cdots$. Also, $f'(t) \equiv 1$ wherever the derivative is defined. Hence Eq. (21) in this section gives

$$
\frac{1}{s} = sF(s) + \sum_{n=1}^{\infty} e^{-ns} = sF(s) - 1 + \sum_{n=0}^{\infty} e^{-ns} = sF(s) - 1 + \frac{1}{1 - e^{-ns}},
$$

using the geometric series $\sum_{n=0}^{\infty} x^n = 1/(1-x)$ with $x = e^{-s}$. Solution for $F(s)$ gives

$$F(s) = \frac{1}{s^2} + \frac{1}{s} - \frac{1}{s\left(1-e^{-s}\right)} = \frac{1}{s^2} - \frac{e^{-s}}{s\left(1-e^{-s}\right)}.$$

SECTION 10.3

TRANSLATION AND PARTIAL FRACTIONS

This section is devoted to the computational nuts and bolts of the staple technique for the inversion of Laplace transforms — partial fraction decompositions. If time does not permit going further in this chapter, Sections 10.1–10.3 provide a self-contained introduction to Laplace transforms that suffices for the most common elementary applications.

1. $\mathcal{L}\{t^4\} = \dfrac{24}{s^5}$, so $\mathcal{L}\{t^4 e^{\pi t}\} = \dfrac{24}{(s-\pi)^5}$

2. $\mathcal{L}\{t^{3/2}\} = \dfrac{3\sqrt{\pi}}{4s^{5/2}}$, so $\mathcal{L}\{t^{3/2}\, e^{-4t}\} = \dfrac{3\sqrt{\pi}}{4(s+4)^{5/2}}$.

3. $\mathcal{L}\{\sin 3\pi t\} = \dfrac{3\pi}{s^2+9\pi^2}$, so $\mathcal{L}\{e^{-2t}\sin 3\pi t\} = \dfrac{3\pi}{(s+2)^2+9\pi^2}$.

4.
$$\cos 2\left(t-\frac{\pi}{8}\right) = \cos\left(2t-\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}\left(\cos 2t + \sin 2t\right)$$
$$\mathcal{L}\left\{\cos 2\left(t-\frac{\pi}{8}\right)\right\} = \frac{1}{\sqrt{2}}\frac{s+2}{s^2+4}$$
$$\mathcal{L}\left\{e^{-t/2}\cos 2\left(t-\frac{\pi}{8}\right)\right\} = \frac{1}{\sqrt{2}}\frac{\left(s+1/2\right)+2}{\left(s+1/2\right)^2+4} = \frac{1}{\sqrt{2}}\frac{2s+5}{4s^2+4s+17}$$

5. $F(s) = \dfrac{3}{2s-4} = \dfrac{3}{2}\cdot\dfrac{1}{s-2}$, so $f(t) = \dfrac{3}{2}e^{2t}$

6. $F(s) = \dfrac{(s+1)-2}{(s+1)^3} = \dfrac{1}{(s+1)^2} - \dfrac{2}{(s+1)^3}$, so $f(t) = te^{-t} - t^2e^{-t} = e^{-t}\left(t-t^2\right)$

7. $F(s) = \dfrac{1}{\left(s+2\right)^2}$, so $f(t) = t\,e^{-2t}$

8. $F(s) = \dfrac{s+2}{\left(s+2\right)^2+1}$, so $f(t) = e^{-2t}\cos t$

9. $F(s) = 3\cdot\dfrac{s-3}{(s-3)^2+16} + \dfrac{7}{2}\cdot\dfrac{4}{(s-3)^2+16}$, so $f(t) = e^{3t}[3\cos 4t + (7/2)\sin 4t]$

10. $$F(s) = \frac{2s-3}{(3s-2)^2+16} = \frac{1}{9}\cdot\frac{2s-3}{(s-2/3)^2+16/9}$$
$$= \frac{2}{9}\cdot\frac{s-2/3}{(s-2/3)^2+(4/3)^2} - \frac{5}{36}\cdot\frac{4/3}{(s-2/3)^2+(4/3)^2}$$

$$f(t) = \frac{1}{36}e^{2t/3}\left(8\cos\frac{4t}{3} - 5\sin\frac{4t}{3}\right)$$

11. $F(s) = \dfrac{1}{4}\cdot\dfrac{1}{s-2} - \dfrac{1}{4}\cdot\dfrac{1}{s+2}$, so $f(t) = \dfrac{1}{4}\left(e^{2t} - e^{-2t}\right) = \dfrac{1}{2}\sinh 2t$

12. $F(s) = 2\cdot\dfrac{1}{s} + 3\cdot\dfrac{1}{s-3}$, so $f(t) = 2 + 3e^{3t}$

13. $F(s) = 3\cdot\dfrac{1}{s+2} - 5\cdot\dfrac{1}{s+5}$, so $f(t) = 3e^{-2t} - 5e^{-5t}$

14. $F(s) = 2\cdot\dfrac{1}{s} - 3\cdot\dfrac{1}{s+1} + \dfrac{1}{s-2}$, so $f(t) = 2 - 3e^{-t} + e^{2t}$

15. $F(s) = \dfrac{1}{25}\left(-1\cdot\dfrac{1}{s} - 5\cdot\dfrac{1}{s^2} + \dfrac{1}{s-5}\right)$, so $f(t) = \dfrac{1}{25}\left(-1 - 5t + e^{5t}\right)$

16. $$F(s) = \frac{1}{(s+3)^2(s-2)^2} = \frac{1}{125}\left(\frac{2}{s+3} + \frac{5}{(s+3)^2} - \frac{2}{s-2} + \frac{5}{(s-2)^2}\right)$$

$$f(t) = \frac{1}{125}\left[e^{-3t}(2+5t) + e^{2t}(-2+5t)\right]$$

17. $$F(s) = \frac{1}{8}\left(\frac{1}{s^2-4} - \frac{1}{s^2+4}\right) = \frac{1}{16}\left(\frac{2}{s^2-4} - \frac{2}{s^2+4}\right)$$
$$f(t) = \frac{1}{16}(\sinh 2t - \sin 2t)$$

18. $$F(s) = \frac{1}{s-4} + \frac{1}{(s-4)^2} + \frac{48}{(s-4)^3} + \frac{64}{(s-4)^4}$$

$$f(t) = e^{4t}\left(1+12t+24t^2+\frac{32}{3}t^3\right)$$

19. $$F(s) = \frac{s^2-2s}{\left(s^2+1\right)\left(s^2+4\right)} = \frac{1}{3}\left(\frac{-2s-1}{s^2+1}+\frac{2s+4}{s^2+4}\right)$$

$$f(t) = \frac{1}{3}\left(-2\cos t - \sin t + 2\cos 2t + 2\sin 2t\right)$$

20. $$F(s) = \frac{1}{\left(s^2-4\right)^2} = \frac{1}{(s-2)^2(s+2)^2} = \frac{1}{32}\left(\frac{1}{s+2}+\frac{2}{(s+2)^2}-\frac{1}{s-2}+\frac{2}{(s-2)^2}\right)$$

$$f(t) = \frac{1}{32}\left[e^{-2t}(1+2t)+e^{2t}(-1+2t)\right]$$

21. First we need to find A, B, C, D so that

$$\frac{s^2+3}{\left(s^2+2s+2\right)^2} = \frac{As+B}{s^2+2s+2}+\frac{Cs+D}{\left(s^2+2s+2\right)^2}.$$

When we multiply both sides by the quadratic factor s^2+2s+2 and collect coefficients, we get the linear equations

$$\begin{aligned} -2B-D+3 &= 0 \\ -2A-2B-C &= 0 \\ -2A-B+1 &= 0 \\ -A &= 0 \end{aligned}$$

which we solve for $A=0,\ B=1,\ C=-2,\ D=1$. Thus

$$F(s) = \frac{1}{(s+1)^2+1}+\frac{-2s+1}{\left[(s+1)^2+1\right]^2} = \frac{1}{(s+1)^2+1}-2\cdot\frac{s+1}{\left[(s+1)^2+1\right]^2}+3\cdot\frac{1}{\left[(s+1)^2+1\right]^2}.$$

We now use the inverse Laplace transforms given in Eq. (16) and (17) of Section 10.3 — supplying the factor e^{-t} corresponding to the translation $s \to s+1$ — and get

$$f(t) = e^{-t}\left[\sin t - 2\cdot\frac{1}{2}t\sin t + 3\cdot\frac{1}{2}(\sin t - t\cos t)\right] = \frac{1}{2}e^{-t}\left(5\sin t - 2t\sin t - 3t\cos t\right).$$

22. First we need to find A, B, C, D so that

$$\frac{2s^3 - s^2}{\left(4s^2 - 4s + 5\right)^2} = \frac{As + B}{4s^2 - 4s + 5} + \frac{Cs + D}{\left(4s^2 - 4s + 5\right)^2}.$$

When we multiply each side by the quadratic factor (squared) 5 we get the identity

$$2s^3 - s^2 = (As + B)(4s^2 - 4s + 5) + Cs + D.$$

When we substitute the root $s = 1/2 + i$ of the quadratic into this identity, we find that $C = -3/2$ and $D = -5/4$. When we first differentiate each side of the identity and then substitute the root, we find that $A = 1/2$ and $B = 1/4$. Writing

$$4s^2 - 4s + 5 = 4[(s - 1/2)^2 + 1],$$

it follows that

$$F(s) = \frac{1}{8} \cdot \frac{\left(s - \frac{1}{2}\right) + 1}{\left(s - \frac{1}{2}\right)^2 + 1} - \frac{1}{32} \cdot \frac{3\left(s - \frac{1}{2}\right) + 4}{\left[\left(s - \frac{1}{2}\right)^2 + 1\right]^2}.$$

Finally the results

$$\mathcal{L}^{-1}\{2s/(s^2 + 1)^2\} = t \sin t$$

$$\mathcal{L}^{-1}\{2/(s^2 + 1)^2\} = \sin t - t \cos t$$

of Eqs. (16) and (17) in Section 10.3, together with the translation theorem, yield

$$\begin{aligned} f(t) &= e^{t/2}\left[\frac{1}{8} \cdot (\cos t + \sin t) - \frac{3}{32} \cdot \frac{1}{2} t \sin t - \frac{4}{32} \cdot \frac{1}{2}(\sin t - t \cos t)\right] \\ &= \frac{1}{64} e^{t/2}\left[(8 + 4t)\cos t + (4 - 3t)\sin t\right]. \end{aligned}$$

23. $$\frac{s^3}{s^4 + 4a^4} = \frac{1}{2}\left(\frac{s - a}{s^2 - 2as + 2a^2} + \frac{s + a}{s^2 + 2as + 2a^2}\right),$$

and $s^2 \pm 2as + 2a^2 = (s \pm a)^2 + a^2$, so it follows that

$$\mathcal{L}^{-1}\left\{\frac{s^3}{s^4 + 4a^4}\right\} = \frac{1}{2}\left(e^{at} + e^{-at}\right)\cos at = \cosh at \cos at.$$

24. $$\frac{s}{s^4 + 4a^4} = \frac{1}{4a^2}\left(\frac{a}{s^2 - 2as + 2a^2} - \frac{a}{s^2 + 2as + 2a^2}\right),$$

and $s^2 \pm 2as + 2a^2 = (s \pm a)^2 + a^2$, so it follows that

$$\mathcal{L}^{-1}\left\{\frac{s^3}{s^4+4a^4}\right\} = \frac{1}{4a^2}\left(e^{at}-e^{-at}\right)\sin at = \frac{1}{2a^2}\sinh at \sin at.$$

25. $$\frac{s}{s^4+4a^4} = \frac{1}{4a}\left(\frac{s}{s^2-2as+2a^2}-\frac{s}{s^2+2as+2a^2}\right)$$
$$= \frac{1}{4a}\left(\frac{s-a}{s^2-2as+2a^2}+\frac{a}{s^2-2as+2a^2}-\frac{s+a}{s^2+2as+2a^2}+\frac{a}{s^2+2as+2a^2}\right),$$

and $s^2 \pm 2as + 2a^2 = (s \pm a)^2 + a^2$, so it follows that

$$\mathcal{L}^{-1}\left\{\frac{s}{s^4+4a^4}\right\} = \frac{1}{4a}\left[e^{at}(\cos at+\sin at)-e^{-at}\left(\cos at-\sin at\right)\right]$$
$$= \frac{1}{2a}\left[\frac{1}{2}\left(e^{at}+e^{-at}\right)\sin at+\frac{1}{2}\left(e^{at}-e^{-at}\right)\cos at\right]$$
$$= \frac{1}{2a}\left(\cosh at \sin at+\sinh at \cos at\right).$$

26. $$\frac{1}{s^4+4a^4} = \frac{1}{8a^3}\left(\frac{-s+2a}{s^2-2as+2a^2}+\frac{s+2a}{s^2+2as+2a^2}\right)$$
$$= \frac{1}{8a^3}\left(-\frac{s-a}{s^2-2as+2a^2}+\frac{a}{s^2-2as+2a^2}+\frac{s+a}{s^2+2as+2a^2}+\frac{a}{s^2+2as+2a^2}\right),$$

and $s^2 \pm 2as + 2a^2 = (s \pm a)^2 + a^2$, so it follows that

$$\mathcal{L}^{-1}\left\{\frac{s}{s^4+4a^4}\right\} = \frac{1}{8a^3}\left[e^{at}(-\cos at+\sin at)+e^{-at}\left(\cos at+\sin at\right)\right]$$
$$= \frac{1}{4a^3}\left[\frac{1}{2}\left(e^{at}+e^{-at}\right)\sin at-\frac{1}{2}\left(e^{at}-e^{-at}\right)\cos at\right]$$
$$= \frac{1}{4a^3}\left(\cosh at \sin at-\sinh at \cos at\right).$$

In Problems 27–40 we give first the transformed equation, then the Laplace transform $X(s)$ of the solution, and finally the desired solution $x(t)$.

27. $[s^2X(s) - 2s - 3] + 6[sX(s) - 2] + 25X(s) = 0$

$$X(s) = \frac{2s+15}{s^2+6s+25} = 2\cdot\frac{s+3}{(s+3)^2+16}+\frac{9}{4}\cdot\frac{4}{(s+3)^2+16}$$

$x(t) = e^{-3t}[2 \cos 4t + (9/4)\sin 4t]$

28. $s^2X(s) - 6sX(s) + 8X(s) = \dfrac{2}{s}$

$$X(s) = \frac{2}{s(s^2 - 6s + 8)} = \frac{1}{4}\left(\frac{1}{s} + \frac{1}{s-4} - \frac{2}{s-2}\right)$$

$$x(t) = \frac{1}{4}\left(1 + e^{4t} - 2e^{2t}\right)$$

29. $s^2X(s) - 4X(s) = \dfrac{3}{s^2}$

$$X(s) = \frac{3}{s^2\left(s^2 - 4\right)} = \frac{3}{4}\left(\frac{1}{s^2 - 4} - \frac{1}{s^2}\right)$$

$$x(t) = \frac{3}{8}\sinh 2t - \frac{3}{4}t = \frac{3}{8}\left(\sinh 2t - 2t\right)$$

30. $s^2X(s) + 4sX(s) + 8X(s) = \dfrac{1}{s+1}$

$$X(s) = \frac{1}{(s+1)\left(s^2 + 4s + 8\right)} = \frac{1}{5}\left(\frac{1}{s+1} - \frac{s+3}{s^2 + 4s + 8}\right)$$

$$= \frac{1}{5}\left(\frac{1}{s+1} - \frac{s+2}{(s+2)^2 + 4} - \frac{1}{2}\cdot\frac{2}{(s+2)^2 + 4}\right)$$

$$x(t) = \frac{1}{10}\left[2e^{-t} - e^{-2t}(2\cos 2t + \sin 2t)\right]$$

31. $[s^3X(s) - s - 1] + [s^2X(s) - 1] - 6[sX(s)] = 0$

$$X(s) = \frac{s+2}{s^3 + s^2 - 6s} = \frac{1}{15}\left(-\frac{5}{s} - \frac{1}{s+3} + \frac{6}{s-2}\right)$$

$$x(t) = \frac{1}{15}\left(-5 - e^{-3t} + 6e^{2t}\right)$$

32. $\left[s^4X(s) - s^3\right] - X(s) = 0$

$$X(s) = \frac{s^3}{s^4 - 1} = \frac{1}{2}\left(\frac{s}{s^2 + 1} + \frac{s}{s^2 - 1}\right)$$

$$x(t) = \frac{1}{2}\left(\cos t + \cosh t\right)$$

33. $[s^4 X(s) - 1] + X(s) = 0$

$$X(s) = \frac{1}{s^4+1}$$

It therefore follows from Problem 26 with $a = \sqrt[4]{1/4} = 1/\sqrt{2}$ that

$$x(t) = \frac{1}{\sqrt{2}}\left(\cosh\frac{t}{\sqrt{2}}\sin\frac{t}{\sqrt{2}} - \sinh\frac{t}{\sqrt{2}}\cos\frac{t}{\sqrt{2}}\right).$$

34. $[s^4 X(s) - 2s^2 + 13] + 13[s^2 X(s) - 2] + 36\,X(s) = 0$

$$X(s) = \frac{2s^2+13}{s^4+13s^2+36} = \frac{1}{s^2+4} + \frac{1}{s^2+9}$$

$$x(t) = \frac{1}{2}\sin 2t + \frac{1}{3}\sin 3t$$

35. $\left[s^4 X(s) - 1\right] + 8s^2 X(s) + 16X(s) = 0$

$$X(s) = \frac{1}{s^4+8s^2+16} = \frac{1}{\left(s^2+4\right)^2}$$

$$x(t) = \frac{1}{16}\left(\sin 2t - 2t\cos 2t\right) \qquad \text{(by Eq. (17) in Section 10.3)}$$

36. $s^4 X(s) + 2s^2 X(s) + X(s) = \dfrac{1}{s-2}$

$$X(s) = \frac{1}{(s-2)\left(s^4+2s^2+1\right)} = \frac{1}{25}\left(\frac{1}{s-2} - \frac{s+2}{s^2+1} - \frac{5(s+2)}{\left(s^2+1\right)^2}\right)$$

$$x(t) = \frac{1}{25}\left(e^{-2t} - \cos t - 2\sin t - 5\cdot\frac{1}{2}t\sin t - 10\cdot\frac{1}{2}\left(\sin t - t\cos t\right)\right)$$

$$= \frac{1}{50}\left[2e^{2t} + (10t-2)\cos t - (5t+14)\sin t\right]$$

37. $\left[s^2 X(s) - 2\right] + 4sX(s) + 13X(s) = \dfrac{1}{(s+1)^2}$

$$X(s) = \frac{2 + 1/(s+1)^2}{s^2+4s+13} = \frac{2s^2+4s+13}{(s+1)^2\left(s^2+4s+13\right)}$$

$$= \frac{1}{50}\left[-\frac{1}{s+1}+\frac{5}{(s+1)^2}+\frac{s+98}{(s+2)^2+9}\right]$$

$$= \frac{1}{50}\left[-\frac{1}{s+1}+\frac{5}{(s+1)^2}+\frac{s+2}{(s+2)^2+9}+32\cdot\frac{3}{(s+2)^2+9}\right]$$

$$x(t) = \frac{1}{50}\left[(-1+5t)e^{-t}+e^{-2t}(\cos 3t+32\sin 3t)\right]$$

38. $\left[s^2X(s)-s+1\right]+6\left[sX(s)-1\right]+18X(s) = \frac{s}{s^2+4}$

$$X(s) = \frac{s+5}{s^2+6s+18}+\frac{s}{\left(s^2+4\right)\left(s^2+6s+18\right)}$$

$$= \frac{s+5}{s^2+6s+18}+\frac{1}{170}\left(\frac{7s+12}{s^2+4}-\frac{7s+54}{s^2+6s+18}\right)$$

$$= \frac{1}{170}\left(\frac{7s+12}{s^2+4}+\frac{163s+796}{s^2+6s+18}\right)$$

$$X(s) = \frac{1}{170}\left(\frac{7s+12}{s^2+4}+\frac{163(s+3)}{\left(s+3\right)^2+9}+\frac{307}{\left(s+3\right)^2+9}\right)$$

$$x(t) = \frac{1}{170}\left(7\cos 2t+6\sin 2t\right)+\frac{1}{510}e^{-3t}\left(489\cos 3t+307\sin 3t\right)$$

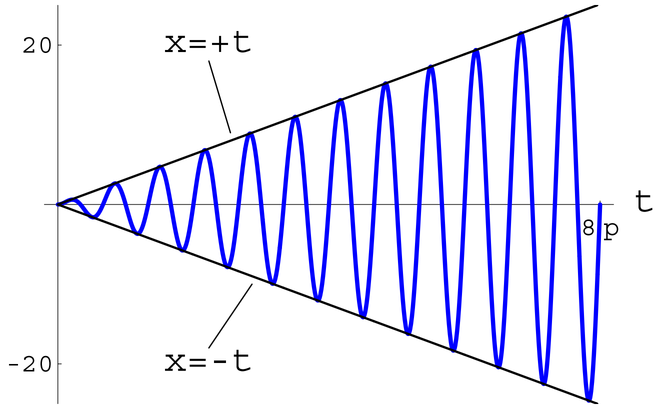
39. $x''+9x = 6\cos 3t, \quad x(0)=x'(0)=0$

$$s^2X(s)+9X(s) = \frac{6s}{s^2+9}$$

$$X(s) = \frac{6s}{\left(s^2+9\right)^2}$$

$$x(t) = 6\cdot\frac{1}{2\cdot 3}t\sin 3t = t\sin 3t \qquad \text{(by Eq. (16) in Section 10.3)}$$

The graph of this resonance is shown in the figure at the top of the next page.



40. $x'' + 0.4x' + 9.04x \;=\; x'' + \dfrac{2}{5}x' + \dfrac{226}{25} \;=\; 6e^{-t/5}\cos 3t$

$$\left(s^2 + \frac{2}{5}s + \frac{226}{25}\right)X(s) \;=\; \frac{6(s+1/5)}{(s+1/5)^2+9}$$

$$X(s) \;=\; \frac{6(s+1/5)}{\left[(s+1/5)^2+9\right]^2}$$

$$x(t) \;=\; t\,e^{-t/5}\sin 3t \qquad \text{(by Eq. (16) in Section 10.3)}$$

SECTION 10.4

DERIVATIVES, INTEGRALS, AND PRODUCTS OF TRANSFORMS

This section completes the presentation of the standard "operational properties" of Laplace transforms, the most important one here being the convolution property $\mathcal{L}\{f*g\} \;=\; \mathcal{L}\{f\}\cdot\mathcal{L}\{g\}$, where the **convolution** $f*g$ is defined by

$$f*g(t) \;=\; \int_0^t f(x)g(t-x)\,dx.$$

Here we use x rather than τ as the variable of integration; compare with Eq. (3) in Section 10.4 of the textbook

1. With $f(t) = t$ and $g(t) = 1$ we calculate

$$t * 1 = \int_0^t x \cdot 1\,dx = \left[\frac{1}{2}x^2\right]_{x=0}^{x=t} = \frac{1}{2}t^2.$$

2. With $f(t) = t$ and $g(t) = e^{at}$ we calculate

$$\begin{aligned}
t * e^{at} &= \int_0^t x \cdot e^{a(t-x)}\,dx = e^{at}\int_0^t x \cdot e^{-ax}\,dx \\
&= e^{at}\int_-^- \left(-\frac{u}{a}\right)e^u\left(-\frac{du}{a}\right) = \frac{e^{at}}{a^2}\int_-^- ue^u\,du \qquad \text{(with } u = -ax\text{)} \\
&= \frac{e^{at}}{a^2}\Big[(u-1)e^u\Big]_-^- \qquad \text{(integral formula \#46 inside back cover)} \\
&= \frac{e^{at}}{a^2}\Big[(-ax-1)e^{-ax}\Big]_{x=0}^{x=t} = \frac{e^{at}}{a^2}\Big[(-at-1)e^{-at}+1\Big] \\
t * e^{at} &= \frac{1}{a^2}\left(e^{at} - at - 1\right).
\end{aligned}$$

3. To compute $(\sin t) * (\sin t) = \int_0^t \sin x \sin(t-x)\,dx,$ we first apply the identity $\sin A \sin B = [\cos(A - B) - \cos(A + B)]/2.$ This gives

$$\begin{aligned}
(\sin t) * (\sin t) &= \int_0^t \sin x \sin(t-x)\,dx \\
&= \frac{1}{2}\int_0^t \big[\cos(2x-t) - \cos t\big]dx \\
&= \frac{1}{2}\left[\frac{1}{2}\sin(2x-t) - x\cos t\right]_{x=0}^{x=t} \\
(\sin t) * (\sin t) &= \frac{1}{2}\left(\sin t - t\cos t\right).
\end{aligned}$$

4. To compute $t^2 * \cos t = \int_0^t x^2 \cos(t-x)\,dx,$ we first substitute

$$\cos(t - x) = \cos t \cos x + \sin t \sin x,$$

and then use the integral formulas

$$\int x^2 \cos x\,dx = x^2 \sin x + 2x\cos x - 2\sin x + C$$

$$\int x^2 \sin x\,dx = -x^2 \cos x + 2x\sin x + 2\cos x + C.$$

from #40 and #41 inside the back cover of the textbook. This gives

$$\begin{aligned}
t^2 * \cos t &= \int_0^t x^2(\cos t\cos x + \sin t \sin x)\,dx \\
&= (\cos t)\int_0^t x^2 \cos x\,dx + (\sin t)\int_0^t x^2 \sin x\,dx \\
&= (\cos t)\Big[x^2 \sin x + 2x\cos x - 2\sin x\Big]_{x=0}^{x=t} \\
&\qquad + (\sin t)\Big[-x^2 \cos x + 2x \sin x + 2\cos x\Big]_{x=0}^{x=t} \\
t^2 * \cos t &= 2(t - \sin t).
\end{aligned}$$

5. $$e^{at} * e^{at} = \int_0^t e^{ax}e^{a(t-x)}dx = \int_0^t e^{at}dx = e^{at}\big[x\big]_{x=0}^{x=t} = t\,e^{at}$$

6. $$\begin{aligned}
e^{at} * e^{bt} &= \int_0^t e^{ax}e^{b(t-x)}dx = e^{bt}\int_0^t e^{(a-b)x}dx \\
&= e^{bt}\left[\frac{e^{(a-b)x}}{a-b}\right]_{x=0}^{x=t} = \frac{e^{bt}\left(e^{(a-b)t}-1\right)}{a-b} = \frac{e^{at}-e^{bt}}{a-b}
\end{aligned}$$

7. $$f(t) = 1 * e^{3t} = e^{3t} * 1 = \int_0^t e^{3x}\cdot 1\,dx = \frac{1}{3}\left(e^{3t}-1\right)$$

8. $$f(t) = 1 * \frac{1}{2}\sin 2t = \int_0^t \frac{1}{2}\sin 2x\,dx = \frac{1}{4}(1-\cos 2t)$$

9. $$\begin{aligned}
f(t) &= \frac{1}{9}\sin 3t * \sin 3t = \frac{1}{9}\int_0^t \sin 3x \sin 3(t-x)\,dx \\
&= \frac{1}{9}\int_0^t \sin 3x\big[\sin 3t\cos 3x - \cos 3t \sin 3x\big]dx \\
&= \frac{1}{9}\sin 3t\int_0^t \sin 3x\cos 3x\,dx - \frac{1}{9}\cos 3t\int_0^t \sin^2 3x\,dx \\
&= \frac{1}{9}\sin 3t\left[\frac{1}{6}\sin^2 3x\right]_{x=0}^{x=t} - \frac{1}{9}\cos 3t\left[\frac{1}{2}\left(x - \frac{1}{6}\sin 6x\right)\right]_{x=0}^{x=t} \\
f(t) &= \frac{1}{54}\left(\sin 3t - 3t\cos 3t\right)
\end{aligned}$$

10. $$f(t) = t * (\sin kt)/k = \frac{1}{k}\int_0^t \sin kx\cdot(t-x)\,dx$$

$$= \frac{t}{k}\int_0^t \sin kx\,dx - \frac{1}{k}\int_0^t x\sin kx\,dx = \frac{kt-\sin kt}{k^3}$$

11. $$f(t) = \cos 2t * \cos 2t = \int_0^t \cos 2x\,\cos 2(t-x)\,dx$$

$$= \int_0^t \cos 2x\left(\cos 2t\cos 2x + \sin 2t\sin 2x\right)dx$$

$$= (\cos 2t)\int_0^t \cos^2 2x\,dx + (\sin 2t)\int_0^t \cos 2x\sin 2x\,dx$$

$$= (\cos 2t)\left[\frac{1}{2}\left(x+\frac{1}{4}\sin 4x\right)\right]_{x=0}^{x=t} + (\sin 2t)\left[\frac{1}{4}\sin^2 2x\right]_{x=0}^{x=t}$$

$$f(t) = \frac{1}{4}\left(\sin 2t + 2t\cos 2t\right)$$

12. $$f(t) = (e^{-2t}\sin t)*(1) = \int_0^t e^{-2x}\sin x\,dx = \frac{1}{5}\left[1-e^{-2t}(\cos t + 2\sin t)\right]$$

13. $$f(t) = e^{3t} * \cos t = \int_0^t (\cos x)e^{3(t-x)}dx$$

$$= e^{3t}\int_0^t e^{-3x}\cos x\,dx$$

$$= e^{3t}\left[\frac{e^{-3x}}{10}\left(-3\cos x + \sin x\right)\right]_{x=0}^{x=t} \qquad \text{(by integral formula \#50)}$$

$$f(t) = \frac{1}{10}\left(3e^{3t} - 3\cos t + \sin t\right)$$

14. $$f(t) = \cos 2t * \sin t = \int_0^t \cos 2x\sin(t-x)\,dx$$

$$= \int_0^t \cos 2x\,(\sin t\cos x - \cos t\sin x)\,dx$$

$$= (\sin t)\int_0^t \cos 2x\cos x\,dx - (\cos t)\int_0^t \cos 2x\sin x\,dx$$

$$= \frac{1}{2}(\sin t)\int_0^t (\cos 3x + \cos x)\,dx - \frac{1}{2}(\cos t)\int_0^t (\sin 3x - \sin x)\,dx$$

$$f(t) = \frac{1}{3}\left(\cos t - \cos 2t\right)$$

15. $$\mathcal{L}\{t\sin t\} = -\frac{d}{ds}\left(\mathcal{L}\{\sin t\}\right) = -\frac{d}{ds}\left(\frac{3}{s^2+9}\right) = \frac{6s}{\left(s^2+9\right)^2}$$

16. $\mathcal{L}\{t^2\cos 2t\} = \dfrac{d^2}{ds^2}\left(\mathcal{L}\{\cos 2t\}\right) = \dfrac{d^2}{ds^2}\left(\dfrac{s}{s^2+4}\right) = \dfrac{2s(s^2-12)}{\left(s^2+4\right)^3}$

17. $\mathcal{L}\{e^{2t}\cos 3t\} = (s-2)/(s^2-4s+13)$

$\mathcal{L}\{te^{2t}\cos 3t\} = -(d/ds)[(s-2)/(s^2-4s+13)] = (s^2-4s-5)/(s^2-4s+13)^2$

18. $\mathcal{L}\{\sin^2 t\} = \mathcal{L}\{(1-\cos 2t)/2\} = 2/s(s^2+4)$

$\mathcal{L}\{e^{-t}\sin^2 t\} = 2/[(s+1)(s^2+2s+5)]$

$\mathcal{L}\{te^{-t}\sin^2 t\} = -(d/ds)[2/((s+1)(\mathrm{s}^2+2\mathrm{s}+5))]$

$= 2(3s^2+6s+7)/[(s+1)^2(s^2+2s+5)^2]$

19. $\mathcal{L}\left\{\dfrac{\sin t}{t}\right\} = \displaystyle\int_s^\infty \frac{ds}{s^2+1} = \left[\tan^{-1}s\right]_s^\infty = \frac{\pi}{2}-\tan^{-1}s = \tan^{-1}\left(\frac{1}{s}\right)$

20. $\mathcal{L}\{1-\cos 2t\} = \dfrac{1}{s}-\dfrac{s}{s^2+4}$, so

$$\mathcal{L}\left\{\frac{1-\cos 2t}{t}\right\} = \int_s^\infty\left(\frac{1}{s}-\frac{s}{s^2+4}\right)ds = \left[\ln\left(\frac{s}{\sqrt{s^2+4}}\right)\right]_s^\infty = \ln\left(\frac{\sqrt{s^2+4}}{s}\right)$$

21. $\mathcal{L}\{e^{3t}-1\} = \dfrac{1}{s-3}-\dfrac{1}{s}$, so

$$\mathcal{L}\left\{\frac{e^{3t}-1}{t}\right\} = \int_s^\infty\left(\frac{1}{s-3}-\frac{1}{s}\right)ds = \left[\ln\left(\frac{s-3}{s}\right)\right]_s^\infty = \ln\left(\frac{s}{s-3}\right)$$

22. $\mathcal{L}\{e^t-e^{-t}\} = \dfrac{1}{s-1}-\dfrac{1}{s+1} = \dfrac{2}{s^2-1}$, so

$$\mathcal{L}\left\{\frac{e^t-e^{-t}}{t}\right\} = \int_s^\infty\left(\frac{1}{s-1}-\frac{1}{s+1}\right)ds = \left[\ln\left(\frac{s-1}{s+1}\right)\right]_s^\infty = \ln\left(\frac{s+1}{s-1}\right)$$

23. $f(t) = -\dfrac{1}{t}\mathcal{L}^{-1}\{F'(s)\} = -\dfrac{1}{t}\mathcal{L}^{-1}\left\{\dfrac{1}{s-2}-\dfrac{1}{s+2}\right\} = -\dfrac{1}{t}\left(e^{2t}-e^{-2t}\right) = -\dfrac{2\sinh 2t}{t}$

24. $f(t) = -\dfrac{1}{t}\mathcal{L}^{-1}\{F'(s)\} = -\dfrac{1}{t}\mathcal{L}^{-1}\left\{\dfrac{2s}{s^2+1}-\dfrac{2s}{s^2+4}\right\} = \dfrac{2}{t}\left(\cos 2t-\cos t\right)$

25. $f(t) = -\frac{1}{t}\mathcal{L}^{-1}\{F'(s)\} = -\frac{1}{t}\mathcal{L}^{-1}\left\{\frac{2s}{s^2+1}-\frac{1}{s+2}-\frac{1}{s-3}\right\} = \frac{1}{t}\left(e^{-2t}+e^{3t}-2\cos t\right)$

26. $f(t) = -\frac{1}{t}\mathcal{L}^{-1}\{F'(s)\} = -\frac{1}{t}\mathcal{L}^{-1}\left\{-\frac{3}{(s+2)^2+9}\right\} = \frac{e^{-2t}\sin 3t}{t}$

27. $f(t) = -\frac{1}{t}\mathcal{L}^{-1}\{F'(s)\} = -\frac{1}{t}\mathcal{L}^{-1}\left\{\frac{-2/s^3}{1+1/s^2}\right\}$

$$= \frac{2}{t}\mathcal{L}^{-1}\left\{\frac{1}{s^3+s}\right\} = \frac{2}{t}\mathcal{L}^{-1}\left\{\frac{1}{s}-\frac{s}{s^2+1}\right\} = \frac{2}{t}(1-\cos t)$$

28. An empirical approach works best with this one. We can construct transforms with powers of (s^2+1) in their denominators by differentiating the transforms of $\sin t$ and $\cos t$. Thus,

$$\mathcal{L}\{t\sin t\} = -\frac{d}{ds}\left(\frac{1}{s^2+1}\right) = \frac{2s}{\left(s^2+1\right)^2}$$

$$\mathcal{L}\{t\cos t\} = -\frac{d}{ds}\left(\frac{s}{s^2+1}\right) = \frac{s^2-1}{\left(s^2+1\right)^2}$$

$$\mathcal{L}\{t^2\cos t\} = -\frac{d}{ds}\left(\frac{s^2-1}{\left(s^2+1\right)^2}\right) = \frac{2s^3-6s}{\left(s^2+1\right)^3}.$$

From the first and last of these formulas it follows readily that

$$\mathcal{L}^{-1}\left\{\frac{s}{\left(s^2+1\right)^3}\right\} = \frac{1}{8}\left(t\sin t - t^2\cos t\right).$$

Alternatively, one could work out the repeated convolution

$$\mathcal{L}^{-1}\left\{\frac{s}{\left(s^2+1\right)^3}\right\} = (\cos t)*(\sin t*\sin t).$$

29. $-[s^2X(s) - x'(0)]' - [s\,X(s)]' - 2[s\,X(s)] + X(s) = 0$

$s(s+1)X'(s) + 4s\,X(s) = 0$ (separable)

$$X(s) = \frac{A}{(s+1)^4} \quad \text{with } A \neq 0$$

$$x(t) = Ct^3 e^{-t} \quad \text{with } C \neq 0$$

30. $-[s^2X(s) - x'(0)]' - 3[s\,X(s)]' - [s\,X(s)] + 3X(s) = 0$

$-(s^2 + 3s)X'(s) - 3s\,X(s) = 0$ (separable)

$$X(s) = \frac{A}{(s+3)^3} \quad \text{with } A \neq 0$$

$$x(t) = Ct^2 e^{-3t} \text{ with } C \neq 0$$

31. $-[s^2X(s) - x'(0)]' + 4[s\,X(s)]' - [s\,X(s)] - 4[X(s)]' + 2X(s) = 0$

$(s^2 - 4s + 4)X'(s) + (3s - 6)X(s) = 0$ (separable)

$(s - 2)X'(s) + 3X(s) = 0$

$$X(s) = \frac{A}{(s-2)^3} \quad \text{with } A \neq 0$$

$$x(t) = Ct^2 e^{2t} \quad \text{with } C \neq 0$$

32. $-[s^2X(s) - x'(0)]' - 2[s\,X(s)]' - 2[s\,X(s)] - 2X(s) = 0$

$-(s^2 + 2s)X'(s) - (4s + 4)X(s) = 0$ (separable)

$$X(s) = \frac{A}{s^2(s+2)^2} = C\left[\frac{1}{s} - \frac{1}{s^2} - \frac{1}{s+2} - \frac{1}{(s+2)^2}\right]$$

$$x(t) = C(1 - t - e^{-2t} - te^{-2t}) \text{ with } C = -A/4 \neq 0$$

33. $-[s^2X(s) - x(0)]' - 2[s\,X(s)] - [X(s)]' = 0$

$(s^2 + 1)X'(s) + 4s\,X(s) = 0$ (separable)

$$X(s) = \frac{A}{(s^2+1)^2} \quad \text{with } A \neq 0$$

$$x(t) = C(\sin t - t\cos t) \text{ with } C \neq 0$$

34. $-(s^2 + 4s + 13)X'(s) - (4s + 8)X(s) = 0$

$$X(s) = \frac{C}{(s^2+4s+13)^2} = \frac{C}{\left[(s+2)^2+9\right]^2}$$

It now follows from Problem 31 in Section 10.2 that

$$x(t) = Ae^{-2t}(\sin 3t - 3t\cos 3t) \text{ with } A \neq 0.$$

35. $$\mathcal{L}^{-1}\left\{\frac{1}{(s-1)\sqrt{s}}\right\} = e^t * \frac{1}{\sqrt{\pi t}} = \int_0^t \frac{1}{\sqrt{\pi x}} \cdot e^{t-x}\,dx$$

$$= \frac{e^t}{\sqrt{\pi}} \int_0^{\sqrt{t}} \frac{1}{u} \cdot e^{-u^2} \cdot 2u\,du = \frac{2e^t}{\sqrt{\pi}} \int_0^{\sqrt{t}} e^{-u^2}\,du = e^t \operatorname{erf}\left(\sqrt{t}\right)$$

36. $s^2X(s) + 4X(s) = F(s)$

$$X(s) = \frac{1}{2} F(s) \cdot \frac{2}{s^2+4}$$

$$x(t) = \frac{1}{2} f(t) * \sin 2t = \frac{1}{2}\int_0^t f(t-\tau)\sin 2\tau\,d\tau$$

37. $s^2X(s) + 2sX(s) + X(s) = F(s)$

$$X(s) = F(s) \cdot \frac{1}{(s+1)^2}$$

$$x(t) = te^{-t} * f(t) = \int_0^t \tau e^{-\tau} f(t-\tau)\,d\tau$$

38. $s^2X(s) + 4sX(s) + 13X(s) = F(s)$

$$X(s) = \frac{F(s)}{s^2+4s+13} = \frac{1}{3} F(s) \cdot \frac{3}{(s+2)^2+9}$$

$$x(t) = \frac{1}{3} f(t) * e^{-2t}\sin 3t = \frac{1}{3}\int_0^t e^{-2\tau} f\left(t-\tau\right)\sin 3\tau\,d\tau$$

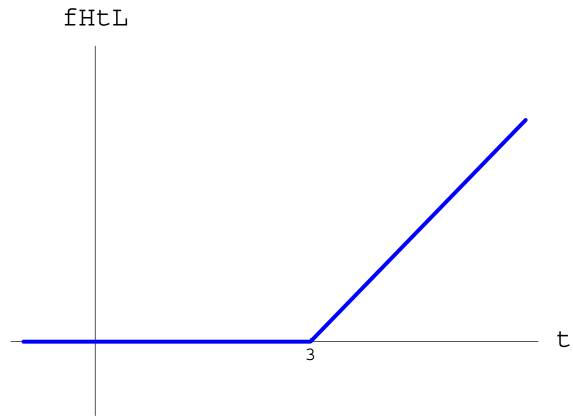
SECTION 10.5

PERIODIC AND PIECEWISE CONTINUOUS INPUT FUNCTIONS

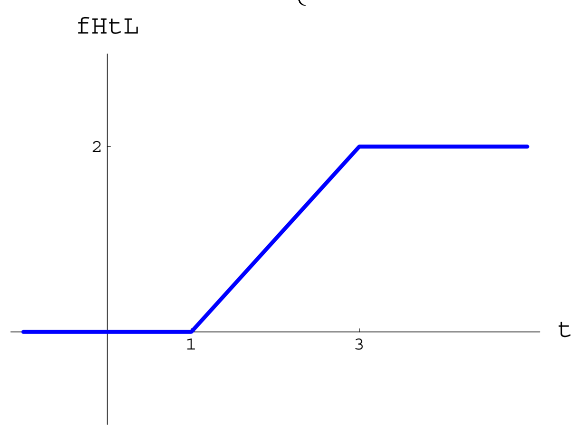
In Problems 1 through 10, we first derive the inverse Laplace transform $f(t)$ of $F(s)$ and then show the graph of $f(t)$.

1. $F(s) = e^{-3s}\mathcal{L}\{t\}$ so Eq. (3b) in Theorem 1 gives

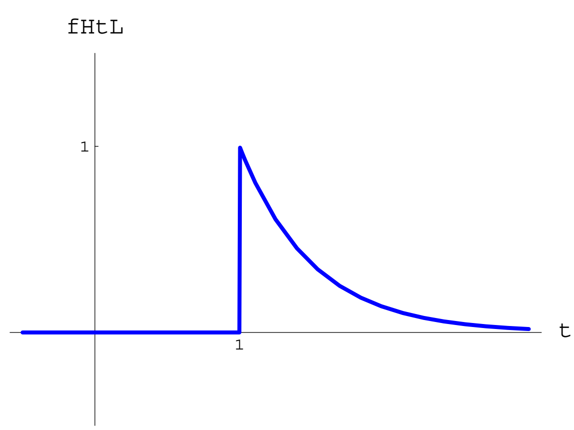
$$f(t) = u(t-3)\cdot(t-3) = \begin{cases} 0 & \text{if } t<3, \\ t-3 & \text{if } t\ge 3. \end{cases}$$



2. $$f(t) = (t-1)u(t-1)-(t-3)u(t-3) = \begin{cases} 0 & \text{if } t<1, \\ t-1 & \text{if } 1\le t<3, \\ 2 & \text{if } t\ge 3. \end{cases}$$

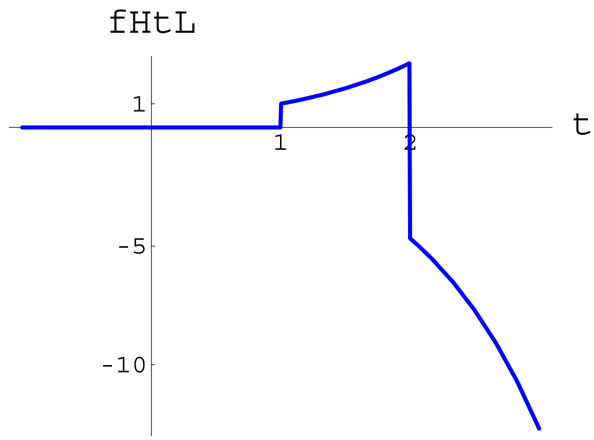


3. $F(s) = e^{-s}\mathcal{L}\{e^{-2t}\}$ so $f(t) = u(t-1)\cdot e^{-2(t-1)} = \begin{cases} 0 & \text{if } t<1, \\ e^{-2(t-1)} & \text{if } t\ge 1. \end{cases}$



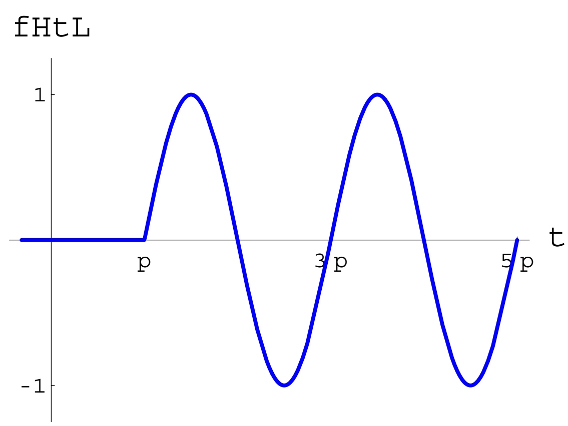
4. $F(s) = e^{-s}\mathcal{L}\{t\} - e^2 e^{-2s}\mathcal{L}\{t\}$ so

$$f(t) = e^{t-1}u(t-1) - e^2 e^{t-2}u(t-2) = \begin{cases} 0 & \text{if } t < 1, \\ e^{t-1} & \text{if } 1 \le t < 2, \\ e^{t-1} - e^t & \text{if } t \ge 2. \end{cases}$$



5. $F(s) = e^{-\pi s}\mathcal{L}\{\sin t\}$ so

$$f(t) = u(t-\pi)\cdot\sin(t-\pi) = -u(t-\pi)\sin t = \begin{cases} 0 & \text{if } t < \pi, \\ -\sin t & \text{if } t \ge \pi. \end{cases}$$



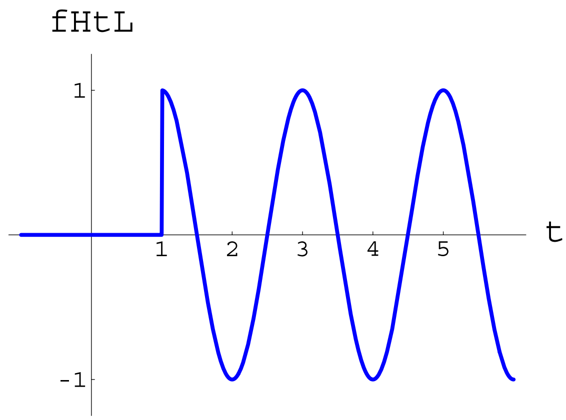
6. $F(s) = e^{-s}\mathcal{L}\{\cos \pi t\}$ so

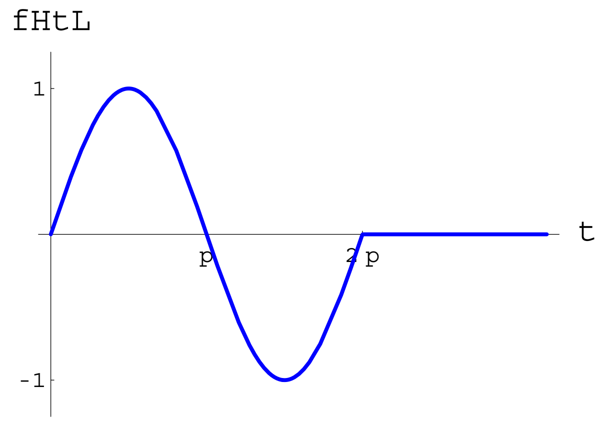
$$f(t) = u(t-1)\cdot\cos\pi(t-1) = -u(t-1)\cos\pi t = \begin{cases} 0 & \text{if } t < 1, \\ -\cos\pi t & \text{if } t \ge 1. \end{cases}$$

7. $F(s) = \mathcal{L}\{\sin t\} - e^{-2\pi s}\mathcal{L}\{\sin t\}$ so

$$f(t) = \sin t - u(t-2\pi)\sin(t-2\pi) = \left[1 - u(t-2\pi)\right]\sin t = \begin{cases} \sin t & \text{if } t < 2\pi, \\ 0 & \text{if } t \ge 2\pi. \end{cases}$$

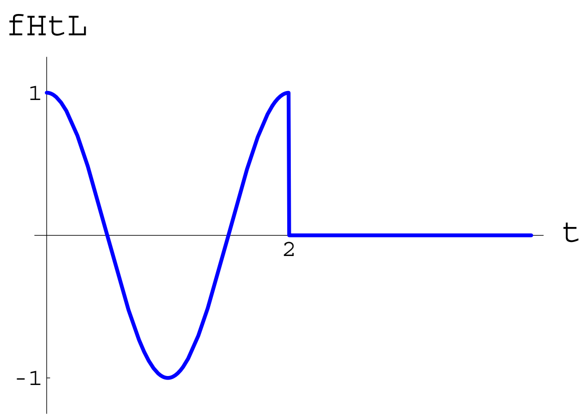
The left-hand figure below is the graph for Problem 6 on the preceding page, and the right-hand figure is the graph for Problem 7.





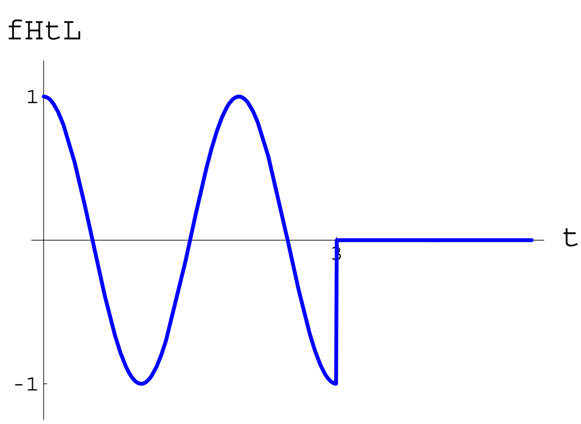
8. $F(s) \;=\; \mathcal{L}\{\cos \pi t\} - e^{-2s}\mathcal{L}\{\cos \pi t\}$ so

$$f(t) \;=\; \cos \pi t - u(t-2)\cos \pi(t-2) = [1-u(t-2)]\cos \pi t \;=\; \begin{cases} \cos \pi t & \text{if } t<2, \\ 0 & \text{if } t \geq 2. \end{cases}$$



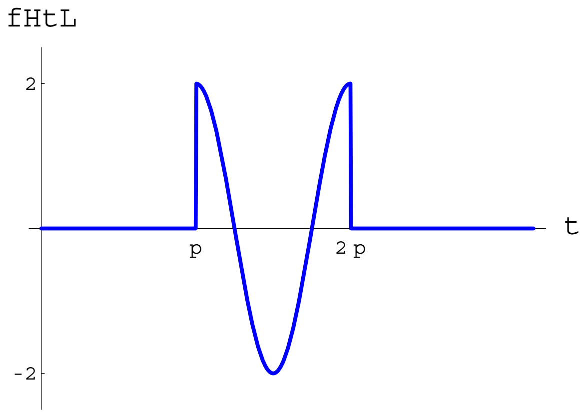
9. $F(s) \;=\; \mathcal{L}\{\cos \pi t\} + e^{-3s}\mathcal{L}\{\cos \pi t\}$ so

$$f(t) \;=\; \cos \pi t + u(t-3)\cos \pi(t-3) = [1-u(t-3)]\cos \pi t \;=\; \begin{cases} \cos \pi t & \text{if } t<3, \\ 0 & \text{if } t \geq 3. \end{cases}$$



10. $F(s) = e^{-\pi s}\mathcal{L}\{2\cos 2t\} + e^{-2\pi s}\mathcal{L}\{2\cos 2t\}$ so

$$\begin{aligned} f(t) &= 2u(t-\pi)\cos 2(t-\pi) - 2u(t-2\pi)\cos 2(t-2\pi) \\ &= 2\left[u(t-\pi) - u(t-2\pi)\right]\cos 2t = \begin{cases} 0 \text{ if } t < \pi \text{ or } t \geq 2\pi, \\ 2\cos 2t \text{ if } \pi \leq t < 2\pi. \end{cases} \end{aligned}$$



11. $f(t) = 2 - u(t-3)\cdot 2$ so $F(s) = \dfrac{2}{s} - e^{-3s}\dfrac{2}{s} = \dfrac{2}{s}\left(1 - e^{-3s}\right).$

12. $f(t) = u(t-1) - u(t-4)$ so $F(s) = \dfrac{e^{-s}}{s} - \dfrac{e^{-3s}}{s} = \dfrac{1}{s}\left(e^{-s} - e^{-3s}\right).$

13. $f(t) = [1 - u(t-2\pi)]\sin t = \sin t - u(t-2\pi)\sin(t-2\pi)$ so

$$F(s) = \frac{1}{s^2+1} - e^{-2\pi s}\cdot\frac{1}{s^2+1} = \frac{1-e^{-2\pi s}}{s^2+1}.$$

14. $f(t) = [1 - u(t-2)]\cos \pi t = \cos \pi t - u(t-2)\cos \pi(t-2)$ so

$$F(s) = \frac{s}{s^2+\pi^2} - e^{-2s}\cdot\frac{s}{s^2+\pi^2} = \frac{s\left(1-e^{-2s}\right)}{s^2+\pi^2}.$$

15. $f(t) = [1 - u(t-3\pi)]\sin t = \sin t + u(t-3\pi)]\sin(t-3\pi)$ so

$$F(s) = \frac{1}{s^2+1} + \frac{e^{-3\pi s}}{s^2+1} = \frac{1+e^{-3\pi s}}{s^2+1}.$$

16. $f(t) = [u(t-\pi) - u(t-2\pi)]\sin 2t = u(t-\pi)\sin 2(t-\pi) - u(t-2\pi)\sin 2(t-2\pi)$ so

$$F(s) = \left(e^{-\pi s} - e^{-2\pi s}\right)\cdot\frac{2}{s^2+4} = \frac{2\left(e^{-\pi s} - e^{-2\pi s}\right)}{s^2+4}.$$

17. $f(t) = [u(t-2)-u(t-3)]\sin\pi t = u(t-2)\sin\pi(t-2)+u(t-3)\sin\pi(t-3)$ so

$$F(s) = \left(e^{-2s}+e^{-3s}\right)\cdot\frac{\pi}{s^2+\pi^2} = \frac{\pi\left(e^{-2s}+e^{-3s}\right)}{s^2+\pi^2}.$$

18. $f(t) = [u(t-3)-u(t-5)]\cos\dfrac{\pi t}{2} = u(t-3)\sin\dfrac{\pi}{2}(t-3)+u(t-5)\sin\dfrac{\pi}{2}(t-5)$ so

$$F(s) = \left(e^{-3s}+e^{-5s}\right)\cdot\frac{\pi/2}{s^2+\pi^2/4} = \frac{2\pi\left(e^{-3s}+e^{-5s}\right)}{4s^2+\pi^2}.$$

19. If $g(t) = t+1$ then $f(t) = u(t-1)\cdot t = u(t-1)\cdot g(t-1)$ so

$$F(s) = e^{-s}G(s) = e^{-s}L\{t+1\} = e^{-s}\cdot\left(\frac{1}{s^2}+\frac{1}{s}\right) = \frac{e^{-s}(s+1)}{s^2}.$$

20. If $g(t)=t+1$ then $f(t) = [1-u(t-1)]t+u(t-1) = t-u(t-1)g(t-1)+u(t-1)$ so

$$F(s) = \frac{1}{s^2}-e^{-s}\cdot G(s)+\frac{e^{-s}}{s} = \frac{1}{s^2}-e^{-s}\cdot\left(\frac{1}{s^2}+\frac{1}{s}\right)+\frac{e^{-s}}{s} = \frac{1-e^{-s}}{s^2}.$$

21. If $g(t)=t+1$ and $h(t)=t+2$ then

$$\begin{aligned} f(t) &= t[1-u(t-1)]+(2-t)[u(t-1)-u(t-2]\\ &= t-2t\,u(t-1)+2u(t-1)-2u(t-2)+t\,u(t-2)\\ &= t-2u(t-1)g(t-1)+2u(t-1)-2u(t-2)+u(t-2)h(t-2) \end{aligned}$$

so

$$F(s) = \frac{1}{s^2}-2e^{-s}\left(\frac{1}{s^2}+\frac{1}{s}\right)+\frac{2e^{-s}}{s}-\frac{2e^{-2s}}{s}+e^{-2s}\left(\frac{1}{s^2}+\frac{2}{s}\right) = \frac{\left(1-e^{-s}\right)^2}{s^2}.$$

22. $f(t) = [u_1(t) - u_2(t)]\,t^3 = u_1(t)g(t-1) - u_2(t)h(t-2)$ where

$$g(t) = (t+1)^3 = t^3+3t^2+3t+1,$$
$$h(t) = (t+2)^3 = t^3+6t^2+12t+8.$$

It follows that

$$\begin{aligned} F(s) &= e^{-s}G(s) - e^{-2s}H(s)\\ &= [(s^3+3s^2+6s+6)e^{-s} - (8s^3+12s^2+12s+6)e^{-2s}]/s^4. \end{aligned}$$

23. With $f(t) = 1$ and $p=1$, Formula (6) in the text gives

$$\mathcal{L}\{1\} = \frac{1}{1-e^{-s}}\int_0^1 e^{-st}\cdot 1\,dt = \frac{1}{1-e^{-s}}\left[-\frac{e^{-st}}{s}\right]_{t=0}^{t=1} = \frac{1}{s}.$$

24. With $f(t) = \cos kt$ and $p = 2\pi/k$, Formula (6) and the integral formula

$$\int e^{at}\cos bt\,dt = e^{at}\left[\frac{a\cos bt + b\sin bt}{a^2+b^2}\right] + C$$

give

$$\begin{aligned}\mathcal{L}\{\cos kt\} &= \frac{1}{1-e^{-2\pi s/k}}\int_0^{2\pi/k} e^{-st}\cdot\cos kt\,dt \\ &= \frac{1}{1-e^{-2\pi s/k}}\left[e^{-st}\left(\frac{-s\cos kt + k\sin kt}{s^2+k^2}\right)\right]_{t=0}^{t=2\pi/k} \\ &= \frac{1}{1-e^{-2\pi s/k}}\left[e^{-2\pi s/k}\left(\frac{-s}{s^2+k^2}\right) - e^{-0}(-s)\right] = \frac{s}{s^2+k^2}.\end{aligned}$$

25. With $p = 2a$ and $f(t) = 1$ if $0 \le t \le a$, $f(t) = 0$ if $a < t \le 2a$, Formula (6) gives

$$\begin{aligned}\mathcal{L}\{f(t)\} &= \frac{1}{1-e^{-2as}}\int_0^a e^{-st}\cdot 1\,dt = \frac{1}{1-e^{-2as}}\left[-\frac{e^{-st}}{s}\right]_{t=0}^{t=a} \\ &= \frac{1-e^{-as}}{s\left(1-e^{-as}\right)\left(1+e^{-as}\right)} = \frac{1}{s\left(1+e^{-as}\right)}.\end{aligned}$$

26. With $p = a$ and $f(t) = t/a$, Formula (6) and the integral formula $\int ue^u\,du = (u-1)e^u$ (with $u = -st$) give

$$\begin{aligned}\mathcal{L}\{f(t)\} &= \frac{1}{a\left(1-e^{-as}\right)}\int_0^a e^{-st}\cdot t\,dt = \frac{1}{a\left(1-e^{-as}\right)}\int_0^{-as} e^u\cdot\left(-\frac{u}{s}\right)\left(-\frac{du}{s}\right) \\ &= \frac{1}{as^2\left(1-e^{-as}\right)}\int_0^{-as} e^u u\,du = \frac{1}{as^2\left(1-e^{-as}\right)}\Big[(u-1)e^u\Big]_0^{-as} \\ &= \frac{1}{as^2\left(1-e^{-as}\right)}\Big[(-as-1)e^{-as}+1\Big] = \frac{1}{as^2} - \frac{e^{-as}}{s\left(1-e^{-as}\right)}.\end{aligned}$$

27. $G(s) = \mathcal{L}\{t/a - f(t)\} = (1/as^2) - F(s)$. Now substitution of the result of Problem 26 in place of $F(s)$ immediately gives the desired transform.

28. This computation is very similar to the one in Problem 26, except that $p = 2a$:

$$\begin{aligned}\mathcal{L}\{f(t)\} &= \frac{1}{1-e^{-2as}}\int_0^a e^{-st}\cdot t\,dt = \frac{1}{1-e^{-2as}}\int_0^{-as} e^u\cdot\left(-\frac{u}{s}\right)\left(-\frac{du}{s}\right) \\ &= \frac{1}{s^2\left(1-e^{-2as}\right)}\int_0^{-as} e^u u\,du = \frac{1}{s^2\left(1-e^{-2as}\right)}\Big[(u-1)e^u\Big]_0^{-as}\end{aligned}$$

$$= \frac{1}{s^2\left(1-e^{-2as}\right)}\left[(-as-1)e^{-as}+1\right] = \frac{1-e^{-as}(1+as)}{s^2\left(1-e^{-2as}\right)}.$$

29. With $p = 2\pi/k$ and $f(t) = \sin kt$ for $0 \le t \le \pi/k$ while $f(t) = 0$ for $\pi/k \le t \le 2\pi/k$, Formula (6) and the integral formula

$$\int e^{at}\sin bt\,dt = e^{at}\left[\frac{a\sin bt - b\cos bt}{a^2+b^2}\right] + C$$

give

$$\begin{aligned}
\mathcal{L}\{f(t)\} &= \frac{1}{1-e^{-2\pi s/k}}\int_0^{\pi/k} e^{-st}\cdot\sin kt\,dt \\
&= \frac{1}{1-e^{-2\pi s/k}}\left[e^{-st}\left(\frac{-s\sin kt - k\cos kt}{s^2+k^2}\right)\right]_{t=0}^{t=\pi/k} \\
&= \frac{1}{1-e^{-2\pi s/k}}\left[\frac{e^{-\pi s/k}(k)-(-k)}{s^2+k^2}\right] \\
&= \frac{k\left(1+e^{-\pi s/k}\right)}{\left(1-e^{-\pi s/k}\right)\left(1+e^{-\pi s/k}\right)\left(s^2+k^2\right)} = \frac{k}{\left(s^2+k^2\right)\left(1-e^{-\pi s/k}\right)}.
\end{aligned}$$

30. $h(t) = f(t)+g(t) = f(t)+u(t-\pi/k)f(t-\pi/k)$, so Problem 29 gives

$$\begin{aligned}
H(s) &= F(s)+e^{-\pi s/k}F(s) = \left(1+e^{-\pi s/k}\right)F(s) \\
&= \left(1+e^{-\pi s/k}\right)\cdot\frac{k}{\left(s^2+k^2\right)\left(1-e^{-\pi s/k}\right)} = \frac{k}{s^2+k^2}\cdot\frac{1+e^{-\pi s/k}}{1-e^{-\pi s/k}}\cdot\frac{e^{\pi s/2k}}{e^{\pi s/2k}} \\
&= \frac{k}{s^2+k^2}\cdot\frac{e^{\pi s/2k}+e^{-\pi s/2k}}{e^{\pi s/2k}-e^{-\pi s/2k}} = \frac{k}{s^2+k^2}\frac{\cosh(\pi s/2k)}{\sinh(\pi s/2k)} = \frac{k}{s^2+k^2}\coth\frac{\pi s}{2k}.
\end{aligned}$$

In Problems 31–42, we first write and transform the appropriate differential equation. Then we solve for the transform of the solution, and finally inverse transform to find the desired solution.

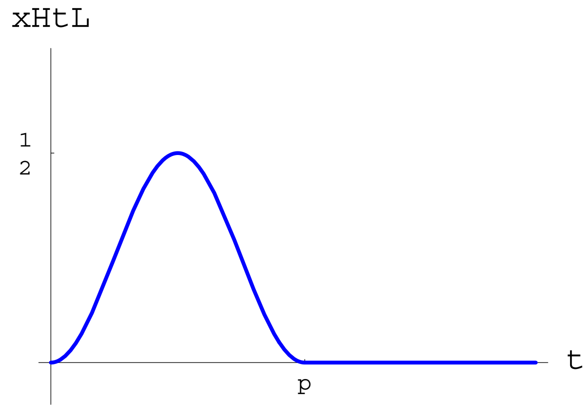
31. $x'' + 4x = 1 - u(t-\pi)$

$$s^2X(s) + 4X(s) = \frac{1-e^{-\pi s}}{s}$$

$$X(s) = \frac{1-e^{-\pi s}}{s\left(s^2+4\right)} = \frac{1}{4}\left(1-e^{-\pi s}\right)\left(\frac{1}{s}-\frac{s}{s^2+4}\right)$$

$$x(t) = (1/4)[1 - u(t-\pi)][1 - \cos 2(t-\pi)] = (1/2)[1 - u(t-\pi)]\sin^2 t$$

The graph of the position function $x(t)$ is shown at the top of the next page.



32. $x'' + 5x' + 4x = 1 - u(t-2)$

$$s^2X(s) + 5s\,X(s) + 4X(s) = \frac{1-e^{-2s}}{s}$$

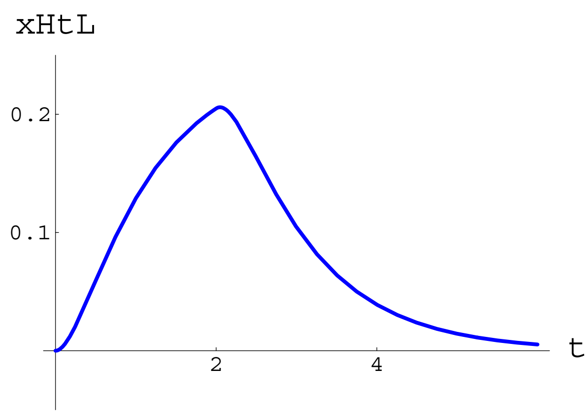
$$X(s) = \frac{1-e^{-2s}}{s\left(s^2+5s+4\right)} = (1 - e^{-2s})G(s)$$

where

$$G(s) = \frac{1}{12}\left(\frac{3}{s} - \frac{4}{s+1} + \frac{1}{s+4}\right), \text{ so } g(t) = \frac{1}{12}\left(3 - 4e^{-t} + e^{-4t}\right).$$

It follows that

$$x(t) = g(t) - u(t-2)g(t-2) = \begin{cases} g(t) & \text{if } t<2, \\ g(t) - g(t-2) & \text{if } t \geq 2. \end{cases}$$

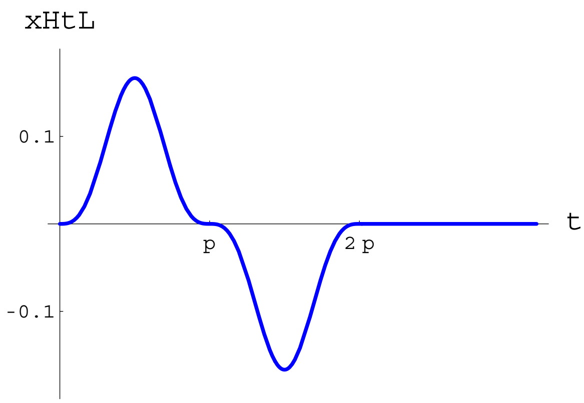


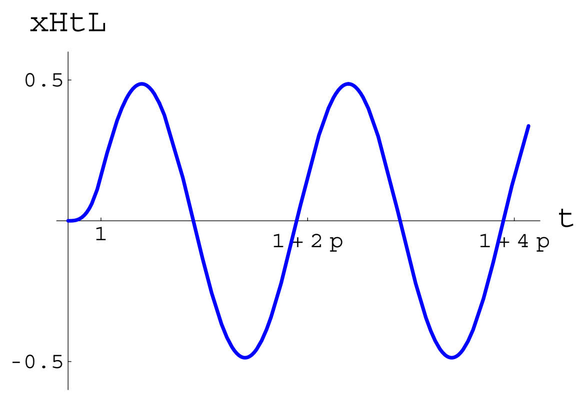
33. $x'' + 9x = [1 - u(t - 2\pi)]\sin t$

$$X(s) = \frac{1-e^{-2\pi s}}{\left(s^2+1\right)\left(s^2+4\right)} = \frac{1}{8}\left(1-e^{-2\pi s}\right)\left(\frac{1}{s^2+1}-\frac{1}{s^2+9}\right)$$

$$x(t) = \frac{1}{8}[1-u(t-2\pi)]\left(\sin t-\frac{1}{3}\sin 3t\right)$$

The left-hand figure below show the graph of this position function.





34. $x''+x = [1 - u(t-1)]\,t = 1-u(t-1)f(t-1)$, where $f(t) = t+1$

$$s^2X(s)+X(s) = \frac{1}{s^2}-e^{-s}G(s) = \frac{1}{s^2}-e^{-s}\left(\frac{1}{s}+\frac{1}{s^2}\right)$$

It follows that

$$X(s) = \frac{1}{s^2(s^2+1)}-\frac{e^{-s}(s+1)}{s^2(s^2+1)}$$
$$= (1-e^{-s})\left(\frac{1}{s^2}-\frac{1}{s^2+1}\right)-e^{-s}\left(\frac{1}{s}-\frac{s}{s^2+1}\right) = (1-e^{-s})G(s)-e^{-s}H(s)$$

where $g(t) = t - \sin t$, $h(t) = 1 - \cos t$. Hence

$$x(t) = g(t) - u(t-1)g(t-1) - u(t-1)h(t-1)$$

and so

$$x(t) = t - \sin t \ \text{ if } t<1,$$
$$x(t) = -\sin t+\sin(t-1)+\cos(t-1) \ \text{ if } t>1.$$

The right-hand figure above shows the graph of this position function.

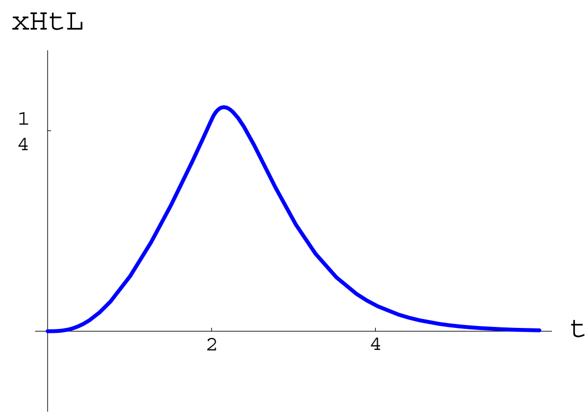
35. $x''+4x'+4x = [1 - u(t-2)]t = t - u(t-2)g(t-2)$ where $g(t)=t+2$

$$(s+2)^2X(s) = \frac{1}{s^2}-e^{-2s}\left(\frac{2}{s}+\frac{1}{s^2}\right)$$

$$X(s) = \frac{1}{s^2(s+2)^2} - e^{-2s}\frac{2s+1}{s^2(s+2)^2}$$

$$= \frac{1}{4}\left(-\frac{1}{s}+\frac{1}{s^2}+\frac{1}{s+2}+\frac{1}{(s+2)^2}\right) - \frac{1}{4}e^{-2s}\left(\frac{1}{s}+\frac{1}{s^2}-\frac{1}{s+2}-\frac{3}{(s+2)^2}\right)$$

$$x(t) = (1/4)\{-1 + t + (1+t)e^{-2t} + u(t-2)[1 - t + (3t - 5)e^{-2(t-2)}]\}$$



36. $x'' + 4x = f(t), \qquad x(0) = x'(0) = 0$

$$\left(s^2+4\right)X(s) = \frac{4\left(1-e^{-\pi s}\right)}{s\left(1+e^{-\pi s}\right)}$$ (by Example 5 of Section 10.5)

$$\left(s^2+4\right)X(s) = \frac{4}{s}+\frac{8}{s}\sum_{n=1}^{\infty}(-1)^n e^{-n\pi s}$$ (as in Eq. (10) of Section 10.5)

Now let

$$g(t) = \mathcal{L}^{-1}\left\{\frac{4}{s(s^2+4)}\right\} = 1-\cos 2t = 2\sin^2 t.$$

Then it follows that

$$x(t) = g(t)+2\sum_{n=1}^{\infty}(-1)^n u_{n\pi}(t)g(t-n\pi) = 2\sin^2 t + 4\sum_{n=1}^{\infty}(-1)^n u_{n\pi}(t)\sin^2 t.$$

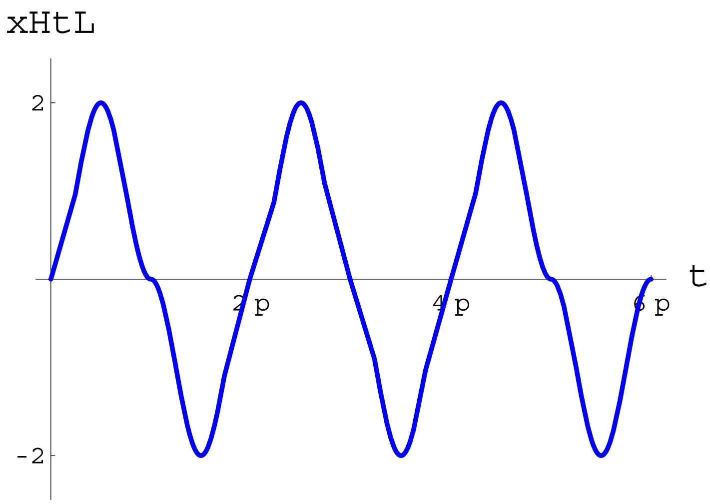
Hence

$$x(t) = \begin{cases} 2\sin^2 t \text{ if } 2n\pi \le t < (2n+1)\pi, \\ -2\sin^2 t \text{ if } (2n-1)\pi \le t < 2n\pi. \end{cases}$$

Consequently the complete solution

$$x(t) = 2|\sin t|\sin t$$

is periodic, so the transient solution is zero. The graph of $x(t)$:



37. $x'' + 2x' + 10x = f(t), \qquad x(0) = x'(0) = 0$

As in the solution of Example 7 we find first that

$$\left(s^2 + 2s + 10\right) X(s) = \frac{10}{s} + \frac{20}{s}\sum_{n=1}^{\infty}(-1)^n e^{-n\pi s},$$

so

$$X(s) = \frac{10}{s(s^2 + 2s + 10)} + 2\sum_{n=1}^{\infty}\frac{10(-1)^n e^{-n\pi s}}{s(s^2 + 2s + 10)}.$$

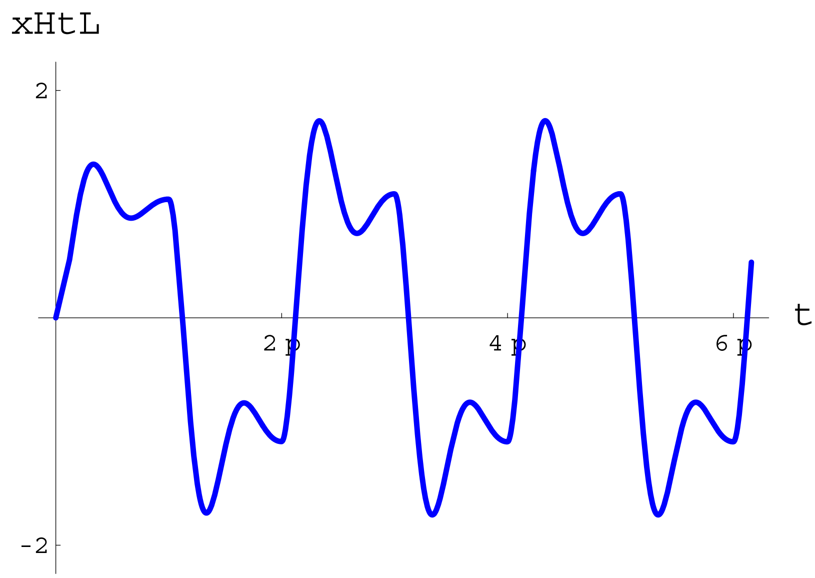
If

$$g(t) = \mathcal{L}^{-1}\left\{\frac{10}{s\left[(s+1)^2 + 9\right]}\right\} = 1 - \frac{1}{3}e^{-t}\left(3\cos 3t + \sin 3t\right),$$

then it follows that

$$x(t) = g(t) + 2\sum_{n=1}^{\infty}(-1)^n u_{n\pi}(t) g(t - n\pi).$$

The graph of $x(t)$ appears at the top of the next page.



38. If the function $x(t)$ satisfies the initial value problem

$$mx'' + cx' + kx = F(t), \qquad x(a) = b_0, \quad x'(a) = b_1$$

for $t \geqq a$ and $x(t) = 0$ for $t < a$, then we may write $x(t) = u(t-a)z(t-a)$ where the function $z(t)$ satisfies the initial value problem

$$mz'' + cz' + kz = F(t+a), \qquad z(0) = b_0, \quad z'(a) = b_1.$$

Then $Z(s) = \mathcal{L}\{z(t)\}$ satisfies the equation

$$m\left(s^2 Z(s) - s\,b_0 - b_1\right) + c\left(sZ(s) - b_0\right) + k\left(Z(s)\right) \;=\; \mathcal{L}\{F(t+a)\}. \qquad (*)$$

Now

$$X(s) \;=\; \mathcal{L}\{x(t)\} \;=\; \mathcal{L}\{u(t-a)z(t-a)\} \;=\; e^{-as}Z(s)$$

by Theorem 1 in Section 10.5. Substitution of $Z(s) = e^{as}X(s)$ into Eq. (*) then gives the desired result.

39. When we substitute the inverse Laplace transforms $a(t),\ b(t),\ c(t)$ given at the beginning of part (c), we get

$$\begin{aligned} v(t) \;&=\; b_0 a(t) + b_1 b(t) + c(t) \\ &=\; \tfrac{1}{4}e^{-2t}\left[b_0\left(4\cos 4t + 2\sin 4t\right) + b_1 \sin 4t + \left(4 - 4\cos 4t - 2\sin 4t\right)\right]. \end{aligned}$$

Similarly, Theorem 1 in this section gives

$$\begin{aligned}
w(t) &= L^{-1}\left\{e^{-\pi s}\left[c_0 A(s) + c_1 B(s) - C(s)\right]\right\} \\
&= u(t-\pi)\left[c_0 a(t-\pi) + c_1 b(t-\pi) - c(t-\pi)\right] \\
&= \tfrac{1}{4} e^{-2(t-\pi)}\Big[c_0\left(4\cos 4(t-\pi) + 2\sin 4(t-\pi)\right) + c_1 \sin 4(t-\pi) - \\
&\qquad\qquad \left(4e^{2(t-\pi)} - 4\cos 4(t-\pi) - 2\sin 4(t-\pi)\right)\Big]\cdot u(t-\pi).
\end{aligned}$$

where as usual $u(t-\pi) = u_\pi(t)$ denotes the unit stop function at π. Then the four continuity equations listed in part (c) yield the equations

$$\begin{aligned}
&e^{-2\pi} b_0 + 1 - e^{-2\pi} = c_0, \qquad && e^{-2\pi} b_1 = c_1, \\
&e^{-2\pi} c_0 - 1 + e^{-2\pi} = b_0, \qquad && e^{-2\pi} c_1 = b_1
\end{aligned}$$

that we solve readily for

$$b_0 = -c_0 = \frac{1-e^{2\pi}}{1+e^{2\pi}} \approx -0.996372, \qquad b_1 = c_1 = 0.$$

Finally, these values for the coefficients yield

$$v(t) = 1 - \frac{e^{-2(t-\pi)}}{1+e^{2\pi}}\left(2\cos 4t + \sin 4t\right) \approx 1 - 0.9981 e^{-2t}\left(2\cos 4t + \sin 4t\right),$$

$$\begin{aligned}
w(t) &= -\left[1 - \frac{e^{-2(t-2\pi)}}{1+e^{2\pi}}\left(2\cos 4t + \sin 4t\right)\right] u(t-\pi) \\
&\approx -\left[1 - 0.9981 e^{-2(t-\pi)}\left(2\cos 4t + \sin 4t\right)\right] u(t-\pi).
\end{aligned}$$

CHAPTER 11

POWER SERIES METHODS

SECTION 11.1

INTRODUCTION AND REVIEW OF POWER SERIES

The power series method consists of substituting a series $y = \Sigma c_n x^n$ into a given differential equation in order to determine what the coefficients $\{c_n\}$ must be in order that the power series will satisfy the equation. It might be pointed out that, if we find a recurrence relation in the form $c_{n+1} = \phi(n)c_n$, then we can determine the radius of convergence ρ of the series solution directly from the recurrence relation,

$$\rho = \lim_{n\to\infty}\left|\frac{c_n}{c_{n+1}}\right| = \lim_{n\to\infty}\left|\frac{1}{\phi(n)}\right|.$$

In Problems 1–10 we give first that recurrence relation that can be used to find the radius of convergence and to calculate the succeeding coefficients $c_1, c_2, c_3, \cdots$ in terms of the arbitrary constant c_0. Then we give the series itself

1. $c_{n+1} = \dfrac{c_n}{n+1}$; it follows that $c_n = \dfrac{c_0}{n!}$ and $\rho = \lim\limits_{n\to\infty}(n+1) = \infty$.

$$y(x) = c_0\left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \cdots\right) = c_0\left(1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots\right) = c_0 e^x$$

2. $c_{n+1} = \dfrac{4c_n}{n+1}$; it follows that $c_n = \dfrac{4^n c_0}{n!}$ and $\rho = \lim\limits_{n\to\infty}\dfrac{n+1}{4} = \infty$.

$$y(x) = c_0\left(1 + 4x + 8x^2 + \frac{32x^3}{3} + \frac{32x^4}{4} + \cdots\right)$$

$$= c_0\left(1 + \frac{4x}{1!} + \frac{4^2x^2}{2!} + \frac{4^3x^3}{3!} + \frac{4^4x^4}{4!} + \cdots\right) = c_0 e^{4x}$$

3. $c_{n+1} = -\dfrac{3c_n}{2(n+1)}$; it follows that $c_n = \dfrac{(-1)^n 3^n c_0}{2^n n!}$ and $\rho = \lim\limits_{n\to\infty}\dfrac{2(n+1)}{3} = \infty$.

$$y(x) = c_0\left(1 - \frac{3x}{2} + \frac{9x^2}{8} - \frac{9x^3}{16} + \frac{27x^4}{128} - \cdots\right)$$

$$= c_0\left(1-\frac{3x}{1!2}+\frac{3^2x^2}{2!2^2}-\frac{3^3x^3}{3!2^3}+\frac{3^4x^4}{4!2^4}-\cdots\right) = c_0e^{-3x/2}$$

4. When we substitute $y = \Sigma c_n x^n$ into the equation $y' + 2xy = 0$, we find that

$$c_1+\sum_{n=0}^{\infty}\left[(n+2)c_{n+2}+2c_n\right]x^{n+1} = 0.$$

Hence $c_1 = 0$ — which we see by equating constant terms on the two sides of this equation — and $c_{n+2} = -\dfrac{2c_n}{n+2}$. It follows that

$$c_1 = c_3 = c_5 = \cdots = c_{\text{odd}} = 0 \quad \text{and} \quad c_{2k} = \frac{(-1)^k c_0}{k!}.$$

Hence

$$y(x) = c_0\left(1-x^2+\frac{x^4}{2}-\frac{x^6}{3}+\cdots\right) = c_0\left(1-\frac{x^2}{1!}+\frac{x^4}{2!}-\frac{x^6}{3!}+\cdots\right) = c_0e^{-x^2}$$

and $\rho = \infty$.

5. When we substitute $y = \Sigma c_n x^n$ into the equation $y' = x^2 y$, we find that

$$c_1+2c_2x+\sum_{n=0}^{\infty}\left[(n+3)c_{n+3}-c_n\right]x^{n+1} = 0.$$

Hence $c_1 = c_2 = 0$ — which we see by equating constant terms and x-terms on the two sides of this equation — and $c_3 = \dfrac{c_n}{n+3}$. It follows that

$$c_{3k+1} = c_{3k+2} = 0 \quad \text{and} \quad c_{3k} = \frac{c_0}{3\cdot 6\cdot\cdots\cdot(3k)} = \frac{c_0}{k!3^k}.$$

Hence

$$y(x) = c_0\left(1+\frac{x^3}{3}+\frac{x^6}{18}+\frac{x^9}{162}+\cdots\right) = c_0\left(1+\frac{x^3}{1!3}+\frac{x^6}{2!3^2}+\frac{x^9}{3!3^3}+\cdots\right) = c_0e^{(x^3/3)}.$$

and $\rho = \infty$.

6. $c_{n+1} = \dfrac{c_n}{2}$; it follows that $c_n = \dfrac{c_0}{2^n}$ and $\rho = \lim\limits_{n\to\infty} 2 = 2$.

$$y(x) = c_0\left(1+\frac{x}{2}+\frac{x^2}{4}+\frac{x^3}{8}+\frac{x^4}{16}+\cdots\right)$$

$$= c_0\left[1+\left(\frac{x}{2}\right)+\left(\frac{x}{2}\right)^2+\left(\frac{x}{2}\right)^3+\left(\frac{x}{2}\right)^4+\cdots\right] = \frac{c_0}{1-\dfrac{x}{2}} = \frac{2c_0}{2-x}$$

7. $c_{n+1} = 2c_n$; it follows that $c_n = 2^n c_0$ and $\rho = \lim_{n\to\infty}\frac{1}{2} = \frac{1}{2}.$

$$y(x) = c_0\left(1+2x+4x^2+8x^3+16x^4+\cdots\right)$$

$$= c_0\left[1+(2x)+(2x)^2+(2x)^3+(2x)^4+\cdots\right] = \frac{c_0}{1-2x}$$

8. $c_{n+1} = -\frac{(2n-1)c_n}{2n+2}$; it follows that $\rho = \lim_{n\to\infty}\frac{2n+2}{2n-1} = 1.$

$$y(x) = c_0\left(1+\frac{x}{2}-\frac{x^2}{8}+\frac{x^3}{16}-\frac{5x^4}{128}+\cdots\right)$$

Separation of variables gives $y(x) = c_0\sqrt{1+x}.$

9. $c_{n+1} = \frac{(n+2)c_n}{n+1}$; it follows that $c_n = (n+1)c_0$ and $\rho = \lim_{n\to\infty}\frac{n+1}{n+2} = 1.$

$$y(x) = c_0\left(1+2x+3x^2+4x^3+5x^4+\cdots\right)$$

Separation of variables gives $y(x) = \frac{c_0}{(1-x)^2}.$

10. $c_{n+1} = \frac{(2n-3)c_n}{2n+2}$; it follows that $\rho = \lim_{n\to\infty}\frac{2n+2}{2n-3} = 1.$

$$y(x) = c_0\left(1-\frac{3x}{2}+\frac{3x^2}{8}+\frac{x^3}{16}+\frac{3x^4}{128}+\cdots\right)$$

Separation of variables gives $y(x) = c_0(1-x)^{3/2}.$

In Problems 11–14 the differential equations are second-order, and we find that the two initial coefficients c_0 and c_1 are both arbitrary. In each case we find the even-degree coefficients in terms of c_0 and the odd-degree coefficients in terms of c_1. The solution series in these problems are all recognizable power series that have infinite radii of convergence.

11. $c_{n+1} = \frac{c_n}{(n+1)(n+2)}$; it follows that $c_{2k} = \frac{c_0}{(2k)!}$ and $c_{2k+1} = \frac{c_1}{(2k+1)!}.$

$$y(x) = c_0\left(1+\frac{x^2}{2!}+\frac{x^4}{4!}+\frac{x^6}{6!}+\cdots\right)+c_1\left(x+\frac{x^3}{3!}+\frac{x^5}{5!}+\frac{x^7}{7!}+\cdots\right) = c_0\cosh x + c_1\sinh x$$

12. $c_{n+1} = \dfrac{4c_n}{(n+1)(n+2)}$; it follows that $c_{2k} = \dfrac{2^{2k}c_0}{(2k)!}$ and $c_{2k+1} = \dfrac{2^{2k}c_1}{(2k+1)!}$.

$$\begin{aligned} y(x) &= c_0\left(1+2x^2+\frac{2x^4}{3}+\frac{4x^6}{45}+\cdots\right)+c_1\left(x+\frac{2x^3}{3}+\frac{2x^5}{15}+\frac{4x^7}{315}+\cdots\right) \\ &= c_0\left(1+\frac{(2x)^2}{2!}+\frac{(2x)^4}{4!}+\frac{(2x)^6}{6!}+\cdots\right)+\frac{c_1}{2}\left((2x)+\frac{(2x)^3}{3!}+\frac{(2x)^5}{5!}+\frac{(2x)^7}{7!}+\cdots\right) \\ &= c_0\cosh 2x+\frac{c_1}{2}\sinh 2x \end{aligned}$$

13. $c_{n+1} = -\dfrac{9c_n}{(n+1)(n+2)}$; it follows that $c_{2k} = \dfrac{(-1)^k 3^{2k}c_0}{(2k)!}$ and $c_{2k+1} = \dfrac{(-1)^k 3^{2k}c_1}{(2k+1)!}$.

$$\begin{aligned} y(x) &= c_0\left(1-\frac{9x^2}{2}+\frac{27x^4}{8}-\frac{81x^6}{80}+\cdots\right)+c_1\left(x-\frac{3x^3}{2}+\frac{27x^5}{40}-\frac{81x^7}{560}+\cdots\right) \\ &= c_0\left(1-\frac{(3x)^2}{2!}+\frac{(3x)^4}{4!}-\frac{(3x)^6}{6!}+\cdots\right)+\frac{c_1}{3}\left((3x)-\frac{(3x)^3}{3!}+\frac{(3x)^5}{5!}-\frac{(3x)^7}{7!}+\cdots\right) \\ &= c_0\cos 3x+\frac{c_1}{3}\sin x \end{aligned}$$

14. When we substitute $y = \Sigma c_n x^n$ into $y''+y-x = 0$ and split off the terms of degrees 0 and 1, we get

$$(2c_2+c_0)+(6c_3+c_1-1)\,x+\sum_{n=2}^{\infty}[(n+1)(n+2)c_{n+2}+c_n]x^n = 0.$$

Hence $c_2 = -\dfrac{c_0}{2}$, $c_3 = -\dfrac{c_1-1}{6}$, and $c_{n+2} = -\dfrac{c_n}{(n+1)(n+2)}$ for $n \geq 2$. It follows that

$$\begin{aligned} y(x) &= c_0+c_0\left(-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\cdots\right)+c_1x+(c_1-1)\left(-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\cdots\right) \\ &= x+c_0\left(1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\cdots\right)+(c_1-1)\left(x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\cdots\right) \\ &= x+c_0\cos x+(c_1-1)\sin x. \end{aligned}$$

15. Assuming a power series solution of the form $y = \Sigma c_n x^n$, we substitute it into the differential equation $xy'+y=0$ and find that $(n+1)c_n = 0$ for all $n \geq 0$. This implies that $c_n = 0$ for all $n \geq 0$, which means that the only power series solution of our differential equation is the trivial solution $y(x) \equiv 0$. Therefore the equation has no *non-trivial* power series solution.

16. Assuming a power series solution of the form $y = \Sigma c_n x^n$, we substitute it into the differential equation $2xy' = y$ and find that $2nc_n = c_n$ for all $n \geq 0$. This implies that $0c_0 = c_0,\ 2c_1 = c_1,\ 4c_2 = c_2, \cdots,$ and hence that $c_n = 0$ for all $n \geq 0$, which means that the only power series solution of our differential equation is the trivial solution $y(x) \equiv 0$. Therefore the equation has no *non-trivial* power series solution.

17. Assuming a power series solution of the form $y = \Sigma c_n x^n$, we substitute it into the differential equation $x^2 y' + y = 0$. We find that $c_0 = c_1 = 0$ and that $c_{n+1} = -nc_n$ for $n \geq 1$, so it follows that $c_n = 0$ for all $n \geq 0$. Just as in Problems 15 and 16, this means that the equation has no *non-trivial* power series solution.

18. When we substitute and assumed power series solution $y = \Sigma c_n x^n$ into $x^3 y' = 2y$, we find that $c_0 = c_1 = c_2 = 0$ and that $c_{n+2} = nc_n/2$ for $n \geq 1$. Hence $c_n = 0$ for all $n \geq 0$, just as in Problems 15–17.

In Problems 19–22 we first give the recurrence relation that results upon substitution of an assumed power series solution $y = \Sigma c_n x^n$ into the given second-order differential equation. Then we give the resulting general solution, and finally apply the initial conditions $y(0) = c_0$ and $y'(0) = c_1$ to determine the desired particular solution.

19. $c_{n+2} = -\dfrac{2^2 c_n}{(n+1)(n+2)}$ for $n \geq 0$, so $c_{2k} = \dfrac{(-1)^k 2^{2k} c_0}{(2k)!}$ and $c_{2k+1} = \dfrac{(-1)^k 2^{2k} c_1}{(2k+1)!}$.

$$y(x) = c_0\left(1 - \frac{2^2 x^2}{2!} + \frac{2^4 x^4}{4!} - \frac{2^6 x^6}{6!} + \cdots\right) + c_1\left(x - \frac{2^2 x^3}{3!} + \frac{2^4 x^5}{5!} - \frac{2^6 x^7}{7!} + \cdots\right)$$

$c_0 = y(0) = 0$ and $c_1 = y'(0) = 3,$ so

$$\begin{aligned} y(x) &= 3\left(x - \frac{2^2 x^3}{3!} + \frac{2^4 x^5}{5!} - \frac{2^6 x^7}{7!} + \cdots\right) \\ &= \frac{3}{2}\left[(2x) - \frac{(2x)^3}{3!} + \frac{(2x)^5}{5!} - \frac{(2x)^7}{7!} + \cdots\right] = \frac{3}{2}\sin 2x. \end{aligned}$$

20. $c_{n+2} = \dfrac{2^2 c_n}{(n+1)(n+2)}$ for $n \geq 0$, so $c_{2k} = \dfrac{2^{2k} c_0}{(2k)!}$ and $c_{2k+1} = \dfrac{2^{2k} c_1}{(2k+1)!}$.

$$y(x) = c_0\left(1 + \frac{2^2 x^2}{2!} + \frac{2^4 x^4}{4!} + \frac{2^6 x^6}{6!} + \cdots\right) + c_1\left(x + \frac{2^2 x^3}{3!} + \frac{2^4 x^5}{5!} + \frac{2^6 x^7}{7!} + \cdots\right)$$

$c_0 = y(0) = 2$ and $c_1 = y'(0) = 0,$ so

$$y(x) = 2\left(1 + \frac{(2x)^2}{2!} + \frac{(2x)^4}{4!} + \frac{(2x)^6}{6!} + \cdots\right) = 2\cosh 2x.$$

21. $c_{n+1} = \dfrac{2nc_n - c_{n-1}}{n(n+1)}$ for $n \geq 1$; with $c_0 = y(0) = 0$ and $c_1 = y'(0) = 1$, we obtain

$c_2 = 1,\ c_3 = \dfrac{1}{2},\ c_4 = \dfrac{1}{6} = \dfrac{1}{3!},\ c_5 = \dfrac{1}{24} = \dfrac{1}{4!},\ c_6 = \dfrac{1}{120} = \dfrac{1}{5!}$. Evidently $c_n = \dfrac{1}{(n-1)!}$, so

$$y(x) = x + x^2 + \frac{x^3}{2!} + \frac{x^4}{3!} + \frac{x^5}{4!} + \cdots = x\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots\right) = x e^x.$$

22. $c_{n+1} = -\dfrac{nc_n - 2c_{n-1}}{n(n+1)}$ for $n \geq 1$; with $c_0 = y(0) = 1$ and $c_1 = y'(0) = -2$, we obtain

$c_2 = 2,\ c_3 = -\dfrac{4}{3} = -\dfrac{2^3}{3!},\ c_4 = \dfrac{2}{3} = \dfrac{2^4}{4!},\ c_5 = -\dfrac{4}{15} = -\dfrac{2^5}{5!}$. Apparently $c_n = \pm\dfrac{2^n}{n!}$, so

$$y(x) = 1 - (2x) + \frac{(2x)^2}{2!} - \frac{(2x)^3}{3!} + \frac{(2x)^4}{4!} - \frac{(2x)^5}{5!} + \cdots = e^{-2x}.$$

23. $c_0 = c_1 = 0$ and the recursion relation

$$(n^2 - n + 1)c_n + (n - 1)c_{n-1} = 0$$

for $n \geq 2$ imply that $c_n = 0$ for $n \geq 0$. Thus any assumed power series solution $y = \Sigma c_n x^n$ must reduce to the trivial solution $y(x) \equiv 0$.

24. **(a)** The fact that $y(x) = (1 + x)^\alpha$ satisfies the differential equation $(1+x)y' = \alpha y$ follows immediately from the fact that $y'(x) = \alpha(1+x)^{\alpha-1}$.

(b) When we substitute $y = \Sigma c_n x^n$ into the differential equation $(1+x)y' = \alpha y$ we get the recurrence formula

$$c_{n+1} = \frac{(\alpha - n)c_n}{n+1}. c_{n+1} = (\alpha - n)c_n/(n + 1).$$

Since $c_0 = 1$ because of the initial condition $y(0) = 1$, the binomial series (Equation (12) in the text) follows.

(c) The function $(1 + x)^\alpha$ and the binomial series must agree on $(-1, 1)$ because of the uniqueness of solutions of linear initial value problems.

25. Substitution of $\sum_{n=0}^{\infty} c_n x^n$ into the differential equation $y'' = y' + y$ leads routinely — via shifts of summation to exhibit x^n-terms throughout — to the recurrence formula

$$(n+2)(n+1)c_{n+2} = (n+1)c_{n+1} + c_n,$$

and the given initial conditions yield $c_0 = 0 = F_0$ and $c_1 = 1 = F_1$. But instead of proceeding immediately to calculate explicit values of further coefficients, let us first multiply the recurrence relation by $n!$. This trick provides the relation

$$(n+2)!c_{n+2} = (n+1)!c_{n+1} + n!c_n,$$

that is, the Fibonacci-defining relation $F_{n+2} = F_{n+1} + F_n$ where $F_n = n!c_n$, so we see that $c_n = F_n / n!$ as desired.

26. This problem is pretty fully outlined in the textbook. The only hard part is squaring the power series:

$$\begin{aligned}
&\left(1 + c_3x^3 + c_5x^5 + c_7x^7 + c_9x^9 + c_{11}x^{11} + \cdots\right)^2 \\
&\quad = x^2 + 2c_3x^4 + \left(c_3^2 + 2c_5\right)x^6 + \left(2c_3c_5 + 2c_7\right)x^8 + \\
&\qquad \left(c_5^2 + 2c_3c_7 + 2c_9\right)x^{10} + \left(2c_5c_7 + 2c_3c_9 + 2c_{11}\right)x^{12} + \cdots
\end{aligned}$$

27. **(b)** The roots of the characteristic equation $r^3 = 1$ are $r_1 = 1$, $r_2 = \alpha = (-1 + i\sqrt{3})/2$, and $r_3 = \beta = (-1 - i\sqrt{3})/2$. Then the general solution is

$$y(x) = Ae^x + Be^{\alpha x} + Ce^{\beta x}. \qquad (*)$$

Imposing the initial conditions, we get the equations

$$\begin{aligned}
A + B + C &= 1 \\
A + \alpha B + \beta C &= 1 \\
A + \alpha^2 B + \beta^2 C &= -1.
\end{aligned}$$

The solution of this system is $A = 1/3$, $B = (1 - i\sqrt{3})/3$, $C = (1 + i\sqrt{3})/3$. Substitution of these coefficients in (*) and use of Euler's relation $e^{i\theta} = \cos\theta + i\sin\theta$ finally yields the desired result.

SECTION 11.2

POWER SERIES SOLUTIONS

Instead of deriving in detail the recurrence relations and solution series for Problems 1 through 15, we indicate where some of these problems and answers originally came from. Each of the differential equations in Problems 1–10 is of the form

$$(Ax^2 + B)y'' + Cxy' + Dy = 0$$

with selected values of the constants A, B, C, D. When we substitute $y = \Sigma c_n x^n$, shift indices where appropriate, and collect coefficients, we get

$$\sum_{n=0}^{\infty}\left[An(n-1)c_n + B(n+1)(n+2)c_{n+2} + Cnc_n + Dc_n\right]x^n = 0.$$

Thus the recurrence relation is

$$c_{n+2} = -\frac{An^2 + (C-A)n + D}{B(n+1)(n+2)}c_n \qquad \text{for } n \geq 0.$$

It yields a solution of the form

$$y = c_0\, y_{\text{even}} + c_1\, y_{\text{odd}}$$

where y_{even} and y_{odd} denote series with terms of even and odd degrees, respectively. The even-degree series $c_0 + c_2x^2 + c_4x^4 + \cdots$ converges (by the ratio test) provided that

$$\lim_{n\to\infty}\left|\frac{c_{n+2}x^{n+2}}{c_nx^n}\right| = \left|\frac{Ax^2}{B}\right| < 1.$$

Hence its radius of convergence is at least $\rho = \sqrt{|B/A|}$, as is that of the odd-degree series $c_1x + c_3x^3 + c_5x^4 + \cdots$. (See Problem 6 for an example in which the radius of convergence is, surprisingly, greater than $\sqrt{|B/A|}$.)

In Problems 1–15 we give first the recurrence relation and the radius of convergence, then the resulting power series solution.

1. $c_{n+2} = c_n$; $\quad \rho = 1$; $\quad c_0 = c_2 = c_4 = \cdots$; $\quad c_1 = c_3 = c_4 = \cdots$

$$y(x) = c_0\sum_{n=0}^{\infty}x^{2n} + c_1\sum_{n=0}^{\infty}x^{2n+1} = \frac{c_0 + c_1x}{1-x^2}$$

2. $c_{n+2} = -\dfrac{1}{2}c_n$; $\quad \rho = 2$; $\quad c_{2n} = \dfrac{(-1)^n c_0}{2^n}$; $\quad c_{2n+1} = \dfrac{(-1)^n c_1}{2^n}$

$$y(x) = c_0\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{2^n} + c_1\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{2^n}$$

3. $c_{n+2} = -\dfrac{c_n}{(n+2)}$; $\quad \rho = \infty$;

$$c_{2n} = \frac{(-1)^n c_0}{(2n)(2n-2)\cdot\cdots\cdot 4\cdot 2} = \frac{(-1)^n c_0}{n!2^n};$$

$$c_{2n+1} = \frac{(-1)^n c_1}{(2n+1)(2n-1)\cdot\cdots\cdot 5\cdot 3} = \frac{(-1)^n c_1}{(2n+1)!!}$$

$$y(x) = c_0\sum_{n=0}^{\infty}(-1)^n \frac{x^{2n}}{n!2^n} + c_1\sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{(2n+1)!!}$$

4. $$c_{n+2} = -\frac{n+4}{n+2}c_n; \qquad \rho = 1$$

$$c_{2n} = \left(-\frac{2n+2}{2n}\right)\left(-\frac{2n}{2n-2}\right)\cdot\cdots\cdot\left(-\frac{6}{4}\right)\left(-\frac{4}{2}\right)c_0 = (-1)^n \frac{2n+2}{2}c_0 = (-1)^n(n+1)c_0$$

$$c_{2n} = \left(-\frac{2n+3}{2n+1}\right)\left(-\frac{2n+1}{2n-1}\right)\cdot\cdots\cdot\left(-\frac{7}{5}\right)\left(-\frac{5}{3}\right)c_0 = (-1)^n \frac{2n+3}{3}c_1$$

$$y(x) = c_0\sum_{n=0}^{\infty}(-1)^n(n+1)x^{2n} + \frac{1}{3}c_1\sum_{n=0}^{\infty}(-1)^n(2n+3)x^{2n+1}$$

5. $$c_{n+2} = \frac{nc_n}{3(n+2)}; \qquad \rho = 3; \qquad c_2 = c_4 = c_6 = \cdots = 0$$

$$c_{2n+1} = \frac{2n-1}{3(2n+1)}\cdot\frac{2n-3}{3(2n-1)}\cdot\cdots\cdot\frac{3}{3(5)}\cdot\frac{1}{3(3)}c_1 = \frac{c_1}{(2n+1)3^n}$$

$$y(x) = c_0 + c_1\sum_{n=0}^{\infty}\frac{x^{2n+1}}{(2n+1)3^n}$$

6. $$c_{n+2} = \frac{(n-3)(n-4)}{(n+1)(n+2)}c_n$$

The factor $(n-3)$ in the numerator yields $c_5 = c_7 = c_9 = \cdots = 0,$ and the factor $(n-4)$ yields $c_6 = c_8 = c_{10} = \cdots = 0.$ Hence y_{even} and y_{odd} are both polynomials with radius of convergence $\rho = \infty.$

$$y(x) = c_0(1+6x^2+x^4) + c_1(x+x^3)$$

7. $$c_{n+2} = -\frac{(n-4)^2}{3(n+1)(n+2)}c_n; \qquad \rho \ge \sqrt{3}$$

The factor $(n-4)$ yields $c_6 = c_8 = c_{10} = \cdots = 0,$ so y_{even} is a 4th-degree polynomial.

We find first that $c_3 = -c_1/2$ and $c_5 = c_1/120$, and then for $n \ge 3$ that

$$c_{2n+1} = \left(-\frac{(2n-5)^2}{3(2n)(2n+1)}\right)\left(-\frac{(2n-7)^2}{3(2n-2)(2n-1)}\right)\cdots\cdots\left(-\frac{1^2}{3(6)(7)}\right)c_5 =$$

$$= (-1)^{n-2}\frac{\left[(2n-5)!!\right]^2}{3^{n-2}(2n+1)(2n-1)\cdots\cdots 7\cdot 6}\cdot\frac{c_1}{120} = 9\cdot(-1)^n\frac{\left[(2n-5)!!\right]^2}{3^n(2n+1)!}c_1$$

$$y(x) = c_0\left(1-\frac{8}{3}x^2+\frac{8}{27}x^4\right)+c_1\left[x-\frac{1}{2}x^3+\frac{1}{120}x^5+9\sum_{n=3}^{\infty}\frac{[(2n-5)!!]^2(-1)^n}{(2n+1)!\,3^n}x^{2n+1}\right]$$

8. $$c_{n+2} = \frac{(n-4)(n+4)}{2(n+1)(n+2)}c_n; \qquad \rho\geq\sqrt{2}$$

We find first that $c_3=-5c_1/4$ and $c_5=7c_1/32$, and then for $n\geq 3$ that

$$c_{2n+1} = \left(\frac{(2n-5)(2n+3)}{2(2n)(2n+1)}\right)\left(\frac{(2n-7)(2n+1)}{2(2n-2)(2n-1)}\right)\cdots\cdots\left(\frac{1\cdot 9}{2(6)(7)}\right)c_5 =$$

$$= \frac{(2n-5)!!(2n+3)(2n+1)\cdots\cdots 9}{2^{n-2}(2n+1)(2n)\cdots\cdots 7\cdot 6}\cdot\frac{7c_1}{32} = 4\cdot\frac{5!}{7\cdot 5\cdot 3}\cdot\frac{7}{32}\frac{(2n-5)!!(2n+3)!!}{2^n(2n+1)!}c_1$$

$$c_{2n+1} = \frac{(2n-5)!!(2n+3)!!}{2^n(2n+1)!}c_1$$

$$y(x) = c_0\left(1-4x^2+2x^4\right)+c_1\left[x-\frac{5}{4}x^3+\frac{7}{32}x^5+\sum_{n=3}^{\infty}\frac{(2n-5)!!(2n+3)!!}{(2n+1)!\,2^n}x^{2n+1}\right]$$

9. $$c_{n+2} = \frac{(n+3)(n+4)}{(n+1)(n+2)}c_n; \qquad \rho=1$$

$$c_{2n} = \frac{(2n+1)(2n+2)}{(2n-1)(2n)}\cdot\frac{(2n-1)(2n)}{(2n-3)(2n-2)}\cdots\cdots\frac{3\cdot 4}{1\cdot 2}c_0 = \frac{1}{2}(n+1)(2n+1)c_0$$

$$c_{2n+1} = \frac{(2n+2)(2n+3)}{(2n)(2n+1)}\cdot\frac{(2n)(2n+1)}{(2n-2)(2n-1)}\cdots\cdots\frac{4\cdot 5}{2\cdot 3}c_1 = \frac{1}{3}(n+1)(2n+3)c_1$$

$$y(x) = c_0\sum_{n=0}^{\infty}(n+1)(2n+1)x^{2n}+\frac{1}{3}c_1\sum_{n=0}^{\infty}(n+1)(2n+3)x^{2n+1}$$

10. $$c_{n+2} = -\frac{(n-4)}{3(n+1)(n+2)}c_n; \qquad \rho=\infty$$

The factor $(n-4)$ yields $c_6=c_8=c_{10}=\cdots=0$, so y_{even} is a 4th-degree polynomial. We find first that $c_3=c_1/6$ and $c_5=c_1/360$, and then for $n\geq 3$ that

$$c_{2n+1} = \frac{-(2n-5)}{3(2n+1)(2n)} \cdot \frac{-(2n-3)}{3(2n-1)(2n-2)} \cdot \cdots \cdot \frac{-1}{3(7)(6)} c_5$$

$$= \frac{(2n-5)!!(-1)^{n-2}}{3^{n-2}(2n+1)(2n)\cdot\cdots\cdot(7)(6)} \cdot \frac{c_1}{360} =$$

$$= \frac{3^2 \cdot 5!}{360} \cdot \frac{(2n-5)!!(-1)^n}{3^n(2n+1)(2n)\cdot\cdots\cdot(7)(6)\cdot 5!} \cdot c_1 = 3 \cdot \frac{(2n-5)!!(-1)^n}{3^n(2n+1)!} c_1$$

$$y(x) = c_0\left(1 + \frac{2}{3}x^2 + \frac{1}{27}x^4\right) + c_1\left[x + \frac{1}{6}x^3 + \frac{1}{360}x^5 + 3\sum_{n=3}^{\infty} \frac{(2n-5)!!(-1)^n}{(2n+1)!\,3^n} x^{2n+1}\right]$$

11. $c_{n+2} = \dfrac{2(n-5)}{5(n+1)(n+2)} c_n; \qquad \rho = \infty$

The factor $(n-5)$ yields $c_7 = c_9 = c_{11} = \cdots = 0,$ so y_{odd} is a 5th-degree polynomial. We find first that $c_2 = -c_1,$ $c_4 = c_0/10$ and $c_6 = c_0/750,$ and then for $n \geq 4$ that

$$c_{2n} = \frac{2(2n-7)}{5(2n)(2n-1)} \cdot \frac{2(2n-5)}{5(2n-2)(2n-3)} \cdot \cdots \cdot \frac{2(1)}{5(8)(7)} c_6$$

$$= \frac{2^{n-3}(2n-7)!!}{5^{n-3}(2n)(2n-1)\cdot\cdots\cdot(8)(7)} \cdot \frac{c_0}{750} =$$

$$= \frac{5^3 \cdot 6!}{2^3 \cdot 750} \cdot \frac{2^n(2n-7)!!}{5^n(2n)(2n)\cdot\cdots\cdot(8)(7)\cdot 6!} \cdot c_1 = 15 \cdot \frac{2^n(2n-7)!!}{5^n(2n)!} c_0$$

$$y(x) = c_1\left(x - \frac{4x^3}{15} + \frac{4x^5}{375}\right) + c_0\left[1 - x^2 + \frac{x^4}{10} + \frac{x^6}{750} + 15\sum_{n=4}^{\infty} \frac{(2n-7)!!\,2^n}{(2n)!\,5^n} x^{2n}\right]$$

12. $c_{n+3} = \dfrac{c_n}{n+2}; \qquad \rho = \infty$

When we substitute $y = \Sigma c_n x^n$ into the given differential equation, we find first that $c_2 = 0,$ so the recurrence relation yields $c_5 = c_8 = c_{11} = \cdots = 0$ also.

$$y(x) = c_0\left[1 + \sum_{n=1}^{\infty} \frac{x^{3n}}{2 \cdot 5 \cdot \cdots \cdot (3n-1)}\right] + c_1 \sum_{n=0}^{\infty} \frac{x^{3n+1}}{n!\,3^n}$$

13. $c_{n+3} = -\dfrac{c_n}{n+3}; \qquad \rho = \infty$

When we substitute $y = \Sigma c_n x^n$ into the given differential equation, we find first that $c_2 = 0,$ so the recurrence relation yields $c_5 = c_8 = c_{11} = \cdots = 0$ also.

$$y(x) = c_0 \sum_{n=0}^{\infty} \frac{(-1)^n x^{3n}}{n!\,3^n} + c_1 \sum_{n=0}^{\infty} \frac{(-1)^n x^{3n+1}}{1 \cdot 4 \cdot \cdots \cdot (3n+1)}$$

14. $c_{n+3} = -\dfrac{c_n}{(n+2)(n+3)}; \qquad \rho = \infty$

When we substitute $y = \Sigma c_n x^n$ into the given differential equation, we find first that $c_2 = 0,$ so the recurrence relation yields $c_5 = c_8 = c_{11} = \cdots = 0$ also. Then

$$c_{3n} = \frac{-1}{(3n)(3n-1)} \cdot \frac{-1}{(3n-3)(3n-4)} \cdot \cdots \cdot \frac{-1}{3 \cdot 2} c_0 = \frac{(-1)^n c_0}{3^n n! \cdot (3n-1)(3n-4) \cdot \cdots \cdot 5 \cdot 2},$$

$$c_{3n+1} = \frac{-1}{(3n+1)(3n)} \cdot \frac{-1}{(3n-2)(3n-3)} \cdot \cdots \cdot \frac{-1}{4 \cdot 3} c_1 = \frac{(-1)^n c_1}{3^n n! \cdot (3n+1)(3n-2) \cdot \cdots \cdot 4 \cdot 1}.$$

$$y(x) = c_0\left[1 + \sum_{n=1}^{\infty} \frac{(-1)^n x^{3n}}{3^n n! \cdot 2 \cdot 5 \cdot \cdots \cdot (3n-1)}\right] + c_1 \sum_{n=0}^{\infty} \frac{(-1)^n x^{3n+1}}{3^n n! \cdot 1 \cdot 4 \cdot \cdots \cdot (3n+1)}$$

15. $c_{n+4} = -\dfrac{c_n}{(n+3)(n+4)}; \qquad \rho = \infty$

When we substitute $y = \Sigma c_n x^n$ into the given differential equation, we find first that $c_2 = c_3 = 0,$ so the recurrence relation yields $c_6 = c_{10} = \cdots = 0$ and $c_7 = c_{11} = \cdots = 0$ also. Then

$$c_{4n} = \frac{-1}{(4n)(4n-1)} \cdot \frac{-1}{(4n-4)(4n-5)} \cdot \cdots \cdot \frac{-1}{4 \cdot 3} c_0 = \frac{(-1)^n c_0}{4^n n! \cdot (4n-1)(4n-5) \cdot \cdots \cdot 5 \cdot 3},$$

$$c_{3n+1} = \frac{-1}{(4n+1)(4n)} \cdot \frac{-1}{(4n-3)(4n-4)} \cdot \cdots \cdot \frac{-1}{5 \cdot 4} c_1 = \frac{(-1)^n c_1}{4^n n! \cdot (4n+1)(4n-3) \cdot \cdots \cdot 9 \cdot 5}.$$

$$y(x) = c_0\left[1 + \sum_{n=1}^{\infty} \frac{(-1)^n x^{4n}}{4^n n! \cdot 3 \cdot 7 \cdot \cdots \cdot (4n-1)}\right] + c_1\left[x + \sum_{n=1}^{\infty} \frac{(-1)^n x^{4n+1}}{4^n n! \cdot 5 \cdot 9 \cdot \cdots \cdot (4n+1)}\right]$$

16. The recurrence relation is $c_{n+2} = -\dfrac{n-1}{n+1} c_n$ for $n \geq 1$. The factor $(n-1)$ in the numerator yields $c_3 = c_5 = c_7 = \cdots = 0$. When we substitute $y = \Sigma c_n x^n$ into the given differential equation, we find first that $c_2 = c_0,$ and then the recurrence relation gives

$$c_{2n} = -\frac{2n-3}{2n-1} \cdot -\frac{2n-5}{2n-3} \cdot \cdots \cdot -\frac{3}{5} \cdot -\frac{1}{3} c_2 = \frac{(-1)^{n-1}}{2n-1} c_0.$$

Hence

$$\begin{aligned} y(x) &= c_1 x + c_0\left(1 + x^2 - \frac{x^4}{3} + \frac{x^6}{5} - \frac{x^8}{7} + \cdots\right) \\ &= c_1 x + c_0 + c_0 x\left(x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots\right) = c_1 x + c_0\left(1 + x \tan^{-1} x\right). \end{aligned}$$

With $c_0 = y(0) = 0$ and $c_1 = y'(0) = 1$ we obtain the particular solution $y(x) = x$.

17. The recurrence relation

$$c_{n+2} = -\frac{(n-2)c_n}{(n+1)(n+2)}$$

yields $c_2 = c_0 = y(0) = 1$ and $c_4 = c_6 = \cdots = 0$. Because $c_1 = y'(0) = 0$, it follows also that $c_1 = c_3 = c_5 = \cdots = 0$. Thus the desired particular solution is $y(x) = 1 + x^2$.

18. The substitution $t = x - 1$ yields $y'' + ty' + y = 0$, where primes now denote differentiation with respect to t. When we substitute $y = \Sigma c_n t^n$ we get the recurrence relation

$$c_{n+2} = -\frac{c_n}{n+2}.$$

for $n \geq 0$, so the solution series has radius of convergence $\rho = \infty$.. The initial conditions give $c_0 = 2$ and $c_1 = 0$, so $c_{\text{odd}} = 0$ and it follows that

$$y = 2\left(1 - \frac{t^2}{2} + \frac{t^4}{2\cdot 4} - \frac{t^6}{2\cdot 4\cdot 6} + \cdots\right),$$

$$y(x) = 2\left(1 - \frac{(x-1)^2}{2} + \frac{(x-1)^4}{2\cdot 4} - \frac{(x-1)^6}{2\cdot 4\cdot 6} + \cdots\right) = 2\sum_{n=0}^{\infty}\frac{(-1)^n(x-1)^{2n}}{n!2^n}.$$

19. The substitution $t = x - 1$ yields $(1 - t^2)y'' - 6ty' - 4y = 0$, where primes now denote differentiation with respect to t. When we substitute $y = \Sigma c_n t^n$ we get the recurrence relation

$$c_{n+2} = \frac{n+4}{n+2}c_n.$$

for $n \geq 0$, so the solution series has radius of convergence $\rho = 1$, and therefore converges if $-1 < t < 1$. The initial conditions give $c_0 = 0$ and $c_1 = 1$, so $c_{\text{even}} = 0$ and

$$c_{2n+1} = \frac{2n+3}{2n+1}\cdot\frac{2n+1}{2n-1}\cdot\cdots\cdot\frac{7}{5}\cdot\frac{5}{3}c_1 = \frac{2n+3}{3}.$$

Thus

$$y = \frac{1}{3}\sum_{n=0}^{\infty}(2n+3)t^{2n+1} = \frac{1}{3}\sum_{n=0}^{\infty}(2n+3)(x-1)^{2n+1},$$

and the x-series converges if $0 < x < 2$.

20. The substitution $t = x - 3$ yields $(t^2 + 1)y'' - 4ty' + 6y = 0$, where primes now denote differentiation with respect to t. When we substitute $y = \Sigma c_n t^n$ we get the recurrence relation

$$c_{n+2} = -\frac{(n-2)(n-3)}{(n+1)(n+2)}c_n$$

for $n \geq 0$. The initial conditions give $c_0 = 2$ and $c_1 = 0$. It follows that $c_{\text{odd}} = 0$, $c_2 = -6$ and $c_4 = c_6 = \cdots = 0$, so the solution reduces to

$$y = 2 - 6t^2 = 2 - 6(x-3)^2.$$

21. The substitution $t = x + 2$ yields $(4t^2 + 1)y'' = 8y$, where primes now denote differentiation with respect to t. When we substitute $y = \Sigma c_n t^n$ we get the recurrence relation

$$c_{n+2} = -\frac{4(n-2)}{(n+2)} c_n$$

for $n \geq 0$. The initial conditions give $c_0 = 1$ and $c_1 = 0$. It follows that $c_{\text{odd}} = 0$, $c_2 = 4$ and $c_4 = c_6 = \cdots = 0$, so the solution reduces to

$$y = 2 + 4t^2 = 1 + 4(x+2)^2.$$

22. The substitution $t = x + 3$ yields $(t^2 - 9)y'' + 3ty' - 3y = 0$, with primes now denoting differentiation with respect to t. When we substitute $y = \Sigma c_n t^n$ we get the recurrence relation

$$c_{n+2} = \frac{(n+3)(n-1)}{9(n+1)(n+2)} c_n$$

for $n \geq 0$. The initial conditions give $c_0 = 0$ and $c_1 = 2$. It follows that $c_{\text{even}} = 0$ and $c_3 = c_5 = \cdots = 0$, so

$$y = 2t = 2x + 6.$$

In Problems 23–26 we first derive the recurrence relation, and then calculate the solution series $y_1(x)$ with $c_0 = 1$ and $c_1 = 0$, the solution series $y_2(x)$ with $c_0 = 0$ and $c_1 = 1$.

23. Substitution of $y = \Sigma c_n x^n$ yields

$$c_0 + 2c_2 + \sum_{n=1}^{\infty}\left[c_{n-1} + c_n + (n+1)(n+2)c_{n+2}\right]x^n = 0,$$

so

$$c_2 = -\frac{1}{2}c_0, \qquad c_{n+2} = -\frac{c_{n-1} + c_n}{(n+1)(n+2)} \quad \text{for} \quad n \geq 1.$$

$$y_1(x) = 1 - \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{24} + \cdots; \qquad y_2(x) = x - \frac{x^3}{6} - \frac{x^4}{12} + \frac{x^5}{120} + \cdots$$

24. Substitution of $y = \Sigma c_n x^n$ yields

$$-2c_2 + \sum_{n=1}^{\infty}\left[2c_{n-1} + n(n+1)c_n - (n+1)(n+2)c_{n+2}\right]x^n = 0,$$

so

$$c_2 = 0, \qquad c_{n+2} = \frac{c_{n-1} + n(n+1)c_n}{(n+1)(n+2)} \quad \text{for} \quad n \geq 1.$$

$$y_1(x) = 1 + \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^6}{45} + \cdots; \qquad y_2(x) = x + \frac{x^3}{3} + \frac{x^4}{6} + \frac{x^5}{5} + \cdots$$

25. Substitution of $y = \Sigma c_n x^n$ yields

$$2c_2 + 6c_3 x + \sum_{n=2}^{\infty} \left[c_{n-2} + (n-1)c_{n-1} + (n+1)(n+2)c_{n+2} \right] x^n = 0,$$

so

$$c_2 = c_3 = 0, \qquad c_{n+2} = -\frac{c_{n-2} + (n-1)c_{n-1}}{(n+1)(n+2)} \quad \text{for} \quad n \geq 2.$$

$$y_1(x) = 1 - \frac{x^4}{12} + \frac{x^7}{126} + \frac{x^8}{672} + \cdots; \qquad y_2(x) = x - \frac{x^4}{12} - \frac{x^5}{20} + \frac{x^7}{126} + \cdots$$

26. Substitution of $y = \Sigma c_n x^n$ yields

$$2c_2 + 6c_3 x + 12c_4 x^2 + (2c_2 + 20c_5)x^3 + \sum_{n=4}^{\infty} \left[c_{n-4} + (n-1)(n-2)c_{n-1} + (n+1)(n+2)c_{n+2} \right] x^n = 0,$$

so

$$c_2 = c_3 = c_4 = c_5 = 0, \qquad c_{n+2} = -\frac{c_{n-4} + (n-1)(n-2)c_{n-1}}{(n+1)(n+2)} \quad \text{for} \quad n \geq 4.$$

$$y_1(x) = 1 - \frac{x^6}{30} + \frac{x^9}{72} - \frac{29x^{12}}{3960} + \cdots; \qquad y_2(x) = x - \frac{x^7}{42} + \frac{x^{10}}{90} - \frac{41x^{13}}{6552} + \cdots$$

27. Substitution of $y = \Sigma c_n x^n$ yields

$$c_0 + 2c_2 + (2c_1 + 6c_3)x + \sum_{n=2}^{\infty} \left[2c_{n-2} + (n+1)c_n + (n+1)(n+2)c_{n+2} \right] x^n = 0,$$

so

$$c_2 = -\frac{c_0}{2}, \quad c_3 = -\frac{c_1}{3}, \qquad c_{n+2} = -\frac{2c_{n-2} + (n+1)c_n}{(n+1)(n+2)} \quad \text{for} \quad n \geq 2.$$

With $c_0 = y(0) = 1$ and $c_1 = y'(0) = -1$, we obtain

$$y(x) = 1 - x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{24} + \frac{x^5}{30} + \frac{29x^6}{720} - \frac{13x^7}{630} - \frac{143x^8}{40320} + \frac{31x^9}{22680} + \cdots.$$

Finally, $x = 0.5$ gives

$$\begin{aligned} y(0.5) &= 1 - 0.5 - 0.125 + 0.041667 - 0.002604 + 0.001042 \\ &\qquad + 0.000629 - 0.000161 - 0.000014 + 0.000003 + \cdots \\ y(0.5) &\approx 0.415562 \approx 0.4156. \end{aligned}$$

28. When we substitute $y = \Sigma c_n x^n$ and $e^{-x} = \sum (-1)^n x^n / n!$ and then collect coefficients of the terms involving $1,\ x,\ x^2$, and x^3, we find that

$$c_2 = -\frac{c_0}{2}, \quad c_3 = \frac{c_0 - c_1}{6}, \quad c_4 = \frac{c_1}{12}, \quad c_5 = -\frac{3c_0 + 2c_1}{120}.$$

With the choices $c_0 = 1,\ c_1 = 0$ and $c_0 = 0,\ c_1 = 1$ we obtain the two series solutions

$$y_1(x) = 1 - \frac{x^2}{2} + \frac{x^3}{6} - \frac{x^5}{40} + \cdots \quad \text{and} \quad y_2(x) = x - \frac{x^3}{6} + \frac{x^4}{12} - \frac{x^5}{60} + \cdots.$$

29. When we substitute $y = \Sigma c_n x^n$ and $\cos x = \sum (-1)^n x^{2n} / (2n)!$ and then collect coefficients of the terms involving $1, x, x^2, \cdots, x^6$, we obtain the equations

$$c_0 + 2c_2 = 0, \quad c_1 + 6c_3 = 0, \quad 12c_4 = 0, \quad -2c_3 + 20c_5 = 0,$$

$$\frac{1}{12}c_2 - 5c_4 + 30c_6 = 0, \quad \frac{1}{4}c_3 - 9c_5 + 42c_6 = 0,$$

$$-\frac{1}{360}c_2 + \frac{1}{2}c_4 - 14c_6 + 56c_8 = 0.$$

Given c_0 and c_1, we can solve easily for $c_2, c_3, \cdots, c_8$ in turn. With the choices $c_0 = 1,\ c_1 = 0$ and $c_0 = 0,\ c_1 = 1$ we obtain the two series solutions

$$y_1(x) = 1 - \frac{x^2}{2} + \frac{x^6}{720} + \frac{13x^8}{40320} + \cdots \quad \text{and} \quad y_2(x) = x - \frac{x^3}{6} - \frac{x^5}{60} - \frac{13x^7}{5040} + \cdots.$$

30. When we substitute $y = \Sigma c_n x^n$ and $\sin x = \Sigma (-1)^n x^{2n+1} / (2n+1)!$, and then collect coefficients of the terms involving $1, x, x^2, \cdots, x^5$, we obtain the equations

$$c_0 + c_1 + 2c_2 = 0, \quad c_1 + 2c_2 + 6c_3 = 0, \quad -\frac{c_1}{6} + c_2 + 3c_3 + 12c_4 = 0,$$

$$-\frac{c_2}{3} + c_3 + 4c_4 + 20c_5 = 0, \quad \frac{c_1}{120} - \frac{c_3}{2} + c_4 + 5c_5 + 30c_6 = 0.$$

Given c_0 and c_1, we can solve easily for $c_2, c_3, \cdots, c_6$ in turn. With the choices $c_0 = 1,\ c_1 = 0$ and $c_0 = 0,\ c_1 = 1$ we obtain the two series solutions

$$y_1(x) = 1 - \frac{x^2}{2} + \frac{x^3}{6} - \frac{x^5}{60} + \frac{x^6}{180} + \cdots \text{ and } y_2(x) = x - \frac{x^2}{2} + \frac{x^4}{18} - \frac{7x^5}{360} + \frac{x^6}{900} + \cdots.$$

33. Substitution of $y = \Sigma c_n x^n$ in Hermite's equation leads in the usual way to the recurrence formula

$$c_{n+2} = -\frac{2(\alpha - n)c_n}{(n+1)(n+2)}.$$

Starting with $c_0 = 1$, this formula yields

$$c_2 = -\frac{2\alpha}{2!}, \quad c_4 = +\frac{2^2\alpha(\alpha-2)}{4!}, \quad c_6 = -\frac{2^3\alpha(\alpha-2)(\alpha-4)}{6!}, \ \ldots.$$

Starting with $c_1 = 1$, it yields

$$c_3 = -\frac{2(\alpha-1)}{3!}, \quad c_5 = +\frac{2^2(\alpha-1)(\alpha-3)}{5!}, \quad c_7 = -\frac{2^3(\alpha-1)(\alpha-3)(\alpha-5)}{7!}, \ \ldots.$$

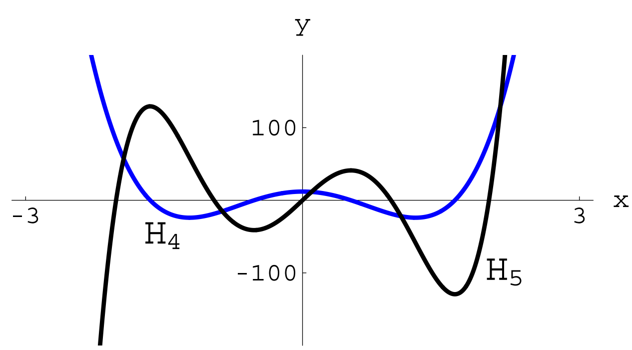
This gives the desired even-term and odd-term series y_1 and y_2. If α is an integer, then obviously one series or the other has only finitely many non-zero terms. For instance, with $\alpha = 4$ we get

$$y_1(x) = 1 - \frac{2\cdot 4}{2}x^2 + \frac{2^2\cdot 4\cdot 2}{24}x^4 = 1 - 4x^2 + \frac{4}{3}x^4 = \frac{1}{12}\left(16x^4 - 48x^2 + 12\right),$$

and with $\alpha = 5$ we get

$$y_2(x) = x - \frac{2\cdot 4}{6}x^3 + \frac{2^2\cdot 4\cdot 2}{120}x^5 = x - \frac{4}{3}x^3 + \frac{4}{15}x^5 = \frac{1}{120}\left(32x^5 - 160x^3 + 120\right).$$

The figure below shows the interlaced zeros of the 4th and 5th Hermite polynomials.



34. Substitution of $y = \Sigma c_n x^n$ in the Airy equation leads upon shift of index and collection of terms to

$$2c_2 + \sum_{n=1}^{\infty} \left[(n+1)(n+2)c_{n+2} - c_{n-1} \right] x^n = 0.$$

The identity principle then gives $c_2 = 0$ and the recurrence formula

$$c_{n+3} = \frac{c_n}{(n+2)(n+3)}.$$

Because of the "3-step" in indices, it follows that $c_2 = c_5 = c_8 = c_{11} = \cdots = 0$. Starting with $c_0 = 1$, we calculate

$$c_3 = \frac{1}{2 \cdot 3} = \frac{1}{3!} =, \quad c_6 = \frac{1}{3! \cdot 5 \cdot 6} = \frac{1 \cdot 4}{6!}, \quad c_9 = \frac{1 \cdot 4}{6! \cdot 8 \cdot 9} = \frac{1 \cdot 4 \cdot 7}{9!}, \; \ldots.$$

Starting with $c_1 = 1$, we calculate

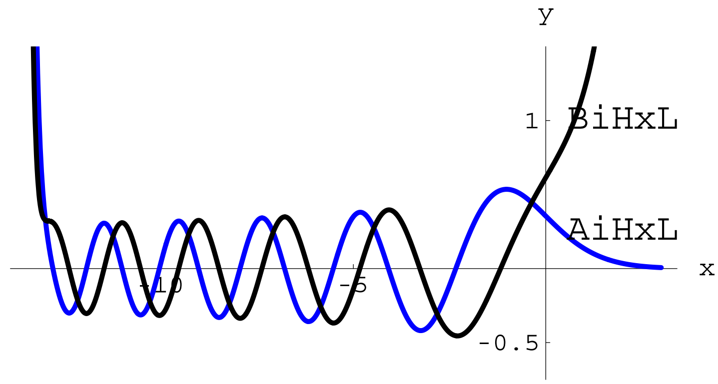
$$c_4 = \frac{1}{3 \cdot 4} = \frac{2}{4!}, \quad c_7 = \frac{2}{4! \cdot 6 \cdot 7} = \frac{2 \cdot 5}{7!}, \quad c_{10} = \frac{2 \cdot 5}{7! \cdot 9 \cdot 10} = \frac{2 \cdot 5 \cdot 8}{10!}, \; \ldots.$$

Evidently we are building up the coefficients

$$c_{3k} = \frac{1 \cdot 4 \cdot \cdots (3k-2)}{(3k)!} \quad \text{and} \quad c_{3k+1} = \frac{2 \cdot 5 \cdot \cdots (3k-1)}{(3k+1)!}$$

that appear in the desired series for $y_1(x)$ and $y_2(x)$. Finally, the Mathematica commands

```
A[1] = 1/6; A[k_] := A[k - 1]/(3 k (3 k - 1))
B[1] = 1/12; B[k_] := B[k - 1]/(3 k (3 k + 1))
n = 40;
y1 = 1 + Sum[A[k] x^(3 k), {k, 1, n}];
y2 = x + Sum[B[k] x^(3 k+1), {k, 1, n}];
yA = y1/(3^(2/3) Gamma[2/3]) - y2/(3^(1/3) Gamma[1/3]);
yB = y1/(3^(1/6) Gamma[2/3]) + y2/(3^(-1/6) Gamma[1/3]);
Plot[ {yA, yB}, {x, -13.5, 3}, PlotRange → {-0.75, 1.5}];
```



produce the figure above. But with $n = 50$ (instead of $n = 40$) terms we get a figure that is visually indistinguishable from Figure 11.2.3 in the textbook.

SECTION 11.3

FROBENIUS SERIES SOLUTIONS

1. Upon division of the given differential equation by x we see that $P(x) = 1 - x^2$ and $Q(x) = (\sin x)/x$. Because both are analytic at $x = 0$ — in particular, $(\sin x)/x \to 1$ as $x \to 0$ because

$$\frac{\sin x}{x} = \frac{1}{x}\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{(2n+1)!} = \sum_{n=1}^{\infty}\frac{(-1)^n x^{2n}}{(2n+1)!} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \cdots$$

— it follows that $x = 0$ is an ordinary point.

2. Division of the differential equation by x yields

$$y'' + xy' + \frac{e^x - 1}{x}y = 0.$$

Because the function

$$\frac{e^x - 1}{x} = \frac{1}{x}\left(\sum_{n=0}^{\infty}\frac{x^n}{n!} - 1\right) = \sum_{n=1}^{\infty}\frac{x^{n-1}}{n!} = 1 + \frac{x}{2!} + \frac{x^2}{3!} + \frac{x^3}{4!} + \cdots$$

is analytic at the origin, we see that $x = 0$ is an ordinary point.

3. When we rewrite the given equation in the standard form of Equation (3) in this section, we see that $p(x) = (\cos x)/x$ and $q(x) = x$. Because $(\cos x)/x \to \infty$ as $x \to 0$ it follows that $p(x)$ is not analytic, so $x = 0$ is an irregular singular point.

4. When we rewrite the given equation in the standard form of Equation (3), we have $p(x) = 2/3$ and $q(x) = (1 - x^2)/3x$. Since $q(x)$ is not analytic at the origin, $x = 0$ is an irregular singular point.

5. In the standard form of Equation (3) we have $p(x) = 2/(1 + x)$ and $q(x) = 3x^2/(1 + x)$. Both are analytic, so $x = 0$ is a regular singular point. The indicial equation is

$$r(r - 1) + 2r = r^2 + r = r(r + 1) = 0,$$

so the exponents are $r_1 = 0$ and $r_2 = -1$.

6. In the standard form of Equation (3) we have $p(x) = 2/(1 - x^2)$ and $q(x) = -2/(1 - x^2)$, so $x = 0$ is a regular singular point with $p_0 = 2$ and $q_0 = -2$. The indicial equation is $r^2 + r - 2 = 0$, so the exponents are $r = -2, 1$.

7. In the standard form of Equation (3) we have $p(x) = (6 \sin x)/x$ and $q(x) = 6$, so $x = 0$ is a regular singular point with $p_0 = q_0 = 6$. The indicial equation is $r^2 + 5r + 6 = 0$, so the exponents are $r_1 = -2$ and $r_2 = -3$.

8. In the standard form of Equation (3) we have $p(x) = 21/(6 + 2x)$ and $q(x) = 9(x^2 - 1)/(6 + 2x)$, so $x = 0$ is a regular singular point with $p_0 = 7/2$ and $q_0 = -3/2$. The indicial equation simplifies to $2r^2 + 5r - 3 = 0$, so the exponents are $r = -3, 1/2$.

9. The only singular point of the differential equation $y'' + \dfrac{x}{1-x}y' + \dfrac{x^2}{1-x}y = 0$ is $x = 1$. Upon substituting $t = x - 1$, $x = t + 1$ we get the transformed equation $y'' - \dfrac{t+1}{t}y' - \dfrac{(t+1)^2}{t}y = 0$, where primes now denote differentiation with respect to t. In the standard form of Equation (3) we have $p(t) = -(1+t)$ and $q(t) = -t(1+t)^2$. Both these functions are analytic, so it follows that $x = 1$ is a regular singular point of the original equation.

10. The only singular point of the differential equation $y'' + \dfrac{2}{x-1}y' + \dfrac{1}{(x-1)^2}y = 0$ is $x = 1$. Upon substituting $t = x - 1$, $x = t + 1$ we get the transformed equation $y'' + \dfrac{2}{t}y' + \dfrac{1}{t^2}y = 0$, where primes now denote differentiation with respect to t. In the standard form of Equation (3) we have $p(t) \equiv 2$ and $q(t) \equiv 1$. Both these functions are analytic, so it follows that $x = 1$ is a regular singular point of the original equation.

11. The only singular points of the differential equation $y'' - \dfrac{2x}{1-x^2}y' + \dfrac{12}{1-x^2}y = 0$ are $x = +1$ and $x = -1$.

$x = +1$: Upon substituting $t = x - 1,\ x = t + 1$ we get the transformed equation $y'' + \dfrac{2(t+1)}{t(t+2)}y' - \dfrac{12}{t(t+2)}y = 0$, where primes now denote differentiation with respect to t. In the standard form of Equation (3) we have $p(t) = \dfrac{2(t+1)}{t+2}$ and $q(t) = -\dfrac{12t}{t+2}$. Both these functions are analytic at $t = 0$, so it follows that $x = +1$ is a regular singular point of the original equation.

$x = -1$: Upon substituting $t = x + 1,\ x = t - 1$ we get the transformed equation $y'' + \dfrac{2(t-1)}{t(t-2)}y' - \dfrac{12}{t(t-2)}y = 0$, where primes now denote differentiation with respect to t. In the standard form of Equation (3) we have $p(t) = \dfrac{2(t-1)}{t-2}$ and $q(t) = -\dfrac{12t}{t-2}$. Both these functions are analytic at $t = 0$, so it follows that $x = -1$ is a regular singular point of the original equation.

12. The only singular point of the differential equation $y'' + \dfrac{3}{x-2}y' + \dfrac{x^3}{(x-2)^3}y = 0$ is $x = 2$. Upon substituting $t = x - 2,\ x = t + 2$ we get the transformed equation $y'' + \dfrac{3}{t}y' + \dfrac{(t+2)^3}{t^3}y = 0$, where primes now denote differentiation with respect to t. In the standard form of Equation (3) we have $p(t) \equiv 3$ and $q(t) = \dfrac{(t+2)^3}{t}$. Because q is *not* analytic at $t = 0$, it follows that $x = 2$ is an irregular singular point of the original equation.

13. The only singular points of the differential equation $y'' + \dfrac{1}{x-2}y' + \dfrac{1}{x+2}y = 0$ are $x = +2$ and $x = -2$.

$x = +2$: Upon substituting $t = x - 2,\ x = t + 2$ we get the transformed equation $y'' + \dfrac{1}{t+4}y' + \dfrac{1}{t}y = 0$, where primes now denote differentiation with respect to t. In the standard form of Equation (3) we have $p(t) = \dfrac{t}{t+4}$ and $q(t) = t$. Both these functions are analytic at $t = 0$, so it follows that $x = +2$ is a regular singular point of the original equation.

$x = -2$: Upon substituting $t = x + 2,\ x = t - 2$ we get the transformed equation $y'' + \dfrac{1}{t}y' + \dfrac{1}{t-4}y = 0$, where primes now denote differentiation with respect to t. In the standard form of Equation (3) we have $p(t) \equiv 1$ and $q(t) = \dfrac{t^2}{t-4}$. Both these functions are analytic at $t = 0$, so it follows that $x = -2$ is a regular singular point of the original equation.

14. The only singular points of the differential equation $y'' + \dfrac{x^2+9}{(x^2-9)^2}y' + \dfrac{x^2+4}{(x^2-9)^2}y = 0$ are $x = +3$ and $x = -3$.

***x* = +3**: Upon substituting $t = x-3$, $x = t+3$ we get the transformed equation $y'' + \dfrac{t^2+6t+13}{t^2(t^2+6)^2}y' + \dfrac{t^2+6t+18}{t^2(t^2+6)^2}y = 0$, where primes now denote differentiation with respect to t. Because $p(t) = \dfrac{t^2+6t+13}{t(t^2+6)^2}$ is *not* analytic at $t = 0$, it follows that $x = 3$ is an irregular singular point of the original equation.

***x* = –3**: Upon substituting $t = x+3$, $x = t-3$ we get the transformed equation $y'' + \dfrac{t^2-6t+13}{t^2(t^2-6)^2}y' + \dfrac{t^2-6t+18}{t^2(t^2-6)^2}y = 0$, where primes now denote differentiation with respect to t. Because $p(t) = \dfrac{t^2-6t+13}{t(t^2-6)^2}$ is *not* analytic at $t = 0$, it follows that $x = -3$ is an irregular singular point of the original equation.

15. The only singular point of the differential equation $y'' - \dfrac{x^2-4}{(x-2)^2}y' + \dfrac{x+2}{(x-2)^2}y = 0$ is $x = 2$. Upon substituting $t = x-2$, $x = t+2$ we get the transformed equation $y'' - \dfrac{t+4}{t}y' + \dfrac{t+4}{t^2}y = 0$, where primes now denote differentiation with respect to t. In the standard form of Equation (3) we have $p(t) = -(t+4)$ and $q(t) = t+4$. Both these functions are analytic, so it follows that $x = 2$ is a regular singular point of the original equation.

16. The only singular points of the differential equation $y'' + \dfrac{3x+2}{x^3(1-x)}y' + \dfrac{1}{x^2(1-x)}y = 0$ are $x = 0$ and $x = 1$.

***x* = 0**: In the standard form of Equation (3) we have $p(x) = \dfrac{3x+2}{x^2(1-x)}$ and $q(x) = \dfrac{1}{1-x}$. Since p is not analytic at $x = 0$, it follows that $x = 0$ is an irregular singular point.

$\boldsymbol{x = 1}$: Upon substituting $t = x - 1,\ x = t + 1$ we get the transformed equation $y'' - \dfrac{3t+5}{(t+1)^3}y' - \dfrac{t}{(t+1)^2}y = 0$, where primes now denote differentiation with respect to t. Both $p(t) \equiv -\dfrac{t(3t+5)}{(t+1)^3}$ and $q(t) = -\dfrac{t^3}{(t+1)^2}$ are analytic at $t = 0$, so it follows that $x = 1$ is a regular singular point of the original equation.

Each of the differential equations in Problems 17–20 is of the form

$$Axy'' + By' + Cy = 0$$

with indicial equation $Ar^2 + (B - A)r = 0$. Substitution of $y = \Sigma c_n x^{n+r}$ into the differential equation yields the recurrence relation

$$c_n = -\frac{C\,c_{n-1}}{A(n+r)^2 + (B-A)(n+r)}$$

for $n \geq 1$. In these problems the exponents $r_1 = 0$ and $r_2 = (A - B)/A$ do *not* differ by an integer, so this recurrence relation yields two linearly independent Frobenius series solutions when we apply it separately with $r = r_1$ and with $r = r_2$.

17. With exponent $r_1 = 0$: $c_n = -\dfrac{c_{n-1}}{4n^2 - 2n}$

$$y_1(x) = x^0\left(1 - \frac{x}{2} + \frac{x^2}{24} - \frac{x^3}{720} + \cdots\right) = \sum_{n=0}^{\infty}\frac{(-1)^n\left(\sqrt{x}\right)^{2n}}{(2n)!} = \cos\sqrt{x}$$

With exponent $r_2 = \dfrac{1}{2}$: $c_n = -\dfrac{c_{n-1}}{4n^2 + 2n}$

$$y_2(x) = x^{1/2}\left(1 - \frac{x}{6} + \frac{x^2}{120} - \frac{x^3}{5040} + \cdots\right) = \sum_{n=0}^{\infty}\frac{(-1)^n\left(\sqrt{x}\right)^{2n+1}}{(2n+1)!} = \sin\sqrt{x}$$

18. With exponent $r_1 = 0$: $c_n = \dfrac{c_{n-1}}{2n^2 + n}$

$$y_1(x) = x^0\left(1 + \frac{x}{3} + \frac{x^2}{30} + \frac{x^3}{630} + \frac{x^4}{22680} + \cdots\right) = \sum_{n=0}^{\infty}\frac{x^n}{n!(2n+1)!!}$$

With exponent $r_2 = -\dfrac{1}{2}$: $c_n = \dfrac{c_{n-1}}{2n^2 - n}$

$$y_2(x) = x^{-1/2}\left(1 + x + \frac{x^2}{6} + \frac{x^3}{90} + \frac{x^4}{2520} + \cdots\right) = \frac{1}{\sqrt{x}}\left[1 + \sum_{n=1}^{\infty}\frac{x^n}{n!(2n-1)!!}\right]$$

19. With exponent $r_1 = 0$: $c_n = \dfrac{c_{n-1}}{2n^2 - 3n}$

$$y_1(x) = x^0\left(1 - x - \frac{x^2}{2} - \frac{x^3}{18} - \frac{x^4}{360} - \cdots\right) = 1 - x - \sum_{n=2}^{\infty}\frac{x^n}{n!(2n-3)!!}$$

With exponent $r_2 = \dfrac{3}{2}$: $c_n = \dfrac{c_{n-1}}{2n^2 + 3n}$

$$y_2(x) = x^{3/2}\left(1 + \frac{x}{5} + \frac{x^2}{70} + \frac{x^3}{1890} + \frac{x^4}{83160} + \cdots\right) = x^{3/2}\left[1 + 3\sum_{n=1}^{\infty}\frac{x^n}{n!(2n+3)!!}\right]$$

20. With exponent $r_1 = 0$: $c_n = -\dfrac{2c_{n-1}}{3n^2 - n}$

$$y_1(x) = x^0\left(1 - x + \frac{x^2}{5} - \frac{x^3}{60} + \frac{x^4}{1320} - \cdots\right) = 1 + \sum_{n=1}^{\infty}\frac{(-1)^n 2^n x^n}{n!\cdot 2\cdot 5\cdot\cdots\cdot(3n-1)}$$

With exponent $r_2 = \dfrac{1}{3}$: $c_n = -\dfrac{2c_{n-1}}{3n^2 + n}$

$$y_2(x) = x^{1/3}\left(1 - \frac{x}{2} + \frac{x^2}{14} - \frac{x^3}{210} + \frac{x^4}{5460} - \cdots\right) = x^{1/3}\sum_{n=0}^{\infty}\frac{(-1)^n 2^n x^n}{n!\cdot 1\cdot 4\cdot\cdots\cdot(3n+1)}$$

The differential equations in Problems 21–24 are all of the form

$$Ax^2y'' + Bxy' + (C + Dx^2)y = 0 \qquad (1)$$

with indical equation

$$\phi(r) = Ar^2 + (B - A)r + C = 0. \qquad (2)$$

Substitution of $y = \Sigma c_n x^{n+r}$ into the differential equation yields

$$\phi(r)c_0x^r + \phi(r+1)c_1x^{r+1} + \sum_{n=2}^{\infty}\left[\phi(r+n)c_n + D\,c_{n-2}\right]x^{n+r} = 0. \qquad (3)$$

In each of Problems 21–24 the exponents r_1 and r_2 do *not* differ by an integer. Hence when we substitute either $r = r_1$ or $r = r_2$ into Equation (*) above, we find that c_0 is arbitrary because $\phi(r)$ is then zero, that $c_1 = 0$ — because its coefficient $\phi(r+1)$ is then nonzero — and that

$$c_n = -\frac{Dc_{n-2}}{\phi(r+n)} = -\frac{Dc_{n-2}}{A(n+r)^2 + (B-A)(n+r) + C} \qquad (4)$$

for $n \geq 2$. Thus this recurrence formula yields two linearly independent Frobenius series solutions when we apply it separately with $r = r_1$ and with $r = r_2$.

21. With exponent $r_1 = 1$: $c_1 = 0, \quad c_n = \dfrac{2c_{n-2}}{n(2n+3)}$

$$y_1(x) = x^1\left(1 + \frac{x^2}{7} + \frac{x^4}{154} + \frac{x^6}{6930} + \cdots\right) = x\left[1 + \sum_{n=1}^{\infty} \frac{x^{2n}}{n! \cdot 7 \cdot 11 \cdot \cdots \cdot (4n+3)}\right]$$

With exponent $r_2 = -\dfrac{1}{2}$: $c_1 = 0, \quad c_n = \dfrac{2c_{n-2}}{n(2n-3)}$

$$y_2(x) = x^{-1/2}\left(1 + x^2 + \frac{x^4}{10} + \frac{x^6}{270} + \cdots\right) = \frac{1}{\sqrt{x}}\left[1 + \sum_{n=1}^{\infty} \frac{x^{2n}}{n! \cdot 1 \cdot 5 \cdot \cdots \cdot (4n-3)}\right]$$

22. With exponent $r_1 = \dfrac{3}{2}$: $c_1 = 0, \quad c_n = -\dfrac{2c_{n-2}}{n(2n+5)}$

$$y_1(x) = x^{3/2}\left(1 - \frac{x^2}{9} + \frac{x^4}{234} - \frac{x^6}{11934} + \cdots\right) = x^{3/2}\left[1 + \sum_{n=1}^{\infty} \frac{x^{2n}}{n! \cdot 9 \cdot 13 \cdot \cdots \cdot (4n+5)}\right]$$

With exponent $r_2 = -1$: $c_1 = 0, \quad c_n = -\dfrac{2c_{n-2}}{n(2n-5)}$

$$y_2(x) = x^{-1}\left(1 + x^2 - \frac{x^4}{6} + \frac{x^6}{126} - \frac{x^8}{5544} + \cdots\right) = \frac{1}{x}\left[1 + x^2 + \sum_{n=2}^{\infty} \frac{(-1)^{n-1} x^{2n}}{n! \cdot 3 \cdot 7 \cdot \cdots \cdot (4n-5)}\right]$$

23. With exponent $r_1 = \dfrac{1}{2}$: $c_1 = 0, \quad c_n = \dfrac{c_{n-2}}{n(6n+7)}$

$$y_1(x) = x^{1/2}\left(1 + \frac{x^2}{38} + \frac{x^4}{4712} + \frac{x^6}{1215696} + \cdots\right) = \sqrt{x}\left[1 + \sum_{n=1}^{\infty} \frac{x^{2n}}{2^n n! \cdot 19 \cdot 31 \cdot \cdots \cdot (12n+7)}\right]$$

With exponent $r_2 = -\dfrac{2}{3}$: $c_1 = 0, \quad c_n = \dfrac{c_{n-2}}{n(6n-7)}$

$$y_2(x) = x^{-2/3}\left(1 + \frac{x^2}{10} + \frac{x^4}{680} + \frac{x^6}{118320} + \cdots\right) = x^{-2/3}\left[1 + \sum_{n=1}^{\infty} \frac{x^{2n}}{2^n n! \cdot 5 \cdot 17 \cdot \cdots \cdot (12n-7)}\right]$$

24. With exponent $r_1 = \dfrac{1}{3}$: $c_1 = 0, \quad c_n = -\dfrac{c_{n-2}}{n(3n+1)}$

$$y_1(x) = x^{1/3}\left(1 - \frac{x^2}{14} + \frac{x^4}{728} - \frac{x^6}{82992} + \cdots\right) = \sqrt[3]{x}\left[1 + \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n}}{2^n n! \cdot 7 \cdot 13 \cdot \cdots \cdot (6n+1)}\right]$$

With exponent $r_2 = 0$: $c_1 = 0, \quad c_n = -\dfrac{c_{n-2}}{n(3n-1)}$

$$y_2(x) = x^0\left(1 - \frac{x^2}{10} + \frac{x^4}{440} - \frac{x^6}{44880} + \cdots\right) = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n}}{2^n n! \cdot 5 \cdot 11 \cdot \cdots \cdot (6n-1)}$$

25. With exponent $r_1 = \dfrac{1}{2}$: $\quad c_n = -\dfrac{c_{n-1}}{2n}$

$$y_1(x) = x^{1/2}\left(1 - \frac{x}{2} + \frac{x^2}{8} - \frac{x^3}{48} + \frac{x^4}{384} - \cdots\right) = \sqrt{x}\sum_{n=0}^{\infty}\frac{(-1)^n x^n}{n!2^n} = \sqrt{x}\, e^{-x/2}$$

With exponent $r_2 = 0$: $\quad c_n = -\dfrac{c_{n-1}}{2n-1}$

$$y_2(x) = x^0\left(1 - x + \frac{x^2}{3} - \frac{x^3}{15} + \frac{x^4}{105} - \cdots\right) = 1 + \sum_{n=1}^{\infty}\frac{(-1)^n x^n}{(2n-1)!!}$$

26. With exponent $r_1 = \dfrac{1}{2}$: $\quad c_1 = 0, \quad c_n = \dfrac{c_{n-2}}{n}$

$$y_1(x) = x^{1/2}\left(1 + \frac{x^2}{2} + \frac{x^4}{8} + \frac{x^6}{48} + \frac{x^8}{384} + \cdots\right) = \sqrt{x}\sum_{n=0}^{\infty}\frac{x^{2n}}{n!2^n} = \sqrt{x}\, e^{x^2/2}$$

With exponent $r_2 = 0$: $\quad c_1 = 0, \quad c_n = \dfrac{2c_{n-2}}{2n-1}$

$$y_2(x) = x^0\left(1 + \frac{2x^2}{3} + \frac{4x^4}{21} + \frac{8x^6}{231} + \frac{16x^8}{3465} + \cdots\right) = 1 + \sum_{n=1}^{\infty}\frac{2^n x^{2n}}{3\cdot 7\cdot\cdots\cdot(4n-1)}$$

The differential equations in Problems 27–29 (after multiplication by x) and the one in Problem 31 are of the same form (1) above as those in Problems 21–24. However, now the exponents r_1 and $r_2 = r_1 - 1$ *do* differ by an integer. Hence when we substitute the smaller exponent $r = r_2$ into Equation (3), we find that c_0 and c_1 are *both* arbitrary, and that c_n is given (for $n \geq 2$) by the recurrence relation in (4). Thus the *smaller* exponent r_2 yields the general solution $y(x) = c_0 y_1(x) + c_1 y_2(x)$ in terms of the two linearly independent Frobenius series solutions $y_1(x)$ and $y_2(x)$.

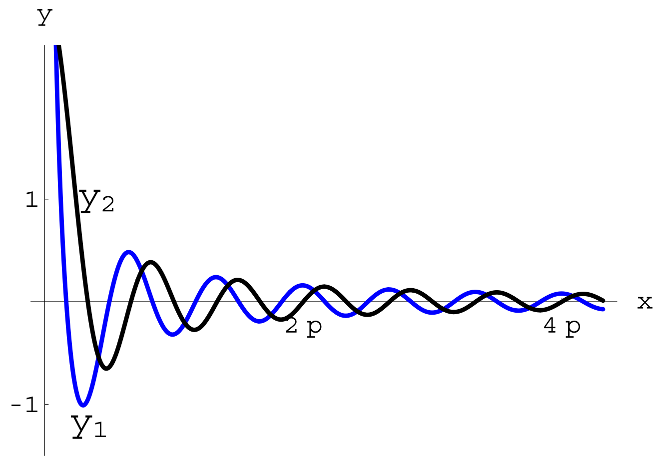
27. Exponents $r_1 = 0$ and $r_2 = -1$; with $r = -1$: $\quad c_n = -\dfrac{9c_{n-2}}{n(n-1)}$

$$y(x) = \frac{c_0}{x}\left(1 - \frac{9x^2}{2} + \frac{27x^4}{8} - \frac{81x^6}{80} + \cdots\right) + \frac{c_1}{x}\left(x - \frac{3x^3}{2} + \frac{27x^5}{40} - \frac{81x^7}{560} + \cdots\right)$$

$$= \frac{c_0}{x}\left(1 - \frac{9x^2}{2} + \frac{81x^4}{24} - \frac{729x^6}{720} + \cdots\right) + \frac{c_1}{3x}\left(3x - \frac{27x^3}{6} + \frac{243x^5}{120} - \frac{2187x^7}{5040} + \cdots\right)$$

$$y(x) = c_0\frac{\cos 3x}{x} + \frac{1}{3}c_1\frac{\sin 3x}{x}$$

The figure at the top of the next page shows the graphs of the independent solutions $y_1(x) = \dfrac{\cos 3x}{x}$ and $y_2(x) = \dfrac{\sin 3x}{x}$.



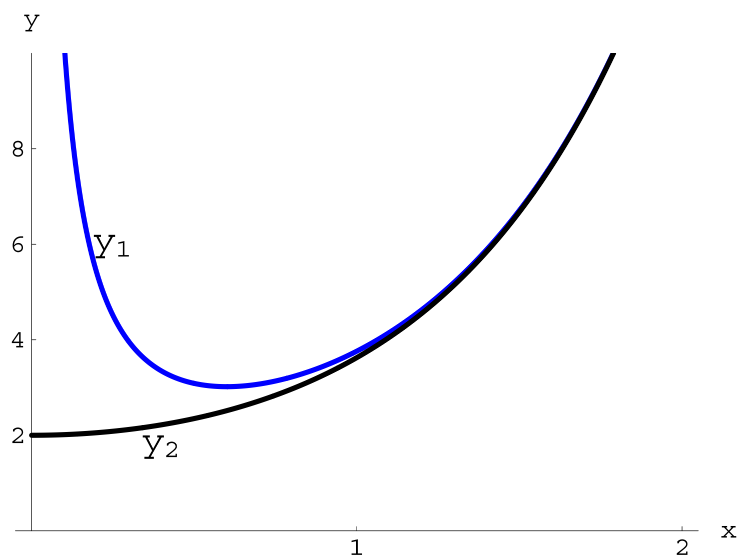
28. Exponents $r_1 = 0$ and $r_2 = -1$; with $r = -1$: $c_n = \dfrac{4c_{n-2}}{n(n-1)}$

$$y(x) = \frac{c_0}{x}\left(1 + 2x^2 + \frac{2x^4}{3} + \frac{4x^6}{45} + \cdots\right) + \frac{c_1}{x}\left(x + \frac{2x^3}{3} + \frac{2x^5}{15} + \frac{4x^7}{315} + \cdots\right)$$

$$= \frac{c_0}{x}\left(1 + \frac{4x^2}{2} + \frac{16x^4}{24} + \frac{96x^6}{720} + \cdots\right) + \frac{c_1}{2x}\left(2x + \frac{8x^3}{6} + \frac{32x^5}{120} + \frac{128x^7}{5040} + \cdots\right)$$

$$y(x) = c_0 \frac{\cosh 2x}{x} + \frac{1}{2}c_1 \frac{\sinh 2x}{x}$$

The figure below shows the graphs of the independent solutions $y_1(x) = \dfrac{\cosh 2x}{x}$ and $y_2(x) = \dfrac{\sinh 2x}{x}$.



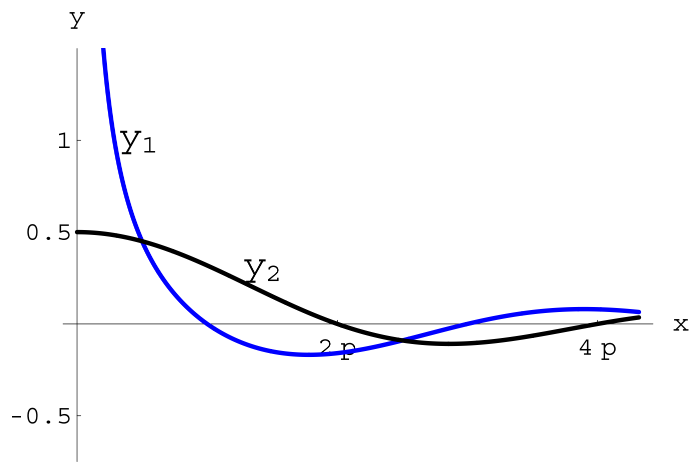
29. Exponents $r_1 = 0$ and $r_2 = -1$; with $r = -1$: $c_n = -\dfrac{c_{n-2}}{4n(n-1)}$

$$y(x) = \frac{c_0}{x}\left(1 - \frac{x^2}{8} + \frac{x^4}{384} - \frac{x^6}{46080} + \cdots\right) + \frac{c_1}{x}\left(x - \frac{x^3}{24} + \frac{x^5}{1920} - \frac{x^7}{322560} + \cdots\right)$$

$$= \frac{c_0}{x}\left(1 - \frac{x^2}{2^2 \cdot 2} + \frac{x^4}{2^4 \cdot 24} - \frac{x^6}{2^6 \cdot 720} + \cdots\right) + \frac{2c_1}{x}\left(\frac{x}{2} - \frac{x^3}{2^3 \cdot 6} + \frac{x^5}{2^5 \cdot 120} - \frac{x^7}{2^7 \cdot 5040} + \cdots\right)$$

$$y(x) = \frac{c_0}{x}\cos\frac{x}{2} + \frac{2c_1}{x}\sin\frac{x}{2}$$

The figure below shows the graphs of the independent solutions
$y_1(x) = \dfrac{\cos x/2}{x}$ and $y_2(x) = \dfrac{\sin x/2}{x}$.



30. The given differential equation $xy'' - y' + 4x^3 y = 0$ has indicial equation $r^2 - 2r = r(r-2) = 0$, so its exponents are $r_1 = 2$ and $r_2 = 0$. Taking $r = 0$, substitution of the power series $y = \sum_{n=0}^{\infty} c_n x^n$ gives

$$-c_1 + 2c_3x^2 + (4c_0 + 8c_4)x^3 + (4c_1 + 15c_5)x^4 + (4c_2 + 24c_6)x^5 +$$
$$(4c_3 + 35c_7)x^6 + (4c_4 + 48c_8)x^7 + (4c_5 + 63c_9)x^8 + \cdots = 0.$$

We see that $c_1 = c_3 = 0$ and

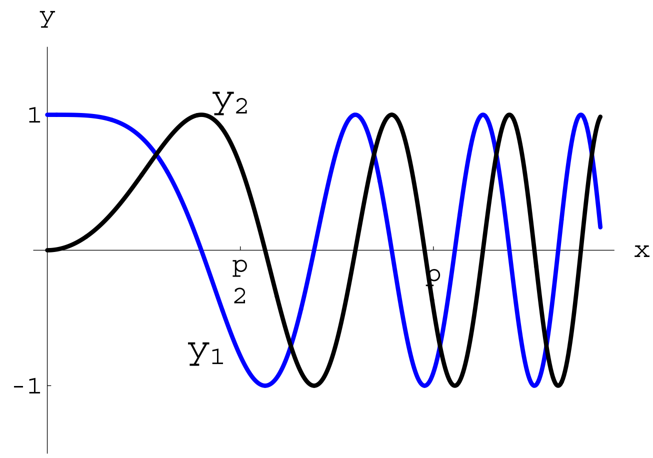
$$c_n = -\frac{4c_{n-4}}{n(n-2)} \quad \text{for} \quad n \geq 4.$$

Hence the odd subscripts all vanish, and we obtain

$$y(x) = c_0 x\left(1 - \frac{x^4}{2} + \frac{x^8}{24} - \frac{x^{12}}{720} + \cdots\right) + c_2\left(x^2 - \frac{x^6}{6} + \frac{x^{10}}{120} - \frac{x^{14}}{5040} + \cdots\right)$$

$$y(x) = c_0 \cos x^2 + c_2 \sin x^2.$$

The figure below shows the graphs of the independent solutions $y_1(x) = \cos x^2$ and $y_2(x) = \sin x^2$.

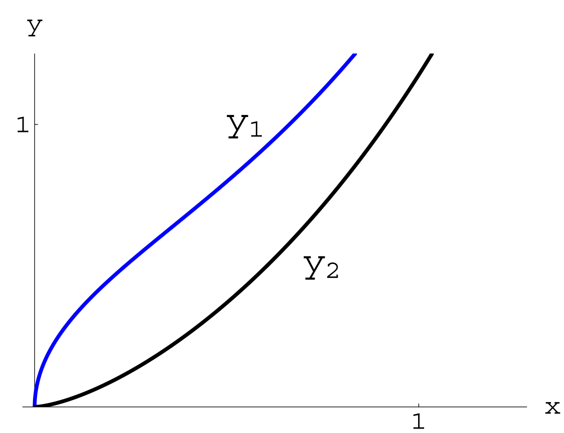


31. The given differential equation $4x^2y'' - 4xy' + (3 - 4x^2)y = 0$ has indicial equation $4r^2 - 8r + 3 = (2r-3)(2r-1) = 0,$ so its exponents are $r_1 = 3/2$ and $r_2 = 1/2.$ With $r = 3/2,$ the recurrence relation $c_n = c_{n-2}/n(n-1)$ yields the general solution

$$y(x) = c_0 x^{1/2}\left(1 + \frac{x^2}{2} + \frac{x^4}{24} + \frac{x^6}{720} + \cdots\right) + c_1 x^{1/2}\left(x + \frac{x^3}{6} + \frac{x^5}{120} + \frac{x^7}{5040} + \cdots\right)$$

$$y(x) = c_0\sqrt{x}\cosh x + c_1\sqrt{x}\sinh x.$$

The figure below shows the graphs of the independent solutions $y_1(x) = \sqrt{x}\cosh x$ and $y_2(x) = \sqrt{x}\sinh x.$



32. The two indicial exponents are $r_1 = 1$ and $r_2 = -1/2$.

With $r_1 = 1$: Substitution of $y = x\sum c_n x^n$ in the differential equation yields

$$(5c_1 - c_0)x^2 + 14c_2x^3 + (c_2 + 27c_3)x^4 + (2c_3 + 44c_4)x^4 + (3c_4 + 65c_6)x^6 + \cdots = 0.$$

Hence we see that $c_1 = c_0/5$ and $c_2 = c_3 = c_4 = c_5 = \cdots 0$. Thus the series terminates and we obtain the polynomial solution

$$y_1(x) = x\left(1 + \frac{x}{5}\right) = x + \frac{x^2}{5}.$$

With $r_2 = -1/2$: We substitute $y = x^{-1/2}\sum c_n x^n$ and obtain the Frobenius solution

$$y_2(x) = \frac{1}{\sqrt{x}}\left(1 - \frac{5x}{2} - \frac{15x^2}{8} - \frac{5x^3}{48} + \frac{x^4}{384} + \cdots\right).$$

Remark: The Mathematica **`DSolve`** function yields the two closed form solutions $y_1(x)$ and

$$y_3(x) = x^{-1/2}e^{-x/2}\left(x^2 + 4x - 2\right) + \sqrt{\frac{\pi}{2}}\, x\left(x+1\right)\operatorname{erf}\sqrt{\frac{x}{2}}.$$

Inquiring minds naturally want to know! The Mathematica **`Series`** command reveals the answer that $y_2(x) = -\frac{1}{2}y_3(x)$.

33. Exponents $r_1 = 1/2$ and $r_2 = -1$. With each exponent we find that c_0 is arbitrary and we can solve recursively for c_n in terms of c_{n-1}.

$$y_1(x) = \sqrt{x}\left(1 + \frac{11x}{20} - \frac{11x^2}{224} + \frac{671x^3}{24192} - \frac{9577x^4}{387072} + \cdots\right)$$

$$y_2(x) = \frac{1}{x}\left(1 + 10x + 5x^2 + \frac{10x^3}{9} - \frac{7x^4}{18} + \cdots\right)$$

34. Exponents $r_1 = 1$ and $r_2 = -1/2$. With each exponent we find that $c_1 = 0$ and we can solve recursively for c_n in terms of c_{n-2}.

$$y_1(x) = x\left(1 - \frac{x^2}{42} + \frac{x^4}{1320} - \frac{37x^6}{2494800} - \cdots\right)$$

$$y_2(x) = \frac{1}{\sqrt{x}}\left(1 - \frac{7x^2}{24} + \frac{19x^4}{3200} - \frac{7661x^6}{43545600} + \cdots\right)$$

35. Substitution of $y = x^r \sum c_n x^n$ into the differential equation yields a result of the form

$$-rc_0 x^{r-1} + (\cdots)x^r + (\cdots)x^{r+1} + \cdots = 0,$$

so we see immediately that $c_0 \neq 0$ implies that $r = 0$. Then substitution of the power series $y = \sum c_n x^n$ yields

$$(c_0 - c_1) + (4c_1 - 2c_2)x + (9c_2 - 3c_3)x^2 + (16c_3 - 4c_4)x^4 + \cdots = 0$$

Evidently $c_n = nc_{n-1}$, so if $c_0 = 1$ it follows that $c_n = n!$ for $n \geq 1$. But the series $\sum n!x^n$ has zero radius of convergence, and hence converges only if $x = 0$. We therefore conclude that the given differential equation has *no* nontrivial Frobenius series solution.

36. **(a)** Substitution of $y = x^r \sum c_n x^n$ into the differential equation $x^2 y'' + Ay' + By = 0$ yields a result of the form

$$Arc_0 x^{r-1} + (\cdots)x^r + (\cdots)x^{r+1} + \cdots = 0,$$

so we see immediately that $A \neq 0$ and $c_0 \neq 0$ imply that $r = 0$.

(b) Substitution of $y = x^r \sum c_n x^n$ into the differential equation $x^3 y'' + Axy' + By = 0$ yields a result of the form

$$(Ar + B)c_0 x^r + (\cdots)x^{r+1} + (\cdots)x^{r+2} + \cdots = 0,$$

so we see immediately that $c_0 \neq 0$ implies that $r = -B/A$.

(c) Substitution of $y = x^r \sum c_n x^n$ into the differential equation $x^3 y'' + Ax^2 y' + By = 0$ yields a result of the form

$$Bc_0 x^r + (\cdots)x^{r+1} + (\cdots)x^{r+2} + \cdots = 0,$$

which is impossible because both $c_0 \neq 0$ and $B \neq 0$. It follows that *no* Frobenius series can satisfy this equation.

37. Substitution of $y = x^r \sum c_n x^n$ into the differential equation $x^3 y'' - y' + y = 0$ yields a result of the form

$$(r-1)^2 c_0 x^r + (\cdots)x^{r+1} + (\cdots)x^{r+2} + \cdots = 0,$$

so it follows that $r = 1$. But then substitution of $y = x \sum c_n x^n$ into the differential equation yields

$$c_1 x^2 + 4c_2 x^3 + 9c_3 x^4 + 16c_4 x^5 + 25c_5 x^6 + \cdots = 0,$$

so it follows that $c_1 = c_2 = c_3 = c_4 = \cdots = 0$. Hence $y(x) = c_0 x$,

38. Exponents $r_1 = 1/2$ and $r_2 = -1/2$; with $r = -1/2$: $c_n = -\dfrac{c_{n-2}}{n(n-1)}$

$$y(x) = \frac{c_0}{\sqrt{x}}\left(1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} + \cdots\right) + \frac{c_1}{\sqrt{x}}\left(x - \frac{x^3}{6} + \frac{x^5}{120} - \frac{x^7}{5040} + \cdots\right)$$

$$= \frac{c_0}{\sqrt{x}}\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\right) + \frac{c_1}{\sqrt{x}}\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\right)$$

$$y(x) = c_0 \frac{\cos x}{\sqrt{x}} + c_1 \frac{\sin 3x}{\sqrt{x}}$$

39. Exponents $r_1 = 1$ and $r_2 = -1$; with $r = +1$: $c_1 = 0$, $c_n = -\dfrac{c_{n-2}}{n(n+2)}$

$$y(x) = c_0 x\left(1 - \frac{x^2}{8} + \frac{x^4}{192} - \frac{x^6}{9216} + \frac{x^8}{737280} - \cdots\right)$$

$$= c_0 x\left(1 - \frac{x^2}{2^2 1!2!} + \frac{x^4}{2^4 2!3!} - \frac{x^6}{2^6 3!4!} + \frac{x^8}{2^8 4!5!} - \cdots\right)$$

If $c_0 = 1/2$, then

$$y(x) = J_1(x) = \frac{x}{2}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!(n+1)}\left(\frac{x}{2}\right)^{2n}.$$

Now, consider the smaller exponent $r_2 = -1$. A Frobenius series with $r = -1$ is of the form $y = x^{-1}\sum_{n=0}^{\infty} c_n x^n$ with $c_0 \neq 0$. However, substitution of this series into Bessel's equation of order 1 gives

$$-c_1 + c_0 x + (c_1 + 3c_3)x^2 + (c_2 + 8c_4)x^3 + (c_3 + 15c_5)x^5 + \cdots = 0,$$

so it follows that $c_0 = 0$, after all. Thus Bessel's equation of order 1 does not have a Frobenius series solution with leading term $c_0 x^{-1}$. However, there is a little more here that meets the eye. We see further that c_2 is arbitrary and that $c_1 = 0$ and $c_n = c_{n-2}/n(n-2)$ for $n > 2$. It follows that our assumed Frobenius series $y = x^{-1}\sum_{n=0}^{\infty} c_n x^n$ actually reduces to

$$y(x) = c_2 x\left(1 - \frac{x^2}{8} + \frac{x^4}{192} - \frac{x^6}{9216} + \frac{x^8}{737280} - \cdots\right).$$

But this is the same as our series solution obtained above using the larger exponent $r = +1$ (calling the arbitrary constant c_2 rather than c_0).

SECTION 11.4

BESSEL'S EQUATION

Of course Bessel's equation is the most important special ordinary differential equation in mathematics, and every student should be exposed at least to Bessel functions of the first kind.

1.
$$\begin{aligned} J_0'(x) &= D_x\left(1+\sum_{m=1}^{\infty}\frac{(-1)^m x^{2m}}{2^{2m}(m!)^2}\right) = \sum_{m=1}^{\infty}\frac{(-1)^m 2m\, x^{2m-1}}{2^{2m}(m!)^2} \\ &= \sum_{m=1}^{\infty}\frac{(-1)^m x^{2m-1}}{2^{2m-1}(m-1)!(m!)} = \sum_{m=0}^{\infty}\frac{(-1)^{m+1} x^{2m+1}}{2^{2m+1}(m)!(m+1!)} \\ &= -\sum_{m=0}^{\infty}\frac{(-1)^m x^{2m+1}}{2^{2m+1}(m)!(m+1!)} = -J_1(x) \end{aligned}$$

2. **(a)**
$$\begin{aligned} \Gamma\left(\frac{2n+1}{2}\right) &= \frac{2n-1}{2}\cdot\Gamma\left(\frac{2n-1}{2}\right) \\ &= \frac{2n-1}{2}\cdot\frac{2n-3}{2}\cdot\Gamma\left(\frac{2n-3}{2}\right) \\ &\vdots \\ &= \frac{2n-1}{2}\cdot\frac{2n-3}{2}\cdot\cdots\cdot\frac{3}{2}\cdot\frac{1}{2}\Gamma\cdot\left(\frac{1}{2}\right) \\ &= \frac{(2n-1)(2n-3)\cdot\cdots\cdot 3\cdot 1}{2^n}\cdot\sqrt{\pi} = \frac{(2n-1)!!}{2^n}\sqrt{\pi} \end{aligned}$$

(b)
$$\begin{aligned} J_{1/2}(x) &= \sum_{m=0}^{\infty}\frac{(-1)^m x^{2m+\frac{1}{2}}}{m!\,\Gamma(m+\frac{3}{2})\,2^{2m+\frac{1}{2}}} = \sqrt{\frac{2}{x}}\sum_{m=0}^{\infty}\frac{(-1)^m x^{2m+1}}{m!\,2^{-m-1}(2m+1)!!\,\sqrt{\pi}\;2^{2m+1}} \\ &= \sqrt{\frac{2}{\pi x}}\sum_{m=0}^{\infty}\frac{(-1)^m x^{2m+1}}{(2\cdot 4\cdot\cdots\cdot 2m)(1\cdot 3\cdot\cdots\cdot(2m+1))} \\ &= \sqrt{\frac{2}{\pi x}}\sum_{m=0}^{\infty}\frac{(-1)^m x^{2m+1}}{(2m+1)!} = \sqrt{\frac{2}{\pi x}}\sin x \end{aligned}$$

$$J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}}\cos x \quad \text{similarly}$$

3. **(a)**
$$\Gamma\left(m+\frac{2}{3}\right) = \Gamma\left(\frac{3m+2}{3}\right) = \frac{3m-1}{3}\cdot\frac{3m-4}{3}\cdot\Gamma\left(\frac{3m-4}{3}\right)$$

$$= \frac{3m-1}{3}\cdot\frac{3m-4}{3}\cdot\dots\cdot\frac{5}{3}\cdot\frac{2}{3}\cdot\Gamma\left(\frac{2}{3}\right) = \frac{2\cdot 5\cdot 8\ \cdots(3m-1)}{3^m}\Gamma\left(\frac{2}{3}\right)$$

(b) $$J_{-1/3}(x) = \sum_{m=0}^{\infty}\frac{(-1)^m}{m!\Gamma(m+2/3)}\left(\frac{x}{2}\right)^{2m-1/3} = \frac{(x/2)^{-1/3}}{\Gamma(2/3)}\sum_{m=0}^{\infty}\frac{(-1)^m 3^m x^{2m}}{m!\cdot 2\cdot 3\cdot 8\cdot\dots\cdot(3m-1)}$$

4. With $p = 1/2$ in Equation (26) in the text we have

$$J_{3/2}(x) = \frac{1}{x}J_{1/2}(x) - J_{-1/2}(x) = \frac{1}{x}\sqrt{\frac{2}{\pi x}}\sin x - \sqrt{\frac{2}{\pi x}}\cos x$$
$$= \frac{1}{x}\sqrt{\frac{2}{\pi x}}\left(\sin x - x\cos x\right) = \sqrt{\frac{2}{\pi x^3}}\left(\sin x - x\cos x\right)$$

5. Starting with $p = 3$ in Equation (26) we get

$$J_4(x) = \frac{6}{x}J_3(x) - J_2(x) = \frac{6}{x}\left[\frac{4}{x}J_2(x) - J_1(x)\right] - J_2(x)$$
$$= \left(\frac{24}{x^2} - 1\right)\left[\frac{2}{x}J_1(x) - J_0(x)\right] - \frac{6}{x}J_1(x)$$
$$= \frac{x^2-24}{x^2}J_0(x) + \frac{8(6-x^2)}{x^3}J_1(x)$$

8. When we carry out the differentiations indicated in Equations (22) and (23) in the text, we get

$$p\,x^{p-1}J_p(x) + x^p J_p'(x) = x^p J_{p-1}(x),$$
$$-p\,x^{-p-1}J_p(x) + x^{-p}J_p'(x) = -x^p J_{p+1}(x).$$

When we solve these two equations for $J_p'(x)$ we get Equations (24) and (25) in the text.

9. $\Gamma(p+m+1) = (p+m)(p+m-1)\cdot\dots\cdot(p+2)(p+1)\Gamma(p+1)$, so

$$J_p(x) = \sum_{m=0}^{\infty}\frac{(-1)^m}{m!\Gamma(p+m+1)}\left(\frac{x}{2}\right)^{2m+p}$$
$$= \frac{(x/2)^p}{\Gamma(p+1)}\sum_{m=0}^{\infty}\frac{(-1)^m}{m!(p+1)(p+2)\cdot\dots\cdot(p+m)}\left(\frac{x}{2}\right)^{2m}.$$

10. Substitution of the power series of Problem 9 yields

$$y(x) = x^2 \cdot \frac{x^{5/2}(A+\cdots)+x^{-5/2}(B+\cdots)}{x^{1/2}(C+\cdots)+x^{-1/2}(D+\cdots)} = \frac{x^5(A+\cdots)+(B+\cdots)}{x(C+\cdots)+(D+\cdots)}$$

where $A = 1/(2^{5/2}\Gamma(7/2))$, $B = 1/(2^{-5/2}\Gamma(-3/2),)$ $C = 1/(2^{1/2}\Gamma(3/2))$, and $D = (1/2^{-1/2})\Gamma(1/2)$. Hence

$$y(0) = \frac{0\cdot(A+\cdots)+(B+\cdots)}{0\cdot(C+\cdots)+(D+\cdots)} = \frac{B}{D} = \frac{2^{-1/2}\Gamma(1/2)}{2^{-5/2}\Gamma(-3/2)} = \frac{2^2\Gamma(1/2)}{(4/3)\Gamma(1/2} = 3.$$

In Problems 11–18 we use a conspicuous dot • to indicate our choice of u and dv in the integration by parts formula $\int u \bullet dv = uv - \int v\,du$. We use repeatedly the facts (from Example 1) that $\int xJ_0(x)\,dx = xJ_1(x)+C$ and $\int J_1(x)\,dx = -J_0(x)+C$.

11.
$$\begin{aligned}\int x^2 J_0(x)\,dx &= \int x \bullet xJ_0(x)\,dx \\ &= x^2J_1(x) - \int x \bullet J_1(x)\,dx \\ &= x^2J_1(x) - \left(-xJ_0(x) + \int J_0(x)\,dx\right) \\ &= x^2J_1(x) + xJ_0(x) - \int J_0(x)\,dx + C\end{aligned}$$

12.
$$\begin{aligned}\int x^3 J_0(x)\,dx &= \int x^2 \bullet xJ_0(x)\,dx \\ &= x^3J_1(x) - 2\int x^2 \bullet J_1(x)\,dx \\ &= x^3J_1(x) - 2\left(-x^2J_0(x) + 2\int xJ_0(x)\,dx\right) \\ &= x^3J_1(x) + 2x^2J_0(x) - 4xJ_1(x) + C = (x^3-4x)J_1(x) + 2x^2J_0(x) + C\end{aligned}$$

13.
$$\begin{aligned}\int x^4 J_0(x)\,dx &= \int x^3 \bullet xJ_0(x)\,dx \\ &= x^4J_1(x) - 3\int x^3 \bullet J_1(x)\,dx \\ &= x^4J_1(x) - 3\left(-x^3J_0(x) + 3\int x \bullet xJ_0(x)\,dx\right) \\ &= x^4J_1(x) + 3x^3J_0(x) - 9\left(x^2J_1(x) - \int x \bullet J_1(x)\,dx\right) \\ &= x^4J_1(x) + 3x^3J_0(x) - 9x^2J_1(x) + 9\left(-xJ_0(x) + \int J_0(x)\,dx\right) \\ &= (x^4-9x^2)J_1(x) + (3x^3-9x)J_0(x) + 9\int J_0(x)\,dx + C\end{aligned}$$

14. $$\int xJ_1(x)\,dx = \int x \bullet J_1(x)\,dx = -xJ_0(x) + \int J_0(x)\,dx + C$$

15. $$\int x^2 J_1(x)\,dx = \int x^2 \bullet J_1(x)\,dx$$

$$= -x^2 J_0(x) + 2\int x J_0(x)\,dx \;=\; -x^2 J_0(x) + 2x J_1(x) + C$$

16.
$$\begin{aligned}
\int x^3 J_1(x)\,dx &= \int x^3 \bullet J_1(x)\,dx \\
&= -x^3 J_0(x) + 3\int x \bullet x J_0(x)\,dx \\
&= -x^3 J_0(x) + 3\left(x^2 J_1(x) - \int x \bullet J_1(x)\,dx\right) \\
&= -x^3 J_0(x) + 3x^2 J_1(x) - 3\left(-x J_0(x) + \int J_0(x)\,dx\right) \\
&= (-x^3 + 3x) J_0(x) + 3x^2 J_1(x) - 3\int J_0(x)\,dx + C
\end{aligned}$$

17.
$$\begin{aligned}
\int x^4 J_1(x)\,dx &= \int x^4 \bullet J_1(x)\,dx \\
&= -x^4 J_0(x) + 4\int x^2 \bullet x J_0(x)\,dx \\
&= -x^4 J_0(x) + 4\left(x^3 J_1(x) - 2\int x^2 \bullet J_1(x)\,dx\right) \\
&= -x^4 J_0(x) + 4x^3 J_1(x) - 8\left(-x^2 J_0(x) + 2\int x J_0(x)\,dx\right) \\
&= (-x^4 + 8x^2) J_0(x) + (4x^3 - 16x) J_1(x) + C
\end{aligned}$$

18. With $p = 1$, Eq. (23) in the text gives $\int x^{-1} J_2(x)\,dx = -x^{-1} J_1(x) + C$. Hence

$$\begin{aligned}
\int J_2(x)\,dx &= \int x \bullet x^{-1} J_2(x)\,dx \\
&= x\left(-x^{-1} J_1(x)\right) + \int x^{-1} J_1(x) \;=\; -J_1(x) + \int x^{-1} J_1(x)\,dx.
\end{aligned}$$

But Eq. (26) with $p = 1$ gives $x^{-1} J_1(x) = \tfrac{1}{2}\left[J_0(x) + J_2(x)\right]$, so

$$\int J_2(x)\,dx = -J_1(x) + \tfrac{1}{2}\int J_0(x)\,dx + \tfrac{1}{2}\int J_2(x)\,dx.$$

Finally, we can solve this last equation for

$$\int J_2(x)\,dx = -2J_1(x) + \int J_0(x)\,dx + C.$$

Problems 19–30 are routine applications of the theorem in this section. In each case it is necessary only to identify the coefficients A, B, C and the exponent q in the differential equation

$$x^2 y'' + Axy' + (B + Cx^q)y = 0. \tag{1}$$

Then we can calculate the values

$$\alpha = \frac{1-A}{2}, \quad \beta = \frac{q}{2}, \quad k = \frac{2\sqrt{C}}{q}, \quad p = \frac{\sqrt{(1-A)^2 - 4B}}{q} \tag{2}$$

and finally write the general solution

$$y(x) \;=\; x^{\alpha}\left[c_1 J_p(kx^{\beta}) + c_2 J_{-p}(kx^{\beta})\right] \qquad (3)$$

specified in Theorem 1 on solutions in terms of Bessel functions. This is a "template procedure" that we illustrate only in a couple of problems.

19. We have $A = -1, B = 1, C = 1, q = 2$ so

$$\alpha = \frac{1-(-1)}{2} = 1, \quad \beta = \frac{2}{2} = 1, \quad k = \frac{2\sqrt{1}}{2} = 1, \quad p = \frac{\sqrt{(1-(-1))^2 - 4(1)}}{2} = 0,$$

so our general solution is $y(x) \;=\; x[c_1 J_0(x) + c_2 Y_0(x)]$, using $Y_0(x)$ because $p = 0$ is an integer.

20. $y(x) \;=\; x^{-1}[c_1 J_1(x) + c_2 Y_1(x)]$

21. $y(x) \;=\; x[c_1 J_{1/2}(3x^2) + c_2 J_{-1/2}(3x^2)]$

22. $y(x) \;=\; x^3[c_1 J_2(2x^{1/2}) + c_2 Y_2(2x^{1/2})]$

23. To match the given equation with Eq. (1) above, we first divide through by the leading coefficient 16 to obtain the equation

$$x^2 y'' + \frac{5}{3}xy' + \left(-\frac{5}{36} + \frac{1}{4}x^3\right)y \;=\; 0$$

with $A = 5/3, B = -5/36, C = 1/4$, and $q = 3$. Then

$$\alpha = \frac{1-5/3}{3} = -\frac{1}{3}, \quad \beta = \frac{3}{2}, \quad k = \frac{2\sqrt{1/4}}{3} = \frac{1}{3}, \quad p = \frac{\sqrt{(1-5/3)^2 - 4(-5/36)}}{3} \;=\; \frac{1}{3},$$

so our general solution is $y(x) \;=\; x^{-1/3}[c_1 J_{1/3}(x^{3/2}/3) + c_2 J_{-1/3}(x^{3/2}/3)]$.

24. $y(x) \;=\; x^{-1/4}[c_1 J_0(2x^{3/2}) + c_2 Y_0(2x^{3/2})]$

25. $y(x) \;=\; x^{-1}[c_1 J_0(x) + c_2 Y_0(x)]$

26. $y(x) \;=\; x^2[c_1 J_1(4x^{1/2}) + c_2 Y_1(4x^{1/2})]$

27. $y(x) \;=\; x^{1/2}[c_1 J_{1/2}(2x^{3/2}) + c_2 J_{-1/2}(2x^{3/2})]$

28. $y(x) = x^{-1/4}[c_1 J_{3/2}(2x^{5/2}/5) + c_2 J_{-3/2}(2x^{5/2}/5)]$

29. $y(x) = x^{1/2}[c_1 J_{1/6}(x^3/3) + c_2 J_{-1/6}(x^3/3)]$

30. $y(x) = x^{1/2}[c_1 J_{1/5}(4x^{5/2}/5) + c_2 J_{-1/5}(4x^{5/2}/5)]$

31. We want to solve the equation $xy'' + 2y' + xy = 0$. If we rewrite it as

$$x^2 y'' + 2xy' + x^2 y = 0$$

then we have the form in Equation (1) with $A = 2$, $B = 0$, $C = 1$, and $q = 2$. Then Equation (2) gives $\alpha = -1/2$, $\beta = 1$, $k = 1$, and $p = 1/2$, so by Equation (3) the general solution is

$$\begin{aligned} y(x) &= x^{-1/2}\left[c_1 J_{1/2}(x) + c_1 J_{-1/2}(x)\right] \\ &= x^{-1/2}\left[c_1\sqrt{\frac{2}{\pi x}}\cos x + c_2\sqrt{\frac{2}{\pi x}}\sin x\right] \\ &= \frac{1}{x}\left(a_1 \cos x + a_2 \sin x\right) \end{aligned}$$

(with $a_i = c_i\sqrt{2/\pi}$), using Equations (19) in Section 11.4.

33. The substitution

$$y = -\frac{u'}{u}, \qquad y' = \frac{(u')^2}{u^2} - \frac{u''}{u}$$

immediately transforms $y' = x^2 + y^2$ to $u'' + x^2 u = 0$. The equivalent equation

$$x^2 u'' + x^4 u = 0$$

is of the form in (1) with $A = B = 0$, $C = 1$, and $q = 4$. Equations (2) give $\alpha = 1/2$, $\beta = 2$, $k = 1/2$, and $p = 1/4$, so the general solution is

$$u(x) = x^{1/2}[c_1 J_{1/4}(x^2/2) + c_2 J_{-1/4}(x^2/2)].$$

To compute $u'(x)$, let $z = x^2/2$ so $x = 2^{1/2}z^{1/2}$. Then Equation (22) in Section 11.4 with $p = 1/4$ yields

$$\frac{d}{dx}\left(x^{1/2}J_{1/4}(x^2/2)\right) = \frac{d}{dz}\left(2^{1/4}z^{1/4}J_{1/4}(z)\right)\cdot\frac{dz}{dx}$$

$$= 2^{1/4} z^{1/4} J_{-3/4}(z) \cdot \frac{dz}{dx}$$

$$= 2^{1/4} \cdot \frac{x^{1/2}}{2^{1/4}} J_{-3/4}(x^2/2) \cdot x \;=\; x^{3/2} J_{-3/4}(x^2/2).$$

Similarly, Equation (23) in Section 11.4 with $p = -1/4$ yields

$$\frac{d}{dx}\left(x^{1/2} J_{-1/4}(x^2/2)\right) = \frac{d}{dz}\left(2^{1/4} z^{1/4} J_{-1/4}(z)\right) \cdot \frac{dz}{dx} = -x^{3/2} J_{3/4}(x^2/2).$$

Therefore

$$u'(x) = x^{3/2}[c_1 J_{-3/4}(x^2/2) - c_2 J_{3/4}(x^2/2)].$$

It follows finally that the general solution of the Riccati equation $y' = x^2 + y^2$ is

$$y(x) = -\frac{u'}{u} = x \cdot \frac{J_{3/4}(\frac{1}{2}x^2) - c\,J_{-3/4}(\frac{1}{2}x^2)}{c\,J_{1/4}(\frac{1}{2}x^2) + J_{-1/4}(\frac{1}{2}x^2)}$$

where the arbitrary constant is $c = c_1/c_2$.

34. Substitution of the series expressions for the Bessel functions in the formula for $y(x)$ in Problem 33 yields

$$y(x) = x \cdot \frac{A\left(\frac{1}{2}x^2\right)^{3/4}(1+\cdots) - c\,B\left(\frac{1}{2}x^2\right)^{-3/4}(1+\cdots)}{c\,C\left(\frac{1}{2}x^2\right)^{1/4}(1+\cdots) + D\left(\frac{1}{2}x^2\right)^{-1/4}(1+\cdots)}$$

where each pair of parentheses encloses a power series in x with constant term 1, and

$$A = 2^{-3/4}/\Gamma(7/4) \qquad B = 2^{3/4}/\Gamma(1/4)$$

$$C = 2^{-1/4}/\Gamma(5/4) \qquad D = 2^{1/4}/\Gamma(3/4).$$

Multiplication of numerator and denominator by $x^{1/2}$ and a bit of simplification gives

$$y(x) = \frac{2^{-3/4} A x^3 (1+\cdots) - 2^{3/4} c\,B(1+\cdots)}{2^{-1/4} c\,C x(1+\cdots) + 2^{1/4} D(1+\cdots)}.$$

It now follows that

$$y(0) = \frac{-2^{3/4} cB}{2^{1/4} D} = \frac{-2^{1/2}\left(2^{3/4}/\Gamma(1/4)\right)}{2^{1/4}/\Gamma(3/4)} = -2c \cdot \frac{\Gamma(3/4)}{\Gamma(1/4)}. \qquad (*)$$

(a) If $y(0) = 0$ then (*) gives $c = 0$ in the general solution formula of Problem 33.

(b) If $y(0) = 1$ then (*) gives $c = -\Gamma(1/4)/2\Gamma(3/4)$. More generally, (*) yields the formula

$$y(x) = x \cdot \frac{2\Gamma\left(\frac{3}{4}\right)J_{3/4}\left(\frac{1}{2}x^2\right) + y_0\,\Gamma\left(\frac{1}{4}\right)J_{-3/4}\left(\frac{1}{2}x^2\right)}{2\Gamma\left(\frac{3}{4}\right)J_{-1/4}\left(\frac{1}{2}x^2\right) - y_0\,\Gamma\left(\frac{1}{4}\right)J_{1/4}\left(\frac{1}{2}x^2\right)}$$

for the solution of the initial value problem

$$y' = x^2 + y^2, \qquad y(0) = y_0.$$

APPENDIX A

EXISTENCE AND UNIQUENESS OF SOLUTIONS

In Problems 1–12 we apply the iterative formula

$$y_{n+1} = b + \int_a^x f(t, y_n(t))\, dt$$

to compute successive approximations $\{y_n(x)\}$ to the solution of the initial value problem

$$y' = f(x, y), \qquad y(a) = b.$$

starting with $y_0(x) = b$.

1.
$y_0(x) = 3$
$y_1(x) = 3 + 3x$
$y_2(x) = 3 + 3x + 3x^2/2$
$y_3(x) = 3 + 3x + 3x^2/2 + x^3/2$
$y_4(x) = 3 + 3x + 3x^2/2 + x^3/2 + x^4/8$
$y(x) = 3 - 3x + 3x^2/2 + x^3/2 + x^4/8 + \cdots = 3e^x$

2.
$y_0(x) = 4$
$y_1(x) = 4 - 8x$
$y_2(x) = 4 - 8x + 8x^2$
$y_3(x) = 4 - 8x + 8x^2 - (16/3)x^3$
$y_4(x) = 4 - 8x + 8x^2 - (16/3)x^3 + (8/3)x^4$
$y(x) = 4 - 8x + 8x^2 - (16/3)x^3 + (8/3)x^4 - \cdots = 4e^{-2x}$

3.
$y_0(x) = 1$
$y_1(x) = 1 - x^2$
$y_2(x) = 1 - x^2 + x^4/2$
$y_3(x) = 1 - x^2 + x^4/2 - x^6/6$
$y_4(x) = 1 - x^2 + x^4/2 - x^6/6 + x^8/24$

$$y(x) = 1 - x^2 + x^4/2 - x^6/6 + x^8/24 - \cdots = \exp(-x^2)$$

4.

$$y_0(x) = 2$$
$$y_1(x) = 2 + 2x^3$$
$$y_2(x) = 2 + 2x^3 + x^6$$
$$y_3(x) = 2 + 2x^3 + x^6 + (1/3)x^9$$
$$y_4(x) = 2 + 2x^3 + x^6 + (1/3)x^9 + (1/12)x^{12}$$
$$y(x) = 2 + 2x^3 + x^6 + (1/3)x^9 + (1/12)x^{12} + \cdots = 2\exp(x^3)$$

5.

$$y_0(x) = 0$$
$$y_1(x) = 2x$$
$$y_2(x) = 2x + 2x^2$$
$$y_3(x) = 2x + 2x^2 + 4x^3/3$$
$$y_4(x) = 2x + 2x^2 + 4x^3/3 + 2x^4/3$$
$$y(x) = 2x + 2x^2 + 4x^3/3 + 2x^4/3 + \cdots = e^{2x} - 1$$

6.

$$y_0(x) = 0$$
$$y_1(x) = (1/2)x^2$$
$$y_2(x) = (1/2)x^2 + (1/6)x^3$$
$$y_3(x) = (1/2)x^2 + (1/6)x^3 + (1/24)x^4$$
$$y_4(x) = (1/2)x^2 + (1/6)x^3 + (1/24)x^4 + (1/120)x^5$$
$$y(x) = (1/2!)x^2 + (1/3!)x^3 + (1/4!)x^4 + (1/5!)x^5 + \cdots = e^x - x - 1$$

7.

$$y_0(x) = 0$$
$$y_1(x) = x^2$$
$$y_2(x) = x^2 + x^4/2$$
$$y_3(x) = x^2 + x^4/2 + x^6/6$$
$$y_4(x) = x^2 + x^4/2 + x^6/6 + x^8/24$$
$$y(x) = x^2 + x^4/2 + x^6/6 + x^8/24 + \cdots = \exp(x^2) - 1$$

8.
$$\begin{aligned}
y_0(x) &= 0\\
y_1(x) &= 2x^4\\
y_2(x) &= 2x^4 + (4/3)x^6\\
y_3(x) &= 2x^4 + (4/3)x^6 + (2/3)x^8\\
y_4(x) &= 2x^4 + (4/3)x^6 + (2/3)x^8 + (4/15)x^{10}\\
y(x) &= 2x^4 + (4/3)x^6 + (2/3)x^8 + (4/15)x^{10} + \cdots = \exp(2x^2) - 2x^2 - 1
\end{aligned}$$

9.
$$\begin{aligned}
y_0(x) &= 1\\
y_1(x) &= (1 + x) + x^2/2\\
y_2(x) &= (1 + x + x^2) + x^3/6\\
y_3(x) &= (1 + x + x^2 + x^3/3) + x^4/24\\
y(x) &= 1 + x + x^2 + x^3/3 + x^4/12 + \cdots = 2e^x - 1 - x
\end{aligned}$$

10.
$$\begin{aligned}
y_0(x) &= 0\\
y_1(x) &= x + (1/2)x^2 + (1/6)x^3 + (1/24)x^4 + \cdots = e^x - 1\\
y_2(x) &= x + x^2 + (1/3)x^3 + (1/12)x^4 + \cdots = 2e^x - x - 2\\
y_3(x) &= x + x^2 + (1/2)x^3 + (1/8)x^4 + \cdots = 3e^x - (1/2)x^2 - 2x - 3\\
y(x) &= x + x^2 + (1/2)x^3 + (1/6)x^4 + \cdots = xe^x
\end{aligned}$$

11.
$$\begin{aligned}
y_0(x) &= 1\\
y_1(x) &= 1 + x\\
y_2(x) &= (1 + x + x^2) + x^3/3\\
y_3(x) &= (1 + x + x^2 + x^3) + 2x^4/3 + x^5/3 + x^6/9 + x^7/63\\
y(x) &= 1 + x + x^2 + x^3 + x^4 + \cdots = 1/(1 - x)
\end{aligned}$$

12.
$$\begin{aligned}
y_0(x) &= 1\\
y_1(x) &= 1 + (1/2)x\\
y_2(x) &= 1 + (1/2)x + (3/8)x^2 + (1/8)x^3 + (1/64)x^4\\
y_3(x) &= 1 + (1/2)x + (3/8)x^2 + (5/16)x^3 + (13/64)x^4 + \cdots\\
y(x) &= 1 + (1/2)x + (3/8)x^2 + (5/16)x^3 + (35/128)x^4 + \cdots = (1 - x)^{-1/2}
\end{aligned}$$

13. $\begin{bmatrix} x_0(t) \\ y_0(t) \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$\begin{bmatrix} x_1(t) \\ y_1(t) \end{bmatrix} = \begin{bmatrix} 1+3t \\ -1+5t \end{bmatrix}$

$\begin{bmatrix} x_2(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} 1+3t+\frac{1}{2}t^2 \\ -1+5t-\frac{1}{2}t^2 \end{bmatrix}$

$\begin{bmatrix} x_3(t) \\ y_3(t) \end{bmatrix} = \begin{bmatrix} 1+3t+\frac{1}{2}t^2+\frac{1}{3}t^3 \\ -1+5t-\frac{1}{2}t^2+\frac{5}{6}t^3 \end{bmatrix}$

14.
$$\begin{aligned}
\mathbf{x}(t) &= \left[\sum_{n=0}^{\infty}\frac{1}{n!}\begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix} t^n\right]\begin{bmatrix} 1 \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} \displaystyle\sum_{n=0}^{\infty}\frac{t^n}{n!} & \displaystyle\sum_{n=1}^{\infty}\frac{t^n}{(n-1)!} \\ 0 & \displaystyle\sum_{n=0}^{\infty}\frac{t^n}{n!} \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} e^t & t\,e^t \\ 0 & e^t \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} \\
\mathbf{x}(t) &= \begin{bmatrix} (1+t)e^t \\ e^t \end{bmatrix}
\end{aligned}$$

16.
$$\begin{aligned}
y_0(x) &= 0 \\
y_1(x) &= (1/3)x^3 \\
y_2(x) &= (1/3)x^3 + (1/63)x^7 \\
y_3(x) &= (1/3)x^3 + (1/63)x^7 + (2/2079)x^{11} + (1/59535)x^{15}
\end{aligned}$$

Then $y_3(1) \approx 0.350185$, which differs by only 0.0134% from the Runge-Kutta approximation $y(1) \approx 0.350232$. As a denouement we may recall from the result of Problem 16 in Section 8.6 that the exact solution of our initial value problem here is

$$y(x) = x \cdot \frac{J_{3/4}\left(\frac{1}{2}x^2\right)}{J_{-1/4}\left(\frac{1}{2}x^2\right)}$$

so the exact value at $x = 1$ is

$$y(1) = \frac{J_{3/4}\left(\frac{1}{2}\right)}{J_{-1/4}\left(\frac{1}{2}\right)} \approx 0.35023\ 18443.$$